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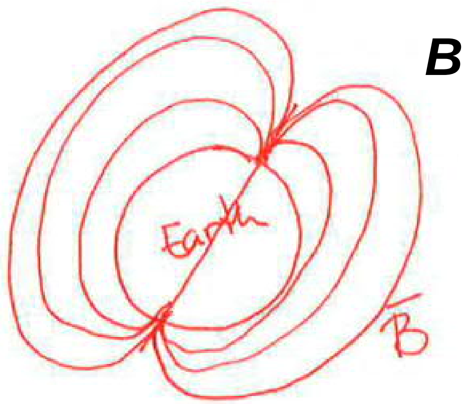
Lecture 2:
Plasma particles with E and B fields

Today’s Menu

- Magnetized plasma & Larmor radius
- Plasma’s diamagnetism
- Charged particle in a multitude of EM fields: ***drift motion***
  - ExB drift, gradient drift, (later: curvature drift, polarization drift, ...)
- Concept of a *guiding center*
- Magnetic moment
- Magnetic mirror & Loss cone
- Adiabatic invariants 1, 2 ,3 and their usefulness

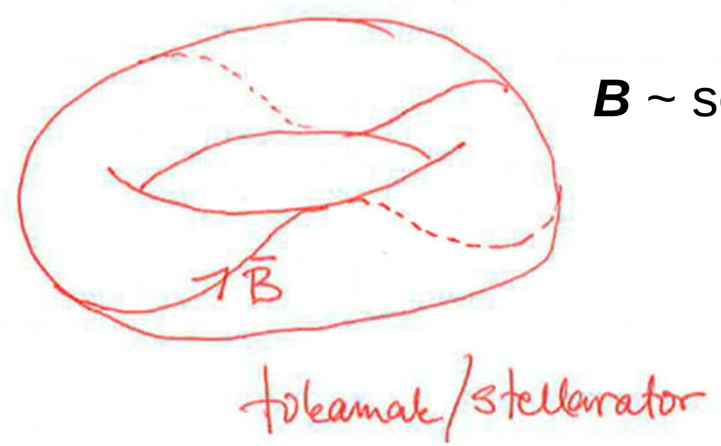
Plasmas of interest

Not only are the plasmas of our interest (space & fusion) weakly coupled, they are also ***magnetized*** … Why?



B ~ tens of *μT*

Earth has its own magnetic field that, in the first approximation, can be considered a *dipole field.*



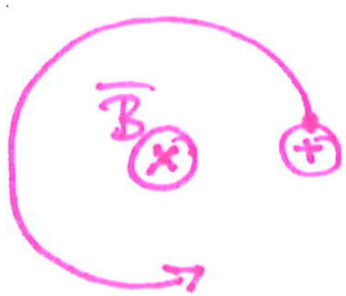
B ~ several *T*

In fusion energy research, the VERY hot plasma is kept away from the vessel walls by magnetic field.

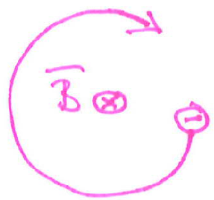
Charged particles in magnetic field

Consider a charge particle (q,m) in uniform magnetic field, $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$.

Lorentz force: $m\frac{d\boldsymbol{v}}{dt} = q\boldsymbol{v} \times \boldsymbol{B}$



$$m\frac{dv_x}{dt} = qv_yB_0$$
$$m\frac{dv_y}{dt} = -qv_xB_0$$
$$m\frac{dv_z}{dt} = 0$$



Collect the constants into $\Omega \equiv qB_0/m$, ***Larmor/cyclotron frequency***

HW ➔ $v_x = v_\perp \sin \Omega t$ with $v_y = v_\perp \cos \Omega t$ (or vice versa), $v_z = v_\parallel$

Larmor motion …

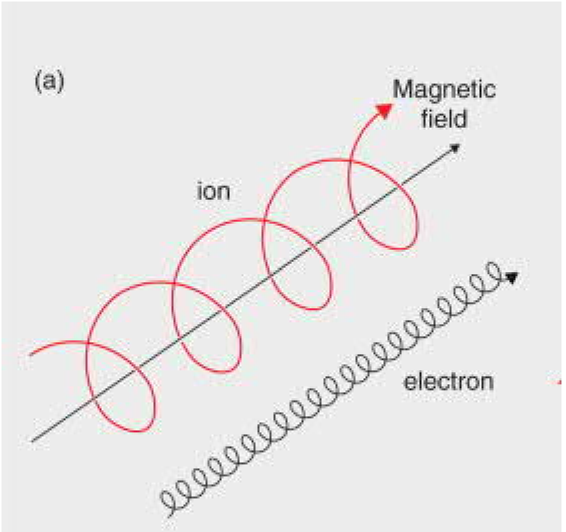
Integrate in time (HW) ➔ $x = \frac{v_\perp}{\Omega} \sin \Omega t$ & y = $-\frac{v_\perp}{\Omega} \cos \Omega t$

➔ charged particles are *gyrating* around the magnetic field line on a circle with the radius defined by their perpendicular velocity and magnetic field strength:

Larmor radius: $r_L = \frac{m v_\perp}{qB}$

Notice rightaway (effects one-by-one):

- Strong field ➔ stick close to field line
- Big charge number ➔ stick close to field line
- Large perpendicular velocity ➔ large gyro radius
- Large mass ➔ large excursions from the field line

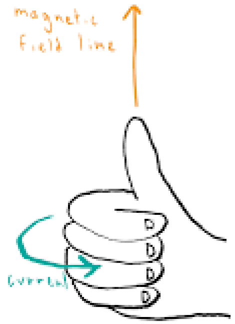


From Science Direct

… and diamagnetism

Particles in plasma thus carry out circular motion around field lines.

A charged particle on a circular path forms a *current ring* .



Ampere's law

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$$ … recall your course in EM

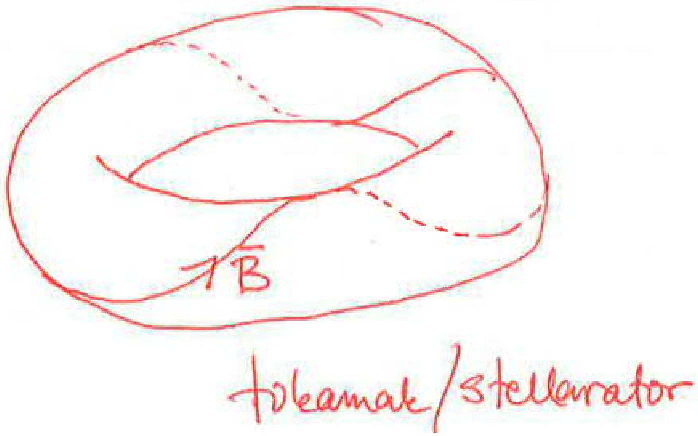
➔ additional magnetic field *opposite* to the background field

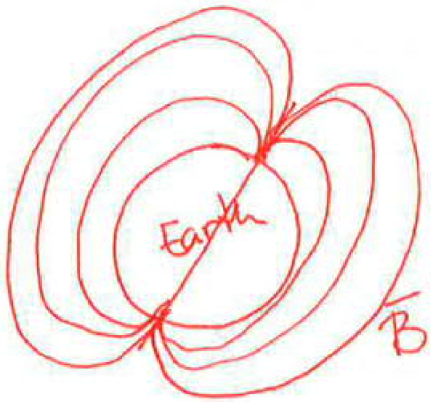
➔ A plasma is ***diamagnetic*** *(… except in some special cases…)*, *i.e.*, tends to *reduce* the imposed magnetic field

Concept of *magnetized* plasma

A plasma is considered ***magnetized*** if the Larmor radius is much much smaller than the *scale length L* over which the magnetic field changes appreciably.

$r_L << L$





Note: not exactly uniform B fields…

Charged particle motion in simple or 'simplish' fields

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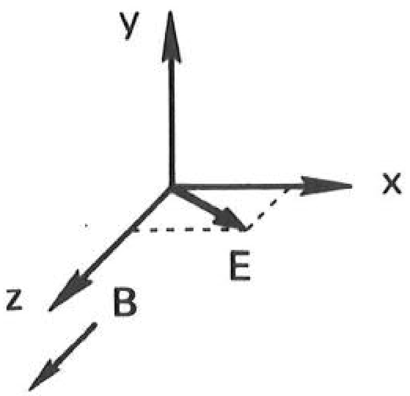
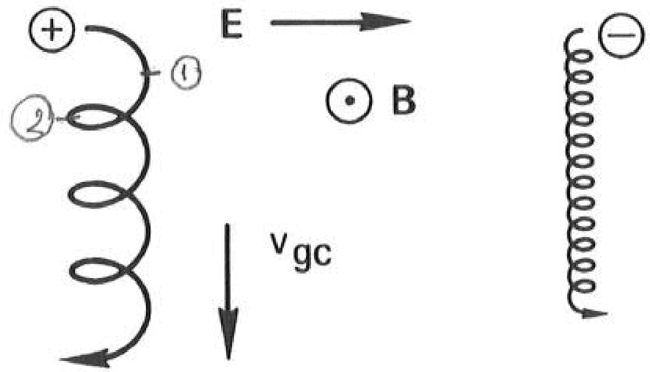
Add a uniform electric field, *E* = *E₀*

$\boldsymbol{E} = E_0\,\hat{\boldsymbol{z}}$ ➔ simply acceleration in the direction of ***B***

Take ***E*** perpendicular to ***B***, e.g., $\boldsymbol{E} = E_0\hat{\boldsymbol{x}}$

Think what happens now during the gyration period …





Can this be true?

Particle seems to move in direction *perpendicular to both E and B fields!!!*

Do the math …

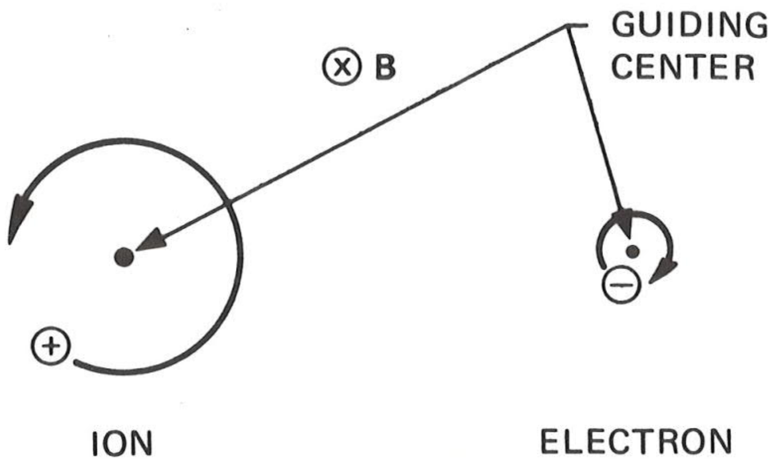
Equations of motion: $$\frac{dv_x}{dt} = \Omega v_y + \frac{qE_0}{m}$$

$$\frac{dv_y}{dt} = -\Omega v_x$$

HW …

➔ $v_x = v_\perp \sin \Omega t$

$v_y = v_\perp \cos \Omega t + \frac{E_0}{B_0}$



Indeed, the particle *drifts* perpendicular to both fields!

Useful concept: the 'center of gyro motion', the ***guiding center***, drifts.

The *ExB* drift

This guiding-center drift is called the $\boldsymbol{E} \times \boldsymbol{B}$ ***drift*** and it has a very important role especially in fusion plasma physics.

General (vector) form: $\boldsymbol{v}_{ExB} = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}$

Things to notice:

- The drift does not depend on the particle – everybody drifts in the same direction with the same velocity!
- This drift is not really specific to just electric field. Any external force, E ➔ F/q, would cause such a drift – but this time depending on the charge!
- e.g., gravitational force

Charged particle motion in non-uniform magnetic field

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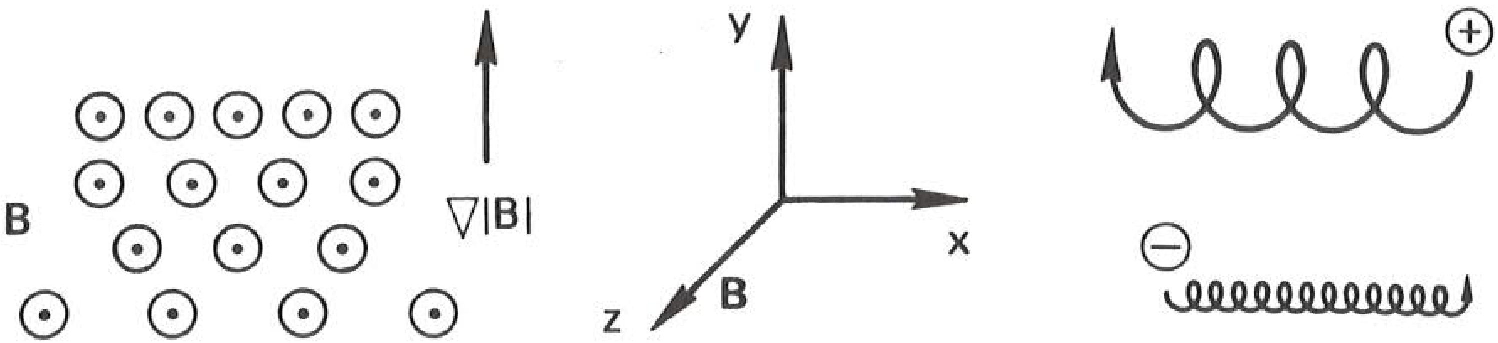
Part I: $\nabla B \perp \boldsymbol{B} = B_0\,\hat{z}$

Choose the axes so that ∇B ∥ $\hat{y}$

What happens now during one gyration period …



The particle is moving (= *drifting*) in direction *perpendicular to both the B field and its gradient!!!*

Do the math …

Taylor expand the magnetic field remembering that $r_L \ll L$

$$B_z = B_o + y\frac{\partial B_z}{\partial y} + \ldots$$

$$F_y = -qv_x B_z(y) \approx -qv_\perp(\sin\Omega t)\left[B_o + r_L(\sin\Omega t)\frac{\partial B_z}{\partial y}\right]$$

where we have also used the *unperturbed* orbit to evaluate the force.

Why? -- Ω gives the shortest time scale ➔ average over one gyro period

$< \sin\Omega t > = 0$, $< (\sin\Omega t)^2 > = \frac{1}{2}$ ➔ $< F_y > = \pm\frac{1}{2}qv_\perp r_L\frac{\partial B_z}{\partial y}$

The gradient drift

So there is an effective net *force* on the particle

➔ obtain GC drift from the generalized *ExB drift:*

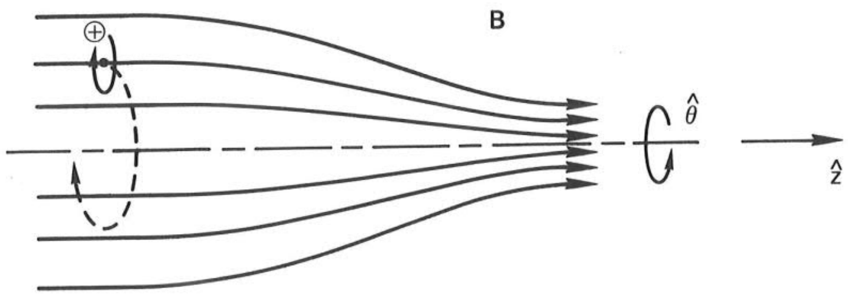
$$v_{GC} = \frac{1}{q}\frac{\boldsymbol{F} \times \boldsymbol{B}}{B^2} = \frac{1}{q}\frac{F_y}{B_0}\hat{x} = \pm\frac{1}{2B_0}v_\perp r_L \frac{\partial B_z}{\partial y}$$

➔ The ***gradient drift*** (∇B-drift) in general vector form

$$\boldsymbol{v}_{\nabla B} = \pm\frac{1}{2}v_\perp r_L \frac{\boldsymbol{B}\times\nabla B}{B^2}$$

This drift *does* depend on the charge, as indicated by the ± sign

Part II: $\nabla B \parallel \mathbf{B} = B_0 \mathbf{z}$



For axial B-field to have parallel gradient means that the field must have also a *radial* component. It can be obtained from $\nabla \cdot \boldsymbol{B} = 0$:

Cylindrical symmetry ➔ cylindrical coordinates: $\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$

Assume *slowly varying* magnetic field ➔

$$rB_r = -\int_0^r r\frac{\partial B_z}{\partial z}dr \approx -\frac{1}{2}r^2\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \Rightarrow B_r \approx -\frac{1}{2}r\left[\frac{\partial B_z}{\partial z}\right]_{r=0}$$

Non-uniformity in r ➔ gradient drift in *poloidal direction.* No problem. ☺

(Radial drift would require non-uniformity in *poloidal* direction)

Full Lorentz force in cylindrical coordinates

$$F_r = q v_\theta B_z$$
$$F_\theta = q(v_z B_r - v_r B_z)$$
$$F_z = -q v_\theta B_r$$

Gyro motion around the fieldline

- The 1st term in F_θ causes a radial drift that forces the particle to follow the bending field lines
- The new physics is brought about by F_z.
- For simplicity, study a particle "on" the axis, $r_{GC} = 0$:

$$F_z = -q v_\perp \frac{1}{2} r_L \left[\frac{\partial B_z}{\partial z}\right]_{r=0}$$

Magnetic force along the field …

$r_L = mv_\perp / qB$ ➔ $F_z = -\frac{1}{2}\frac{mv_\perp^2}{B}\left[\frac{\partial B_z}{\partial z}\right] = -\mu\left[\frac{\partial B_z}{\partial z}\right]$

where $\mu \equiv \frac{1}{2}\frac{mv_\perp^2}{B}$ is the so-called ***magnetic moment*** of the particle.

General (vector) form: $\boldsymbol{F}_\parallel = -\mu \nabla_\parallel B$

Note:

- *μ* can be understood as the magnetic moment due to the current loop created by the gyrating particle (HW)
- The force causes a braking action when particle moves towards higher field …

Now we have a bunch of drifts…
What next?

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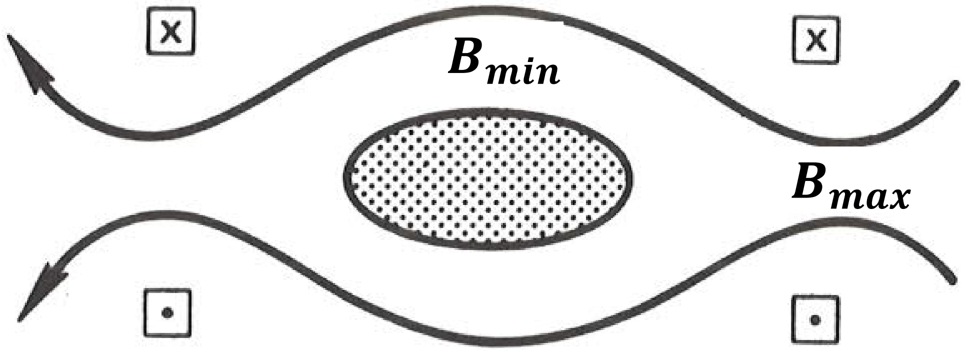
Magnetic mirrors …

”Magnetic bottle”: first attempt to magnetic confinement …

Linear device ➔ $\boldsymbol{B} \approx B(\mathrm{z})\hat{z}$ …

➔ $m\frac{dv_{\parallel}}{dt} = -\mu\frac{\partial B}{\partial s}$ *s = distance along a field line*



Multiply by $v_{\parallel} = \frac{ds}{dt}$

➔ $\frac{m}{2}\frac{d}{dt}\left(v_{\parallel}^{2}\right) = -\mu\frac{\partial B}{\partial s}\frac{\partial s}{\partial t} = -\mu\frac{dB}{dt}$

Note: B does not depend on time, but a *particle* sees it varying ’in time’.

… and invariance of μ

➔ $\frac{d}{dt}\left(\frac{1}{2}mv_{\parallel}^2 + \mu B\right) = B\frac{d\mu}{dt}$

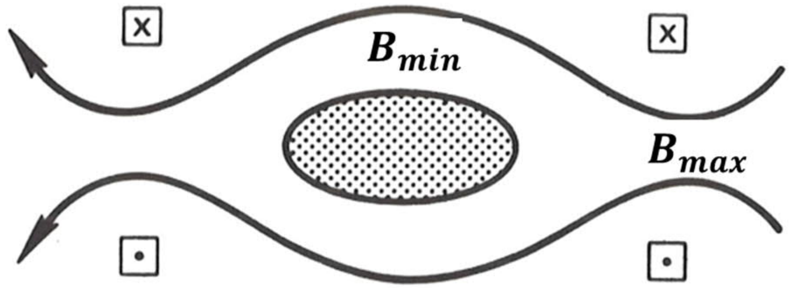
Recall the definition: $\mu \equiv \frac{1}{2}\frac{mv_{\perp}^2}{B}$ ➔ $\frac{1}{2}mv_{\perp}^2 = \mu B$

➔ $E_{tot} = \frac{1}{2}mv_{\parallel}^2 + \mu B$

Total energy is conserved: $\frac{dE_{tot}}{dt} = 0$

➔ $\frac{d\mu}{dt} = 0$ *The magnetic moment is an (adiabatic) invariant !!!*

In the house of mirrors …



$$\mu \equiv \frac{1}{2}\frac{mv_\perp^2}{B} = \text{constant}$$

So what happens if the particle moves to a region with increasing B?

- Perpendicular energy must increase …
- Total energy conserved ➔ $v_\parallel$ must decrease
- B_{max} high enough ➔ Larmor motion eats up all $v_\parallel$ ➔ particle stops
- Now $\boldsymbol{F}_\parallel = -\mu\nabla_\parallel B$ kicks in ➔ particle gets reflected

➔ particle gets trapped in the mirror = particle is *confined*!

This was the idea behind the magnetic bottle.

Magnetic bottle is not plasma-tight…

But we do not get fusion electrons out of our electrical outlets. Why?

There was an 'if' above: **if** B_{max} high enough … What is 'high enough'?

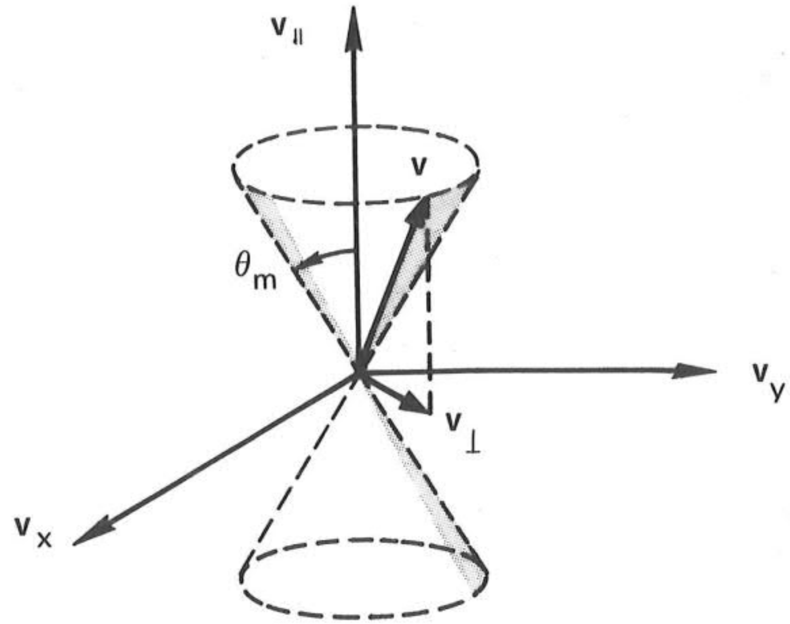
- Let $v_{\parallel,0}$ & $v_{\perp,0}$ correspond to the mid-bottle, i.e., where $B = B_{min}$
- At the (potential) turning point, $B = B_{max}$: $v_{\parallel} = 0$ & $v_{\perp} = v'_{\perp}$
- μ = constant ➔ $\frac{v_{\perp,0}^2}{B_{min}} = \frac{v_{\perp}'^2}{B_{max}}$
- Energy is conserved: $v_{\perp,0}^2 + v_{\parallel,0}^2 = v_{\perp}'^2$

➔ Particle confined only if $v_{\parallel,0}$ is low enough (HW): $\frac{v_{\parallel,0}^2}{v_0^2} < 1 - B_{min}/B_{max}$

The concept of a loss cone

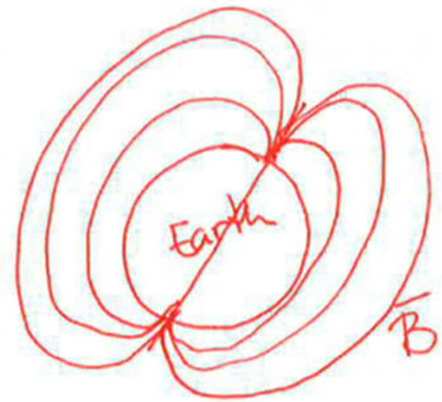
- It is common to denote $\frac{v_\parallel^2}{v^2} \equiv \xi^2$, called the *pitch* of the particle
- Correspondingly, $\theta \equiv \cos^{-1}\xi$ is the *pitch angle*.
- The value of ξ in the weak-field region defines the *loss cone*: $\xi_0^2 > 1 - B_{min}/B_{max}$

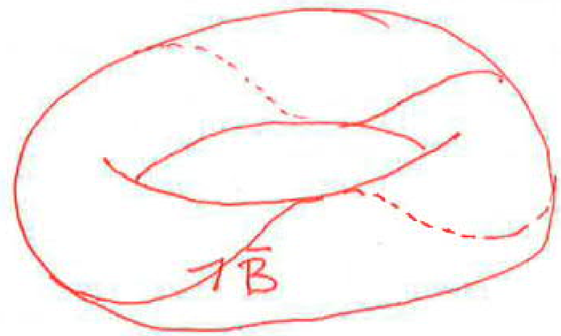
It is clear that for $B_{max} < \infty$, the magnetic bottle leaks and not all the particles are confined. ☹



Things to keep in mind …

- Many interesting plasmas have their mirrors and loss cones …
- In a mirror field, particles with 'small' ξ bounce between the mirror points w/ *bounce frequency* ω_b
- Even though in *uniform* magnetic field particles are stuck with their field line, with additional fields and/or uniformities, the particles will start *drifting* from their mother-fieldline
- More drifts to come in the second period… ;-)





Adiabatic invariants

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Let’s take things a little further …

What is all the fuss about the magnetic moment?

Is it just a fluke of the universe?

Or is there something deep behind its invariance…?

Yes, there is something very fundamental.

And it is not limited just to the magnetic moment…

The idea and use of invariants

Recall basic classical mechanics:

- periodic motion ➔ coordinate q and momentum p that 'oscillate'

  ➔ the action integral $\oint p\, dq$ = constant of motion (CoM)

Introduce a *slow* change in the system.

- Slow = compared to the periodic motion, so that $\oint p\, dq$ can be taken over unperturbed orbit

➔ CoM becomes an *adiabatic invariant*

In plasma physics, three interesting invariants appear…

The 1st adiabatic invariant

In a magnetic field, the periodic motion always present is the *gyration* around the field line

➔ $\oint p\, dq = \oint m v_\perp r_L d\theta = 2\pi r_L m v_\perp = 2\pi \frac{m v_\perp^2}{\Omega} = 4\pi \frac{m}{q} \mu$

➔ Our old friend, the *magnetic moment,* is the related invariant! ☺

Examples of the usefulness of μ

… actually an example of the usefulness of *breaking* μ=const…

Magnetic pumping (= adiabatic compression)

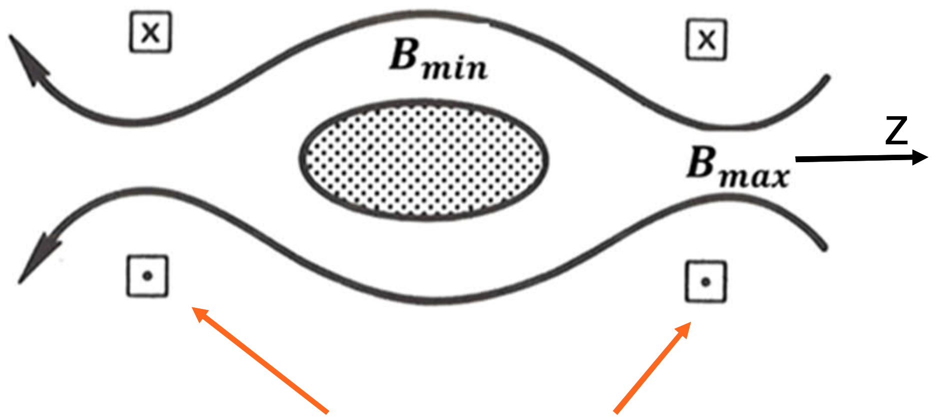
- Vary B sinusoidally

➔ mirror points move back-n-forth in *z*

- Due to μ=const no net heating ☹
- Include collisions

➔ during compression phase, collisions can transfer some $v_\perp$ into $v_\parallel$ which does not care about the expansion phase

➔ net heating!

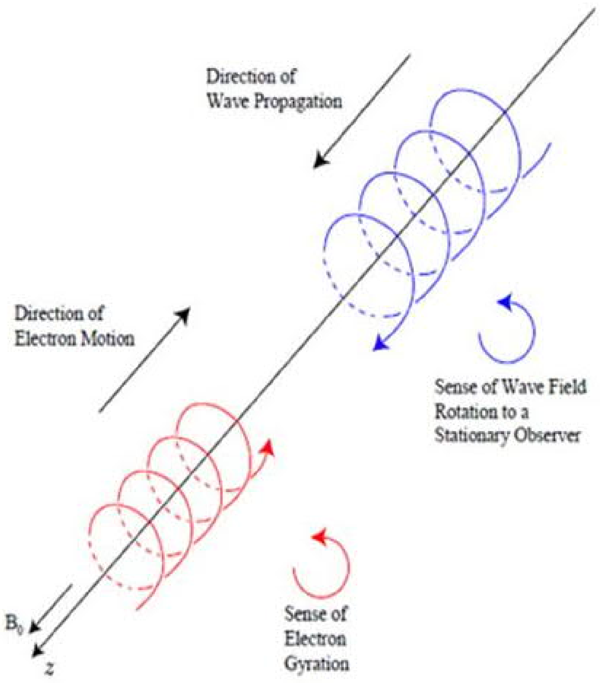


$$I(t) = I_0 \sin \omega t$$

Examples of the usefulness of μ

… again an example of the usefulness of *breaking* μ=const…

Cyclotron heating



- Apply oscillating ***E*** field @$\omega = \Omega$

➔ induced ***E***-field rotates @ $\omega = \Omega$

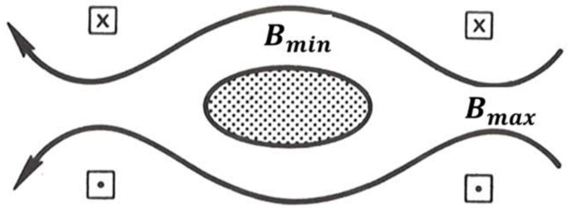
➔ some particles gyrate in phase with ***E*** and get accelerated

- $\omega \ll \Omega$ violated

➔ $\mu \neq$const

➔ net energy increase !

The 2nd adiabatic invariant



We have discovered also another periodic motion:

Magnetic mirror

- ➔ particle with 'small' $v_{\parallel}$ gets trapped and bounces between mirror points at ω_b
- ➔ periodic motion!
- ➔ $\oint p\, dq = \oint m v_{\parallel} ds$, where ds = path length along a field line

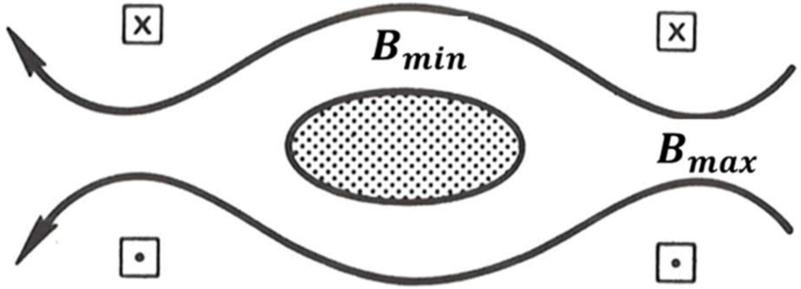
The related CoM, the *longitudinal invariant J*, can be calculated as integral between mirror points: $J = \int_a^b v_{\parallel} ds$.

Lengthy proof ➔ skipped here, but note:

- non-uniform B field ➔ GC drifts across field lines ➔ not exactly periodic

➔ adiabatic invariant !

Application of (non-)invariance of *J* …



Again take a mirror system.

Now apply $I(t) = I_0 \sin \omega t$ w/ $\omega \approx \omega_b$

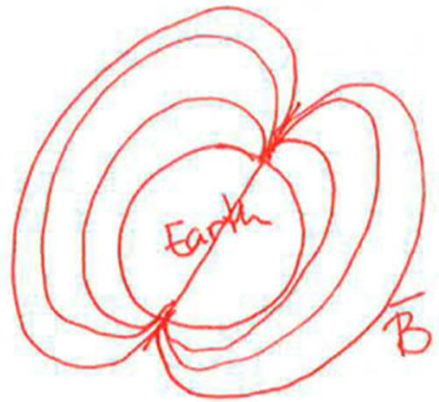
➔ mirrors approach/withdraw from each other

➔ particles with right bounce frequency always see an approaching mirror ➔ will gain *parallel* energy (shorter path length)

Net gain possible because $\omega \ll \omega_b$ violated

➔ *transit-time magnetic pumping*

The third adiabatic invariant



Earth's magnetic field:

- Gyration around field line ➔ μ
- Bounce motion between (polar) mirrors ➔ J
- *Grad-B drift* ➔ particles(= GC's) drift around the Earth ➔ yet another periodic motion!

➔ constant of motion obtained as an integral of the *drift* velocity along the $2\pi\ R_{path}$

➔ … do the math …

➔ total magnetic flux enclosed by the drift surface = const.