Aalto University

Shapes in Action
Sept 18th

Orbifolds and topology

AALTO
MATH&ARTS

Program schedule for Sept 18th

13:15 Where are we ?

13:30 Orbifolds: How to relate topology to patterns ?

14:00 Break

14:15 Magic theorem and its consequences

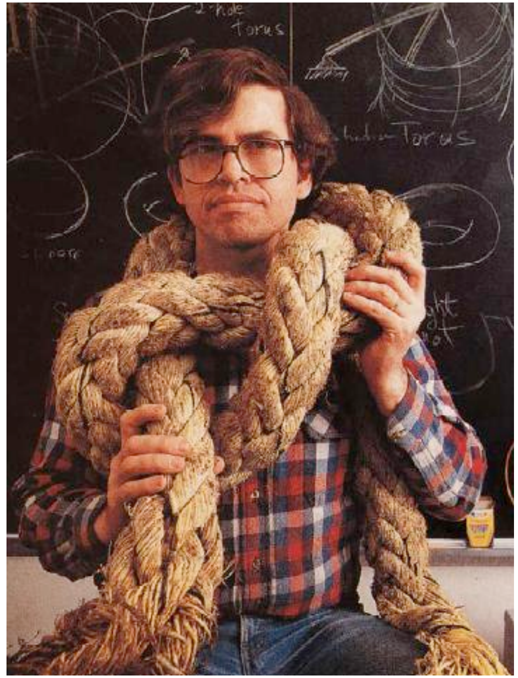
15:00 Break

15:15 Textile analysis in groups

Where are we ?

Goal : Understanding

Signature/Orbifold notation due to
B. Thurston and J.H. Conway (90´)



Done so far:

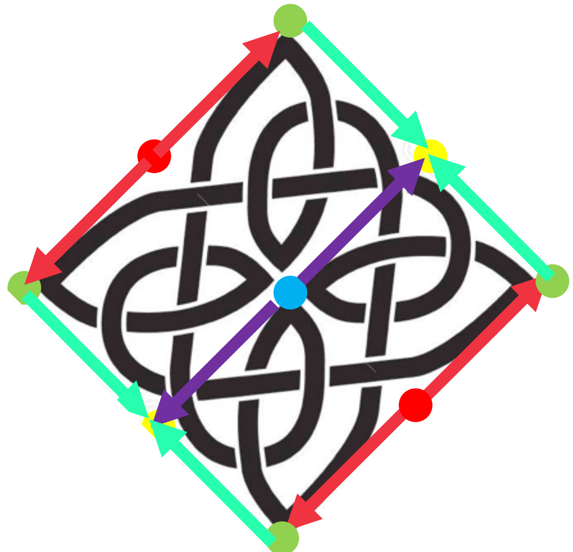
- Basic ideas on planar symmetries,
 signatures (= unique names for patterns)
 fundamental domain (= smallest piece in the pattern, that together with the boundary instructions determine the whole pattern)
- Some examples that make everybody confused…

A fundamental domain of a pattern with

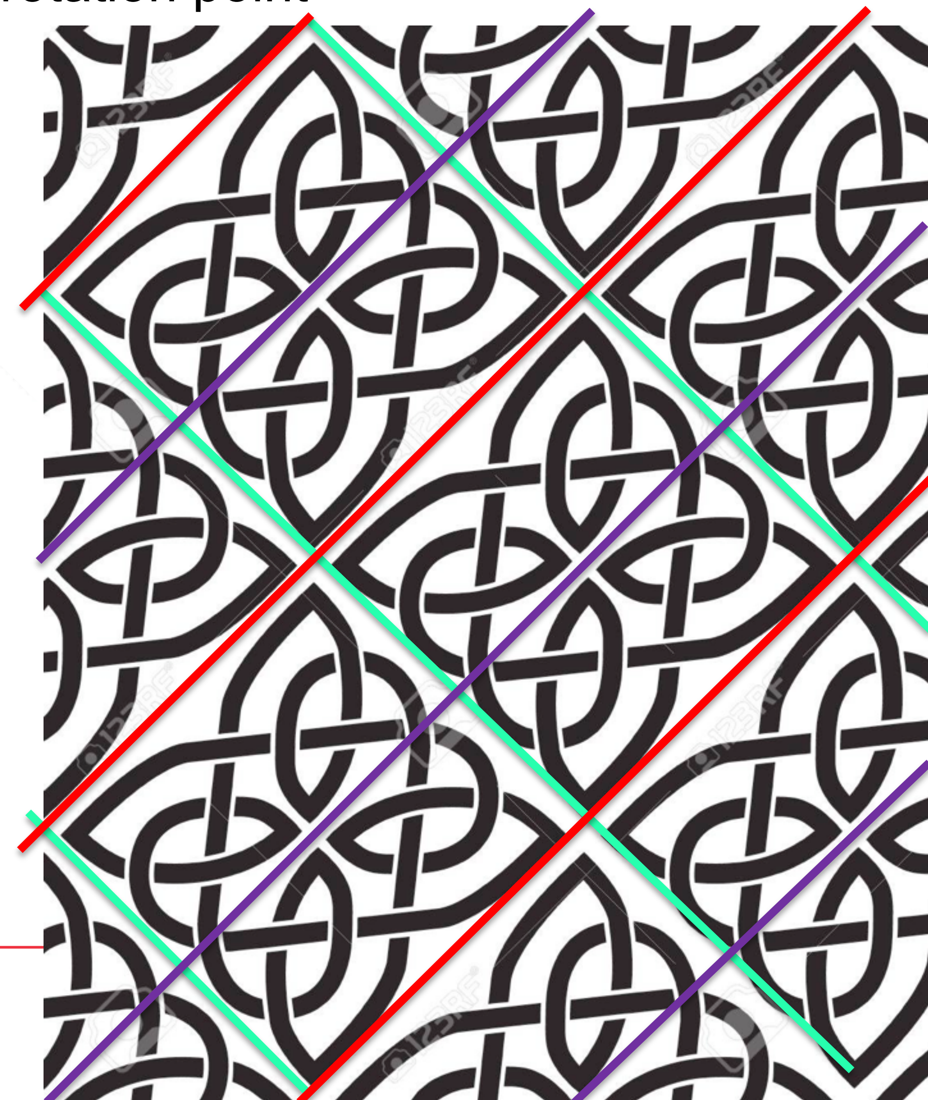
- Four different, order two, rotation points
- Together with the boundary arrows contain all information that is needed to construct the whole pattern

Signature: 2222

Situation after one rotation wrt the blue rotation point



Look at the situation now at the boundary: Rest of the pattern can be created by continuing rotations wrt to other rotation points OR simply by translating over the green& red boundary parts (2 consequent rotations generate a translation)



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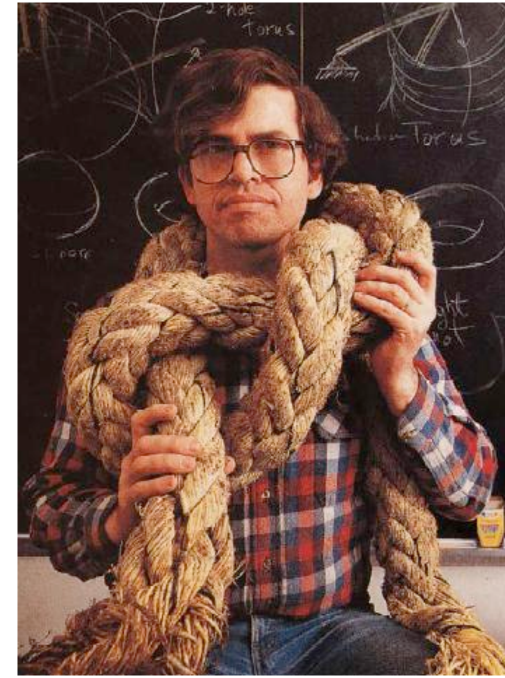
What next?

- Look at the symbols for signatures once more (one symbol was still missing!)

- How many (and which?) symmetries can be present in a same picture

- The Magic theorem will give the answer!



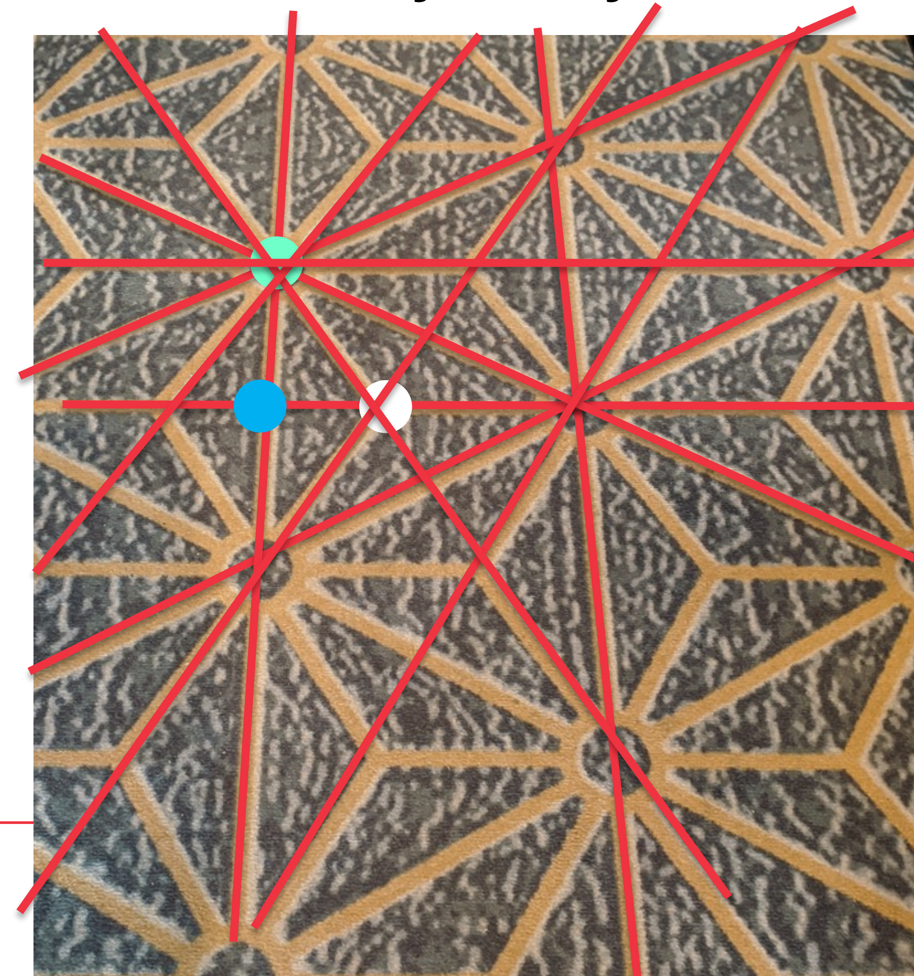
Star *

Star * (in the signature notation) denotes a *mirror* or *kaleidoscopic symmetry* = reflection with respect to a line.

Star alone means: there is one (and only one) single line of mirror symmetry.

Pic by Wanchote Jiamjitrak

*632 symmetry

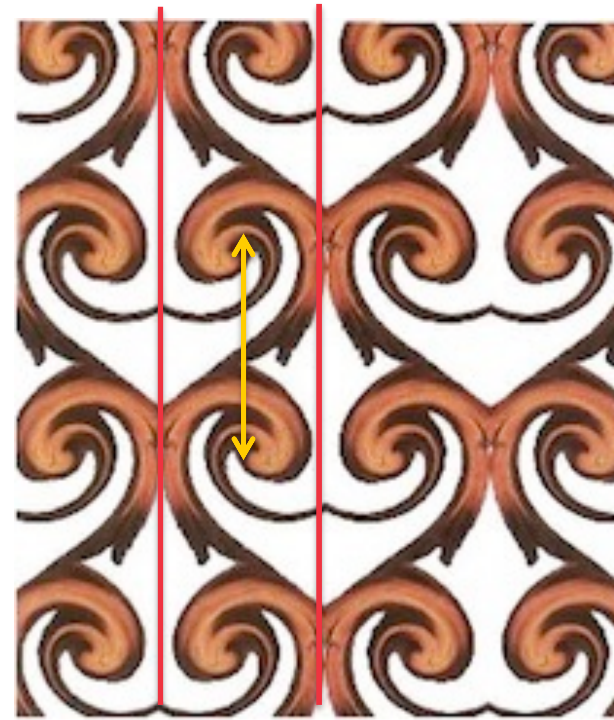


Pic by Anne Kasterpalu

Finite rosette pattern
Signature: 6•

Miracle x

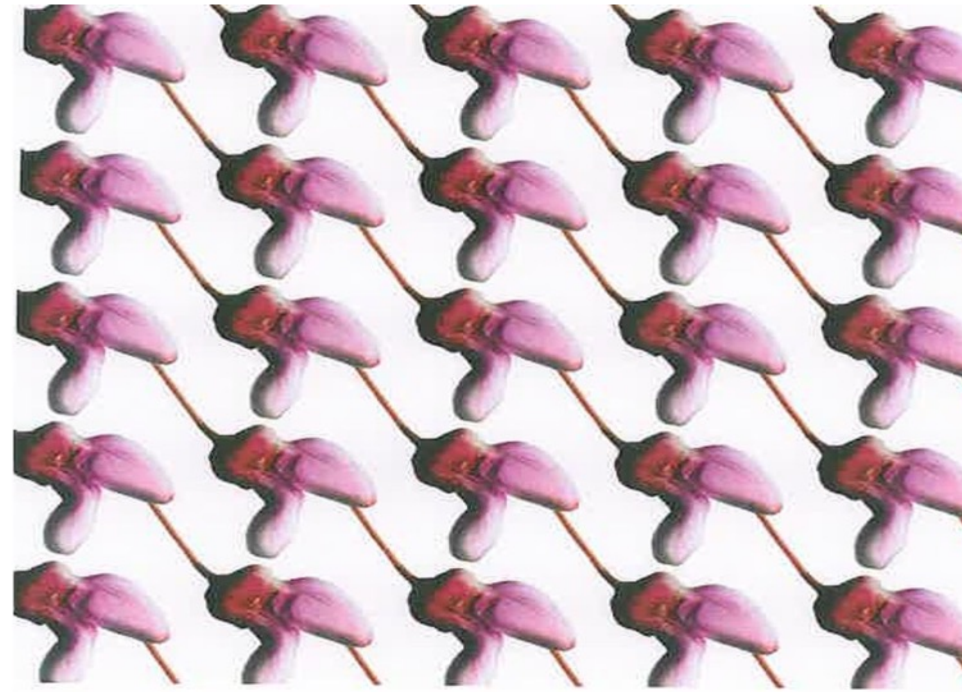
X : between the reflection lines (of the same type) two oppositely oriented patterns that can be connected with a path without crossing the lines



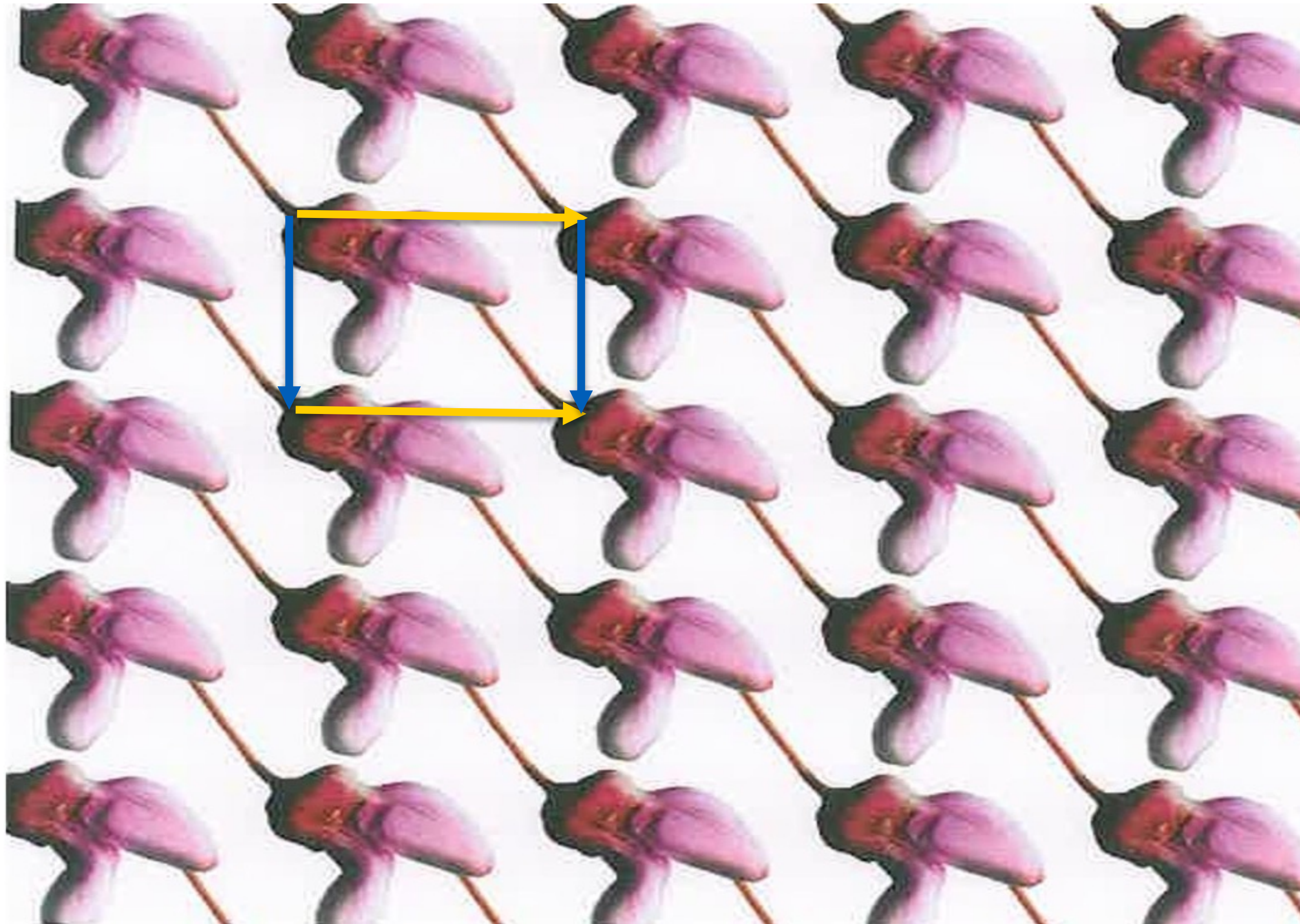
The signature of this pattern
Is *x

Wanderings and Wonder-Rings O

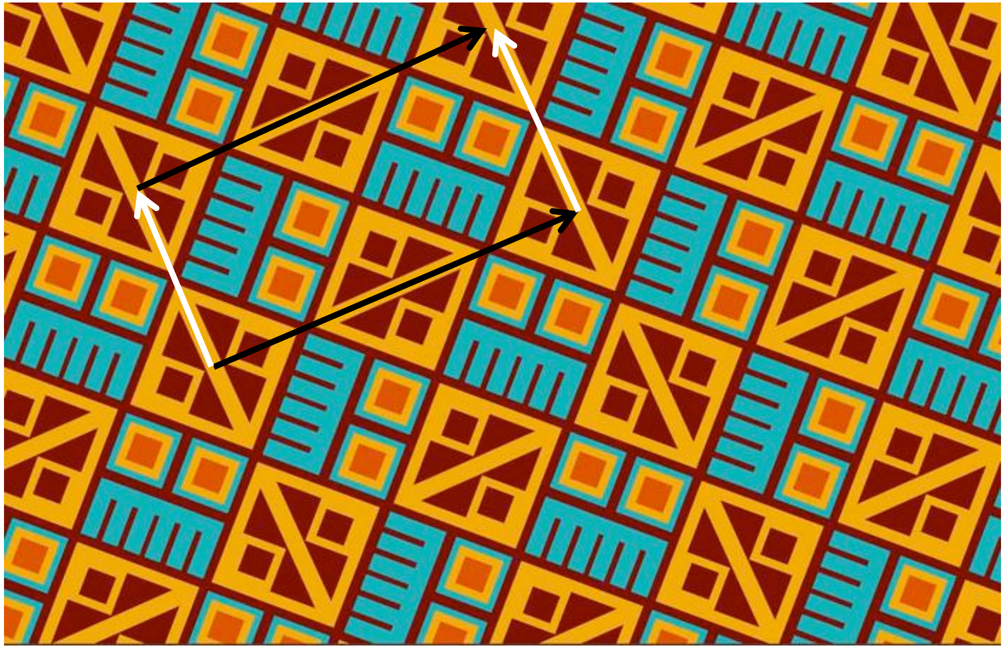
Symmetry that is not explained by mirrors, rotations or miracles



Fundamental domain?



Note again the role of arrows !



Signature

○

Four Fundamental Features

Claim:

- **wonders O…O,**
- **rotations AB…C,**
- **kaleidoscopes *ab…c*de…f... and**
- **miracles x…x**

suffice to describe all repeating patterns of the Euclidean plane. Notational convention: orientation preserving operations first(blue) then orientation reversing (red). Then you need not use different colors to distinguish operations.

Every property has its cost (in euros)

Symbol	Price	Symbol	Price
o	2	* or x	1
2	½	2	¼
3	⅔	3	⅓
4	¾	4	⅜
5	⅘	5	⅖
6	⅚	6	5/12
n	(n-1)/n	n	(n-1)/2n

n- fold rotation point n reflection lines meet at a vertex

The Magic Theorem for plane repeating patterns

The signatures of plane patterns are precisely those with total cost 2 euros.

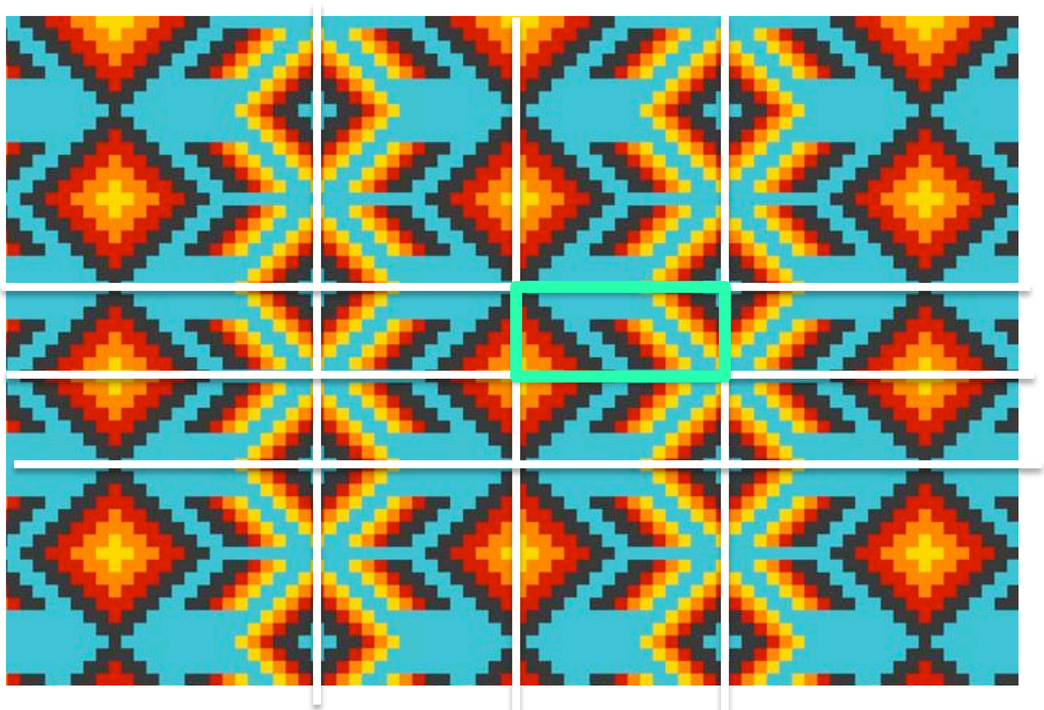
Ingredients for the proof:

- Local geometrical obstructions
- Global topological obstructions

Be patient! Will take a look of the proof a bit later…

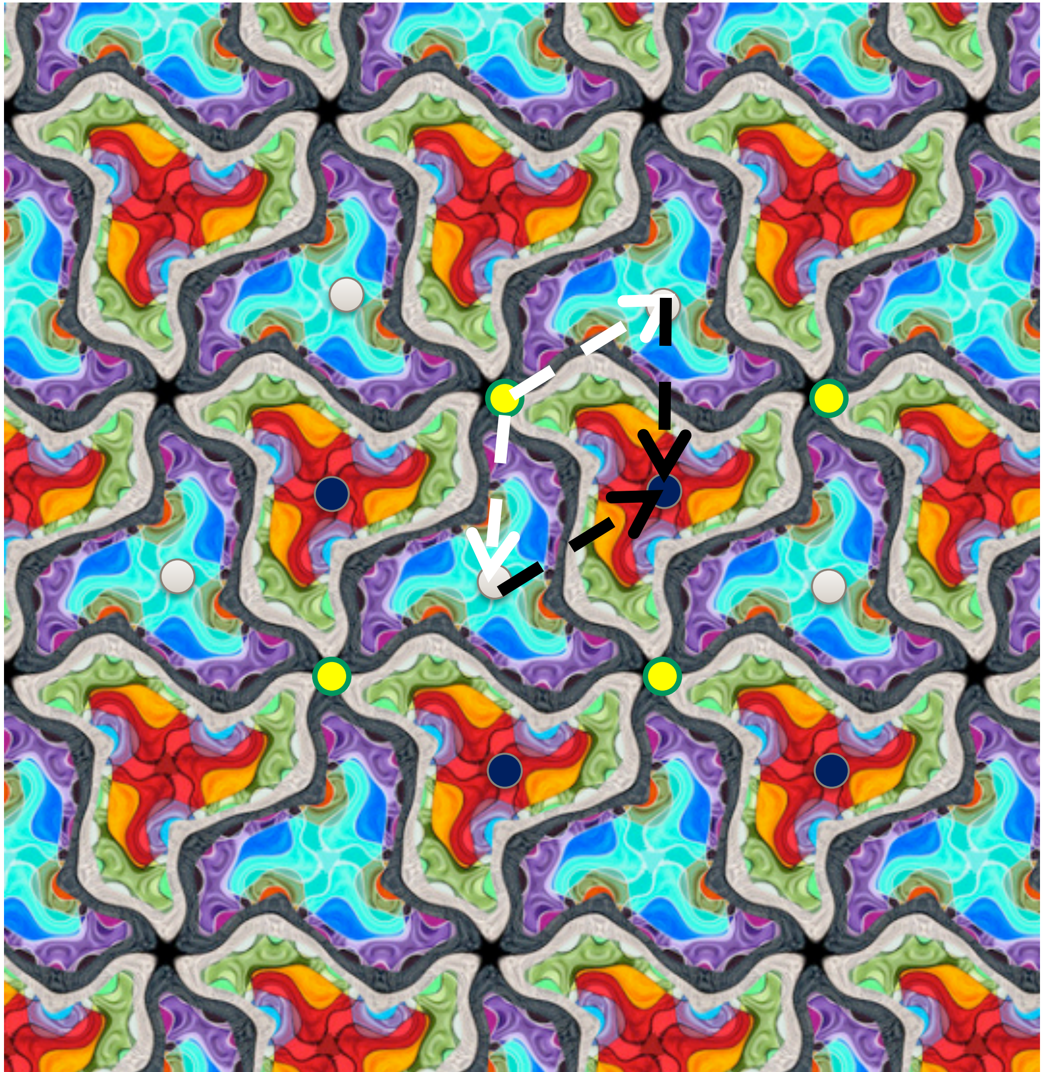
Signature and total price = 2 euro

***2222**
1+1/4+1/4+1/4+1/4=2



- Only reflection lines explain the whole symmetry pattern
- **Fundamental domain:** rectangle with green edges
- Two reflection lines meet in the four **different** vertices

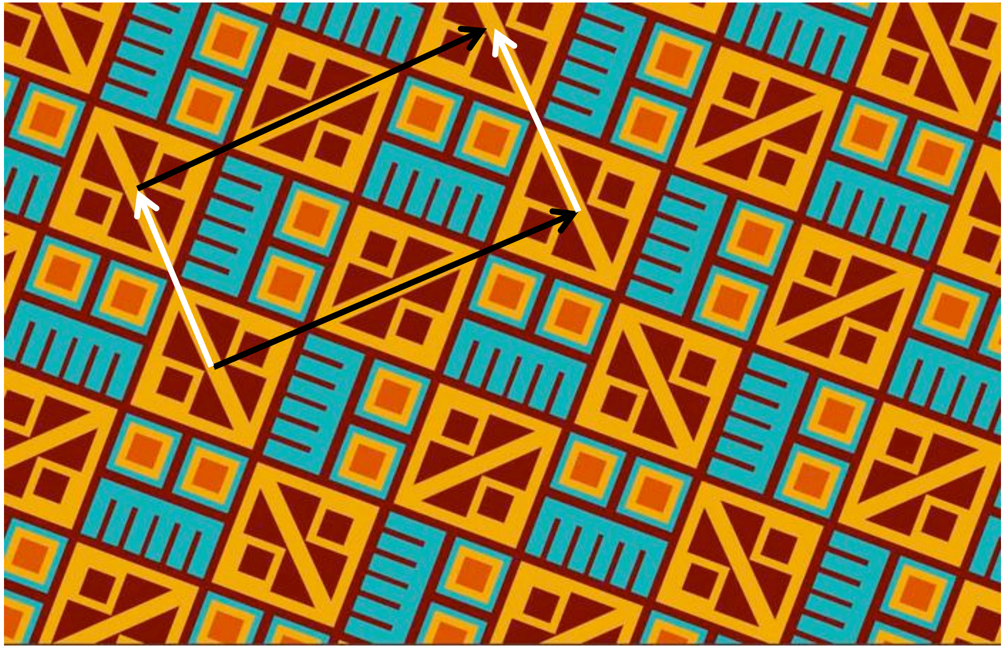
1.



Signature: 333
Price 2/3+2/3+2/3=2

- Three different types of (genuine) rotation points of order three (=120 degree rotations)
- No reflection lines
- **Fundamental domain:** Parallelogram in the picture that has only two different edges

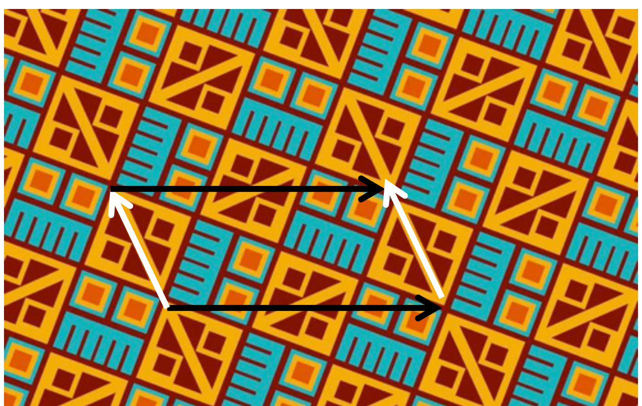
2.



Signature: ○

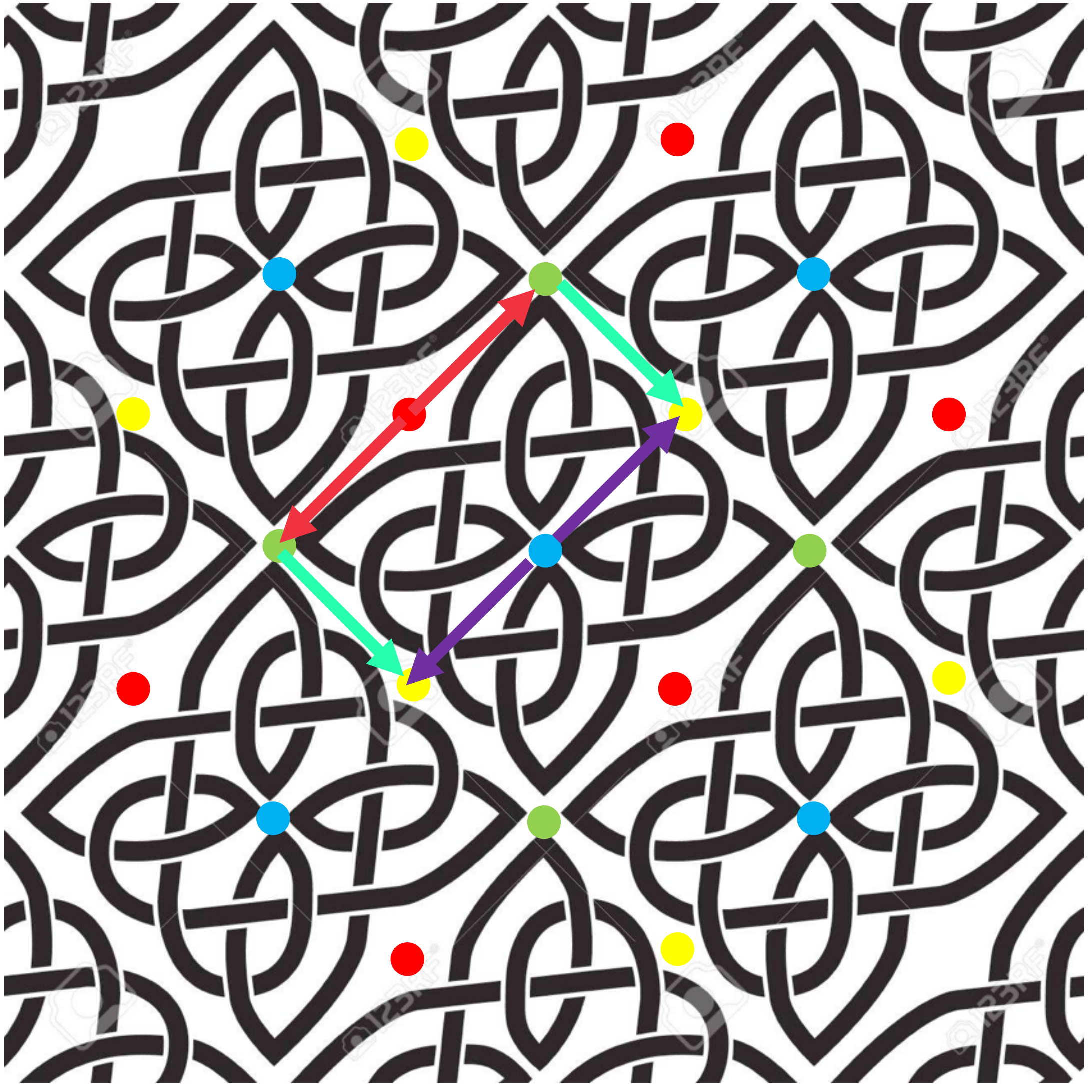
Price: 2

- No rotation points
- No reflection lines
- Two different translations generate the whole pattern
- **Fundamental domain:** Highlighted rectangle, that has only two different edges



- For this type of symmetry no unique way to choose a representative for a parallelogram 'spanning' the pattern

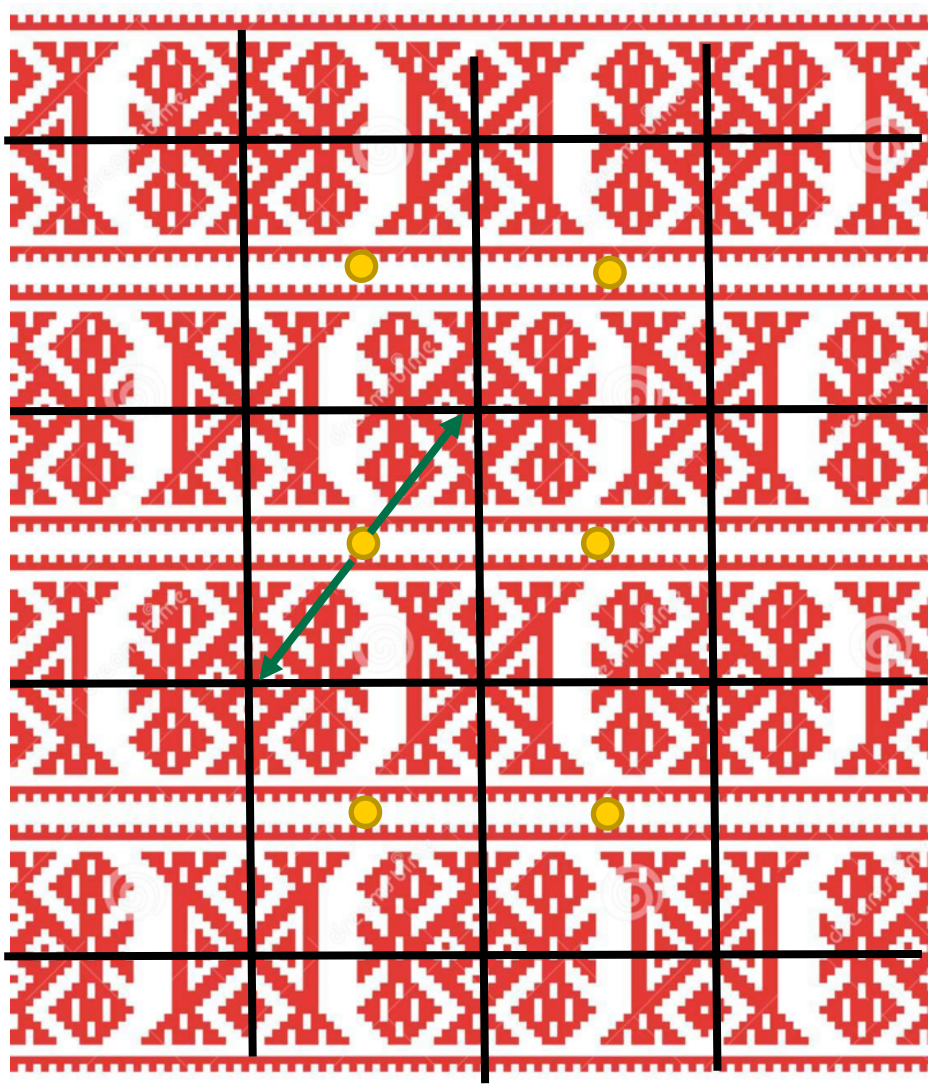
3.



Signature 2222
Price: ½+ ½+ ½+ ½=2
Fundamental domain:
Rectangle bounded by three types of arrows

- Four different rotation points of order two (=180 degree rotations)
- No reflection lines

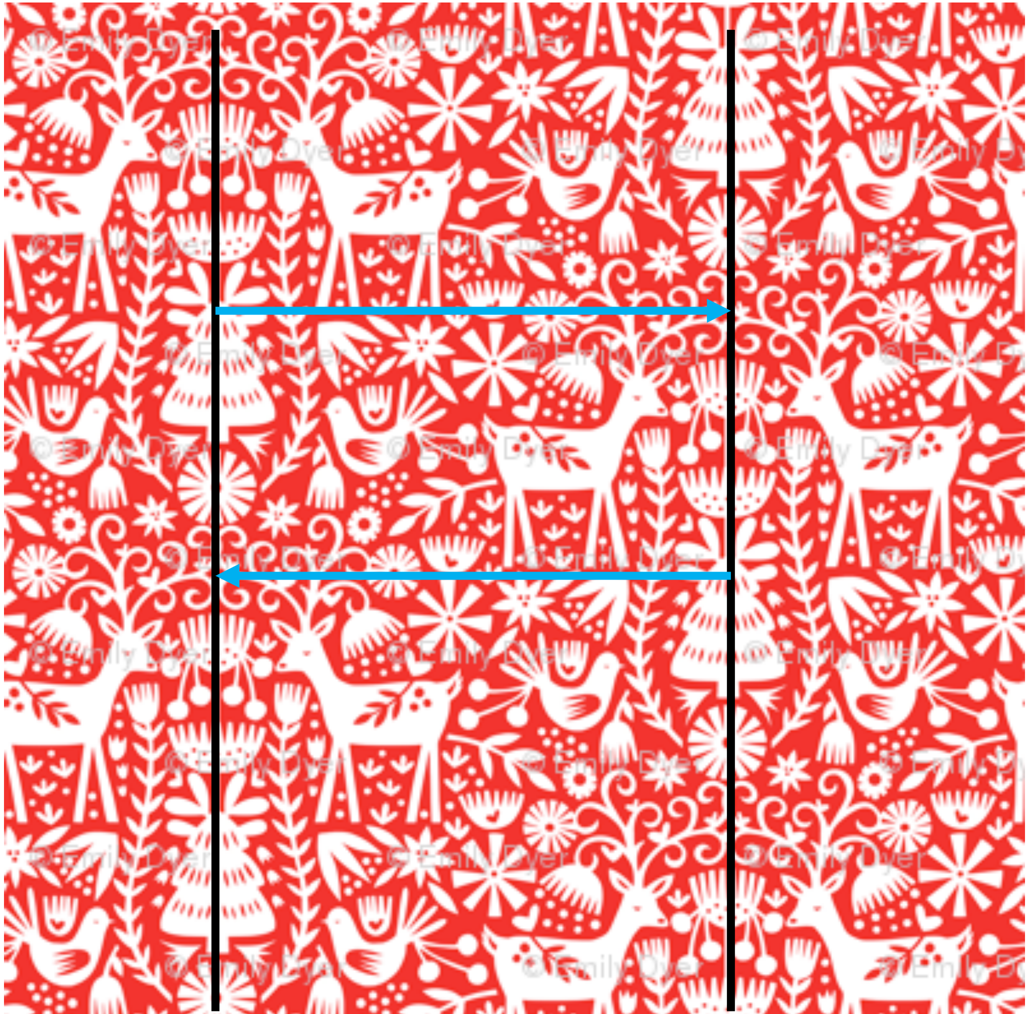
4.



Signature: 2*22
Price: ½+1+¼ +¼=2
Fundamental domain: Triangle
Bounded by two reflection line
 segments and green arrow

- Two types of reflection lines
- One (genuine =not produced by consecutive reflections) rotation point of order two

5.



Signature *x

Price 1+1=2

Fundamental domain:

Rectangle bounded by two different reflection line segments and one type of arrow

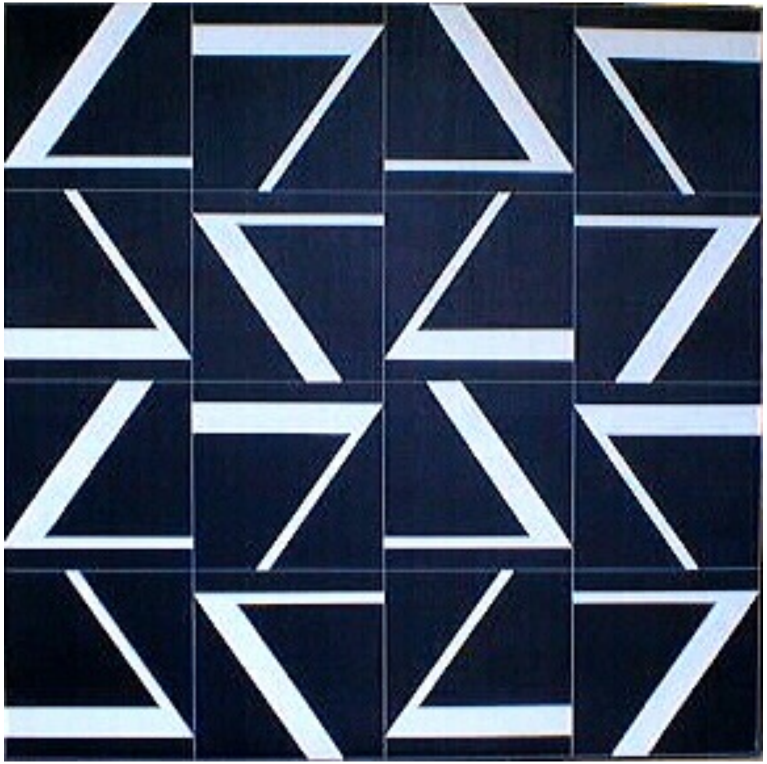
- Only one type of reflection line: Two black lines in the picture are the *same up to rigid motion*
- Between the lines also mirror images that are not caused by a mirror line

=> Blue horizontal arrows cutting the shape have opposite orientation

Perttu Näsänen, 1940-2012

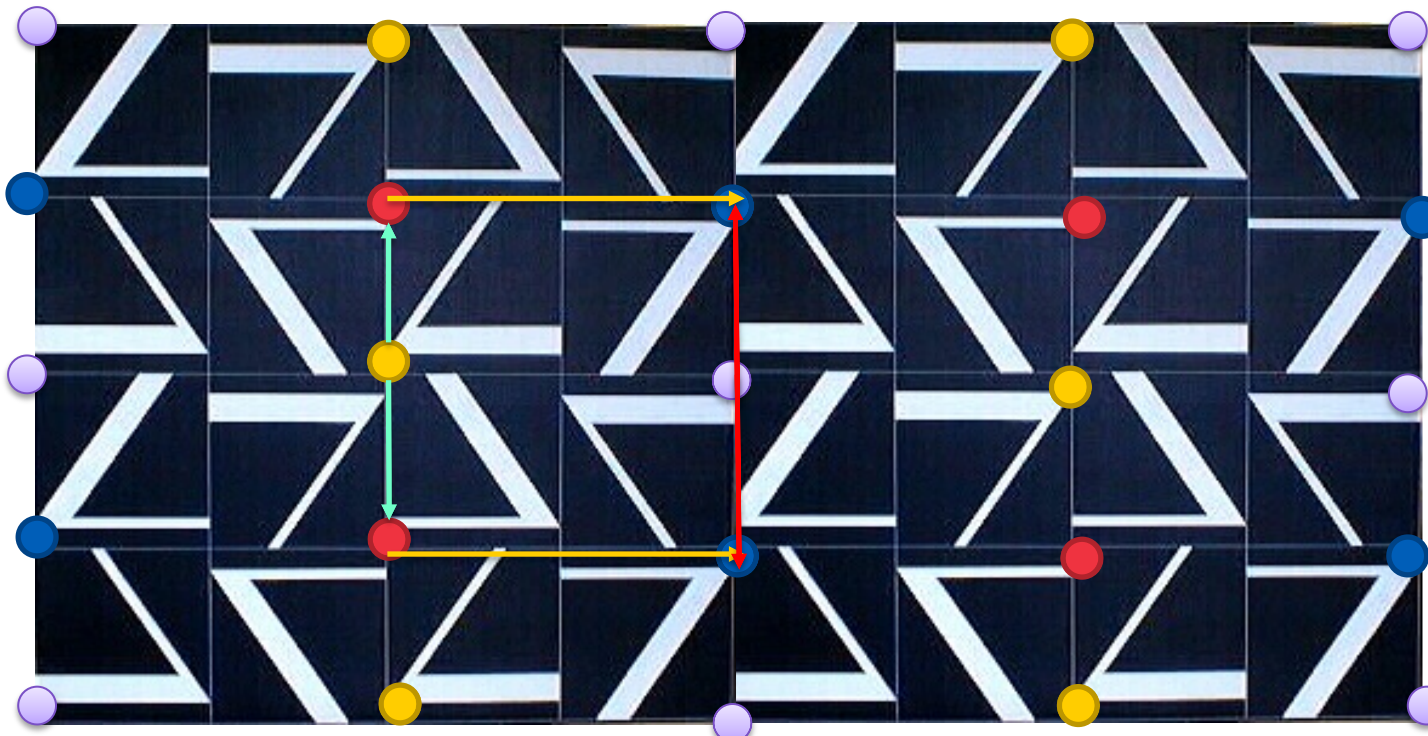
What is the signature of Sik-Sak 1978?

What is the fundamental domain?



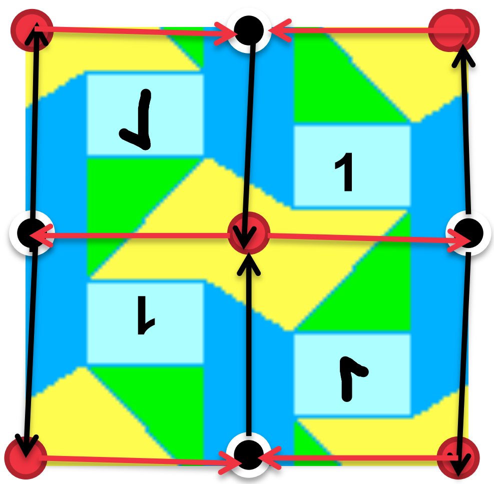
Symmetry 2222

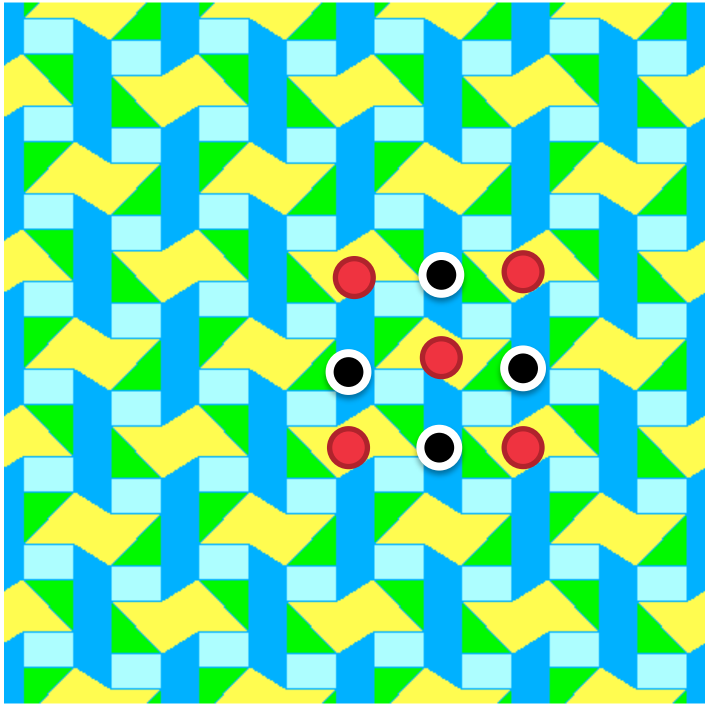
Especially: NO Miracles!



No global *glide reflections*= *reflection*+ *translation*

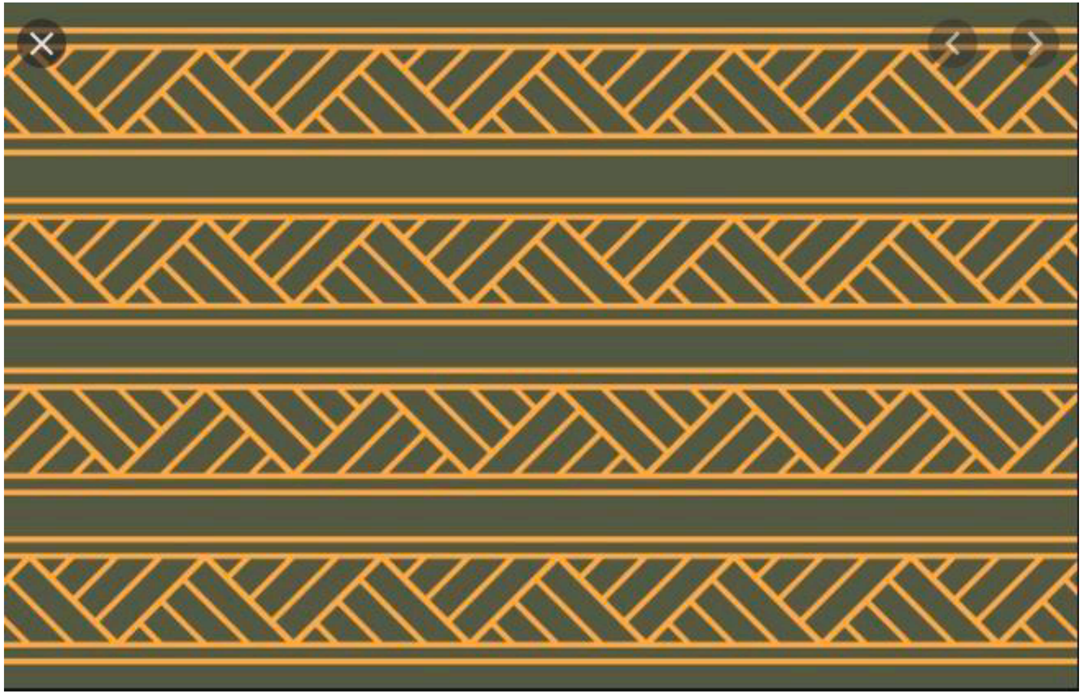
Symmetry type 22x

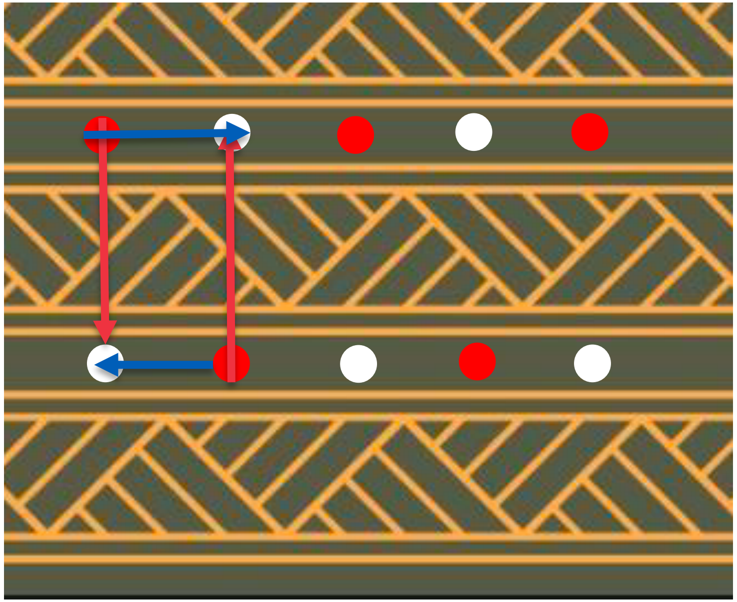




Pic by Tuan Nguyen

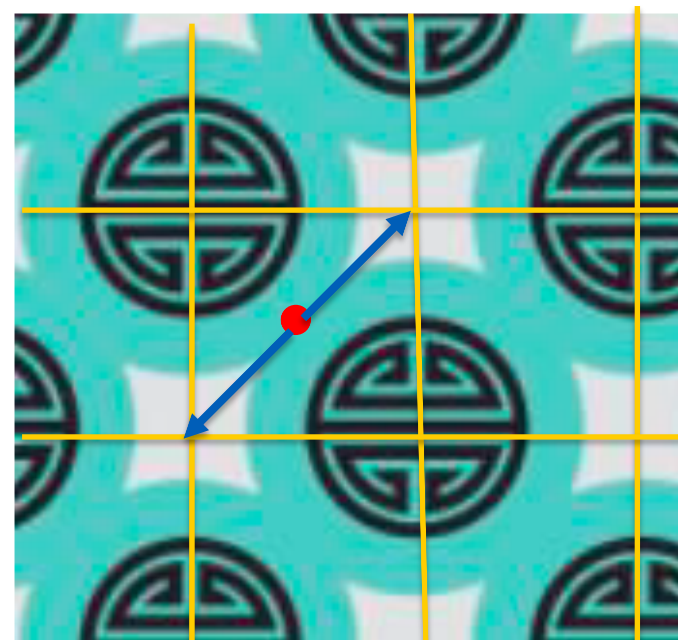
22x symmetry





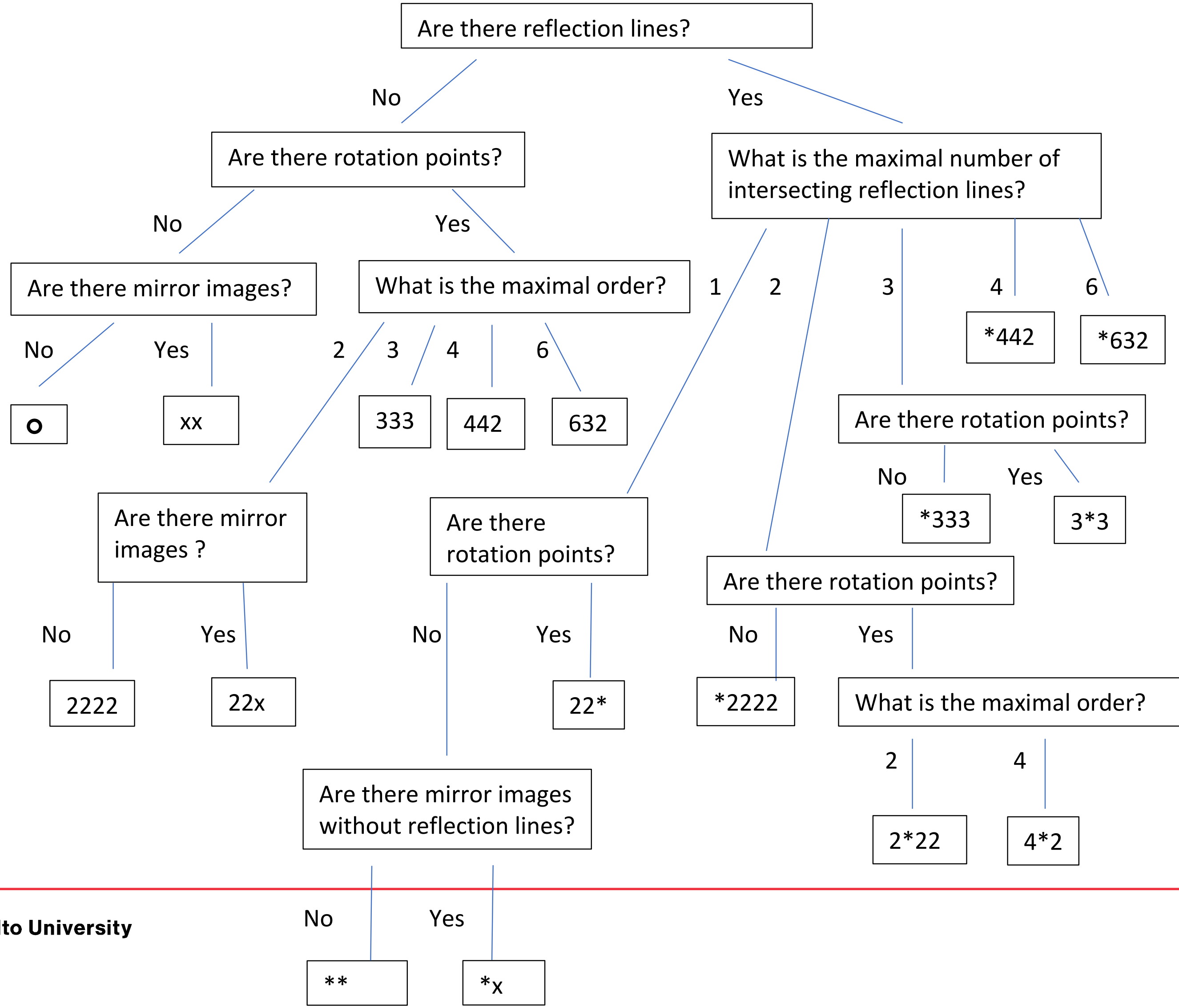
Pic by Tuan Nguyen

2*22 symmetry



Pattern analysis steps

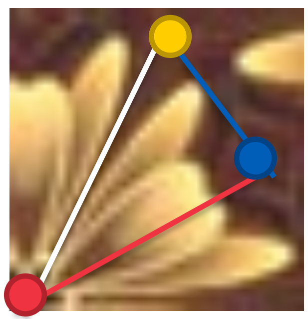
1. Draw all mirror lines (=lines of reflection)
2. Find the fundamental domain of the kaleidoscope
3. How many lines meet on each vertex? => Local symmetries of form *N
4. Find rotationally symmetric points (non-kaleidoscopic)
5. Are there mirror images without mirrors ? Then there must be at least one miracle x.
6. Helpful to look at the price list during the analysis and take the miracle theorem into account
7. If there is only repletion into two directions (nothing from above) then the pattern is ‘wandering’ o



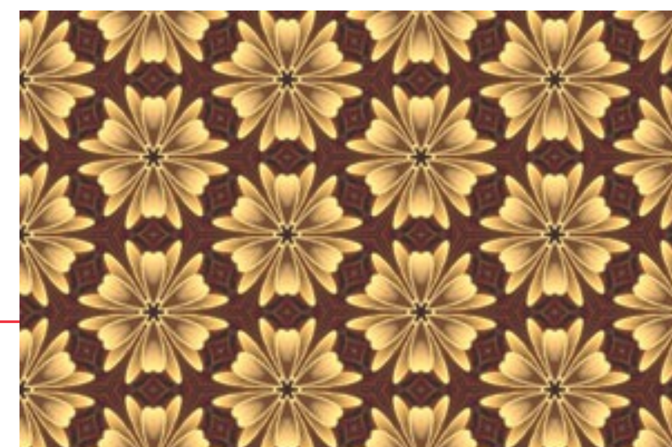
Are there reflection lines?
No
Yes
Are there rotation points?
What is the maximal number of intersecting reflection lines?
No
Yes
Are there mirror images?
What is the maximal order?
1
2
3
4
6
*442
*632
No
Yes
2
3
4
6
o
xx
333
442
632
Are there rotation points?
No
Yes
*333
3*3
Are there mirror images ?
Are there rotation points?
Are there rotation points?
No
Yes
No
Yes
No
Yes
2222
22x
22*
*2222
What is the maximal order?
2
4
2*22
4*2
Are there mirror images without reflection lines?
No
Yes
**
*x

What kind of fundamental domains we have found so far?

Triangle with no identifications on the boundary (different parts coming from reflection lines)



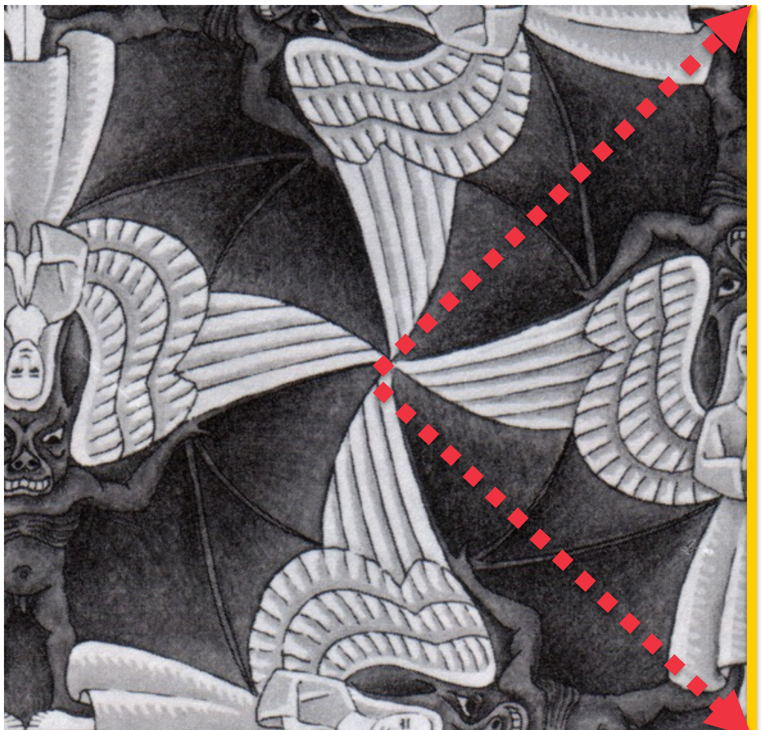
Topologically (= deformations that do not produce new holes are allowed)**: Disk orbifold**



***632**

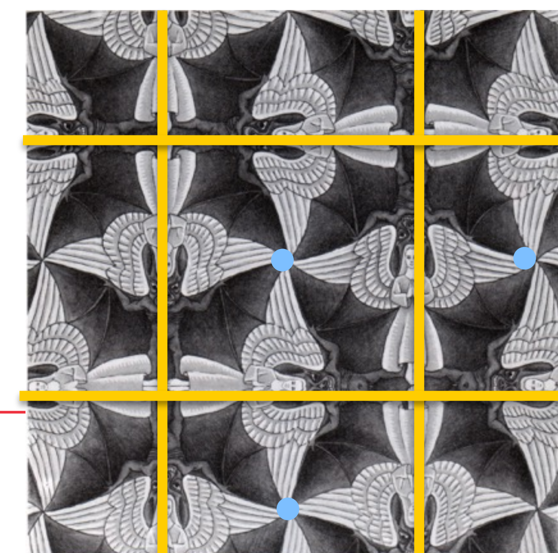
Combination of rotation points and reflection lines

Ex: 4*2

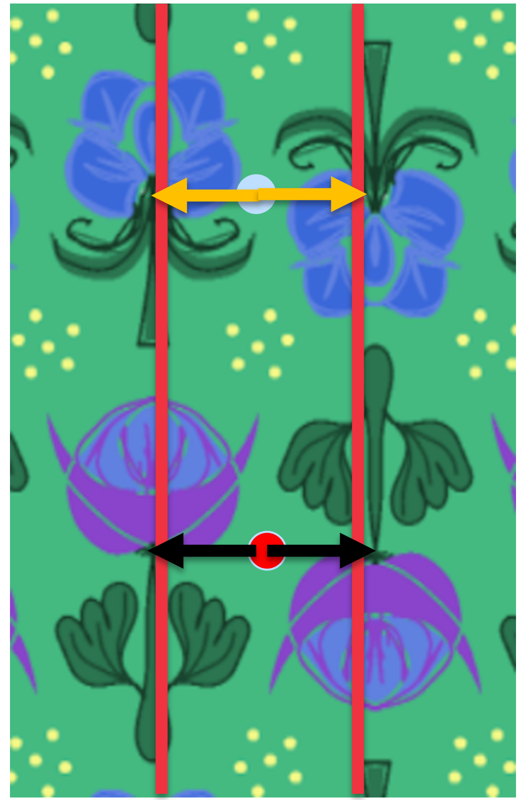


Fundamental domain: **A triangle with some identifications on the boundary** (red arrows due to the presence of a rotation point in the middle)

What is the **topological shape** of the piece **after the identification** (= gluing the red boundary arrows) ?



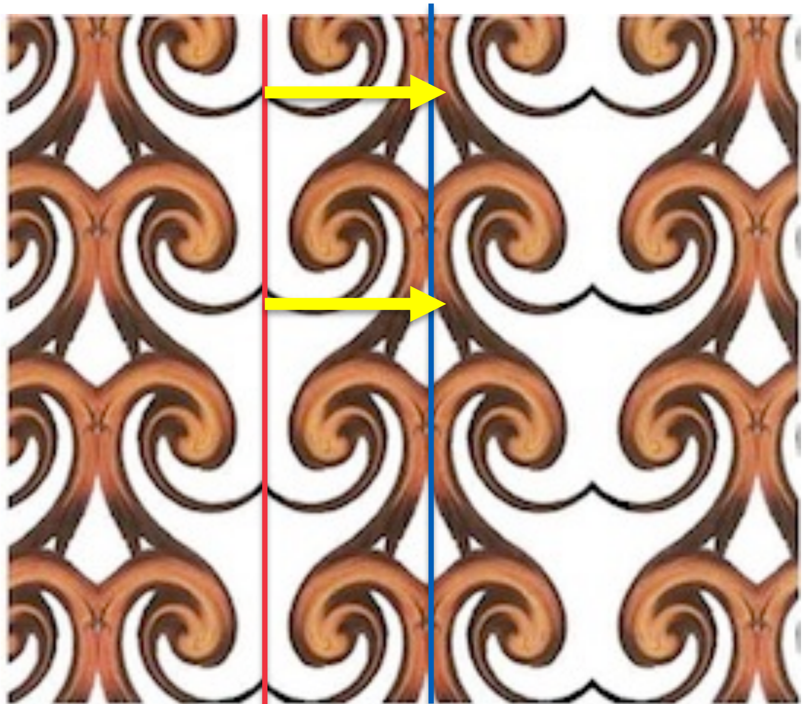
22*



Disk orbifold again ?
Are there other types ?

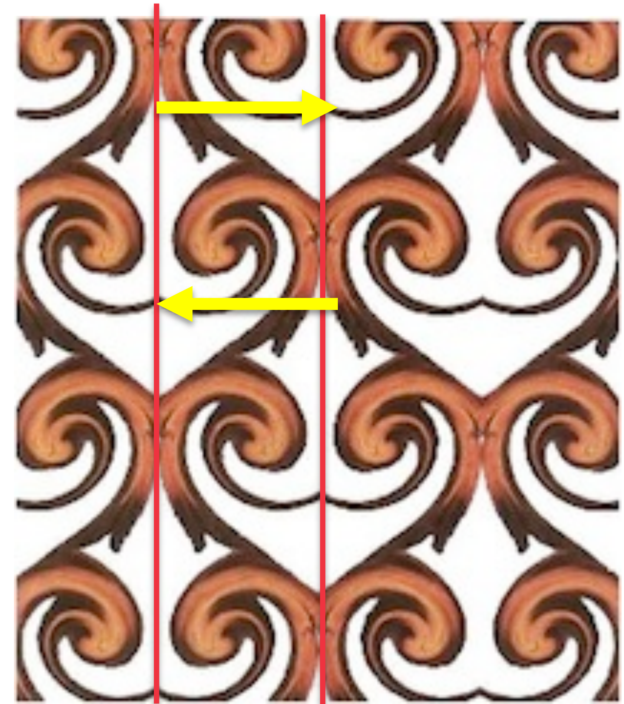
Cost of a miracle (x) = 1euro

Signature ** Annulus orbifold



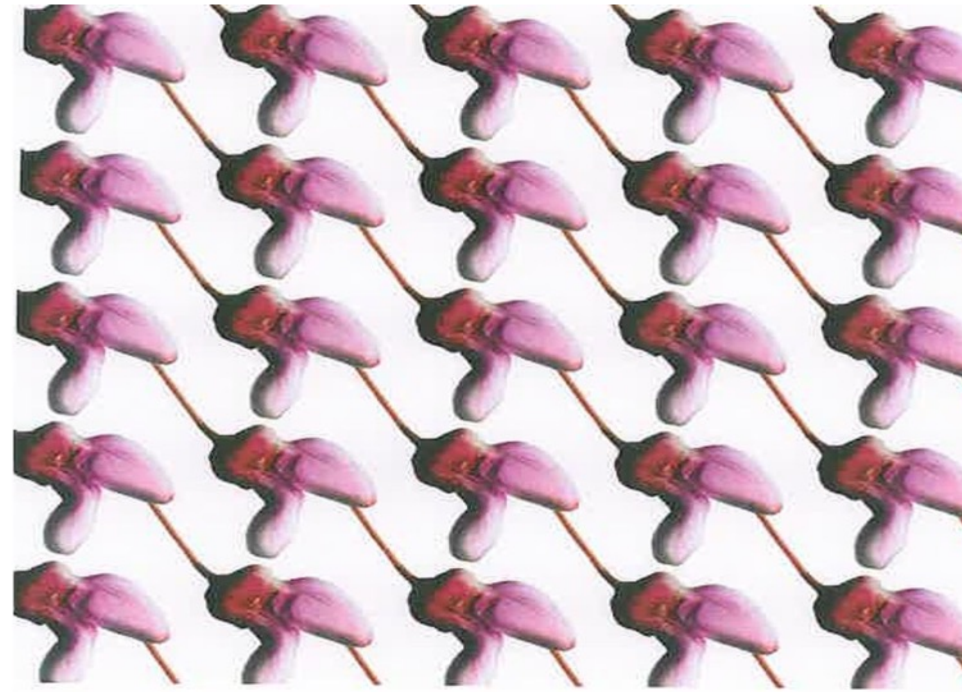
1+1= 2euro

Signature *x Möbius band orbifold



1+1=2euro

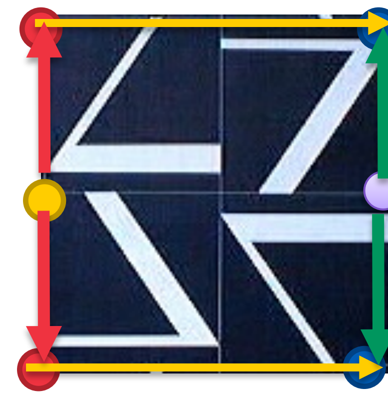
Wanderings O



Torus orbifold !

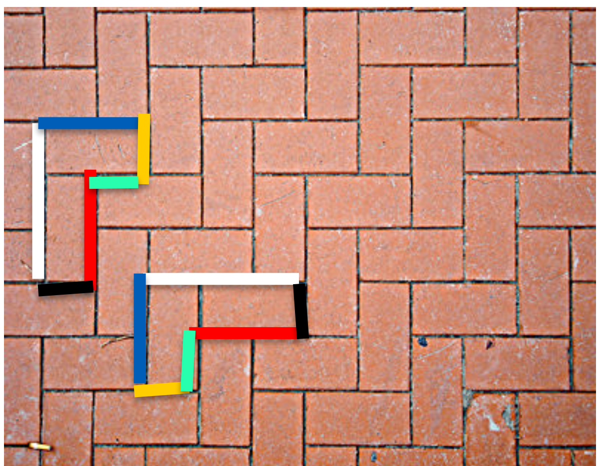
Rotation points only
Ex 2222

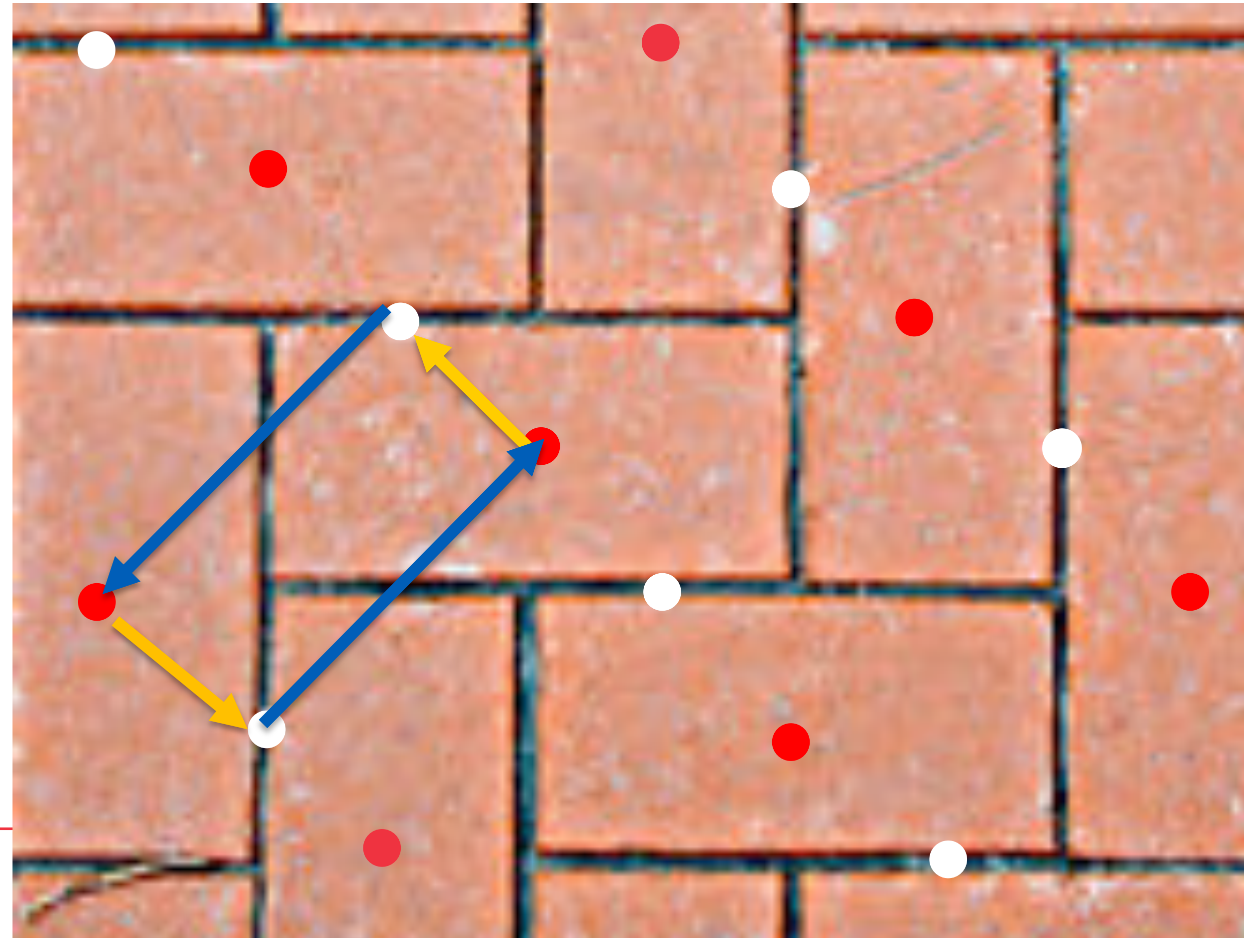
What is this shape after the boundary identifications ?



Ex: Brick walls/pavements

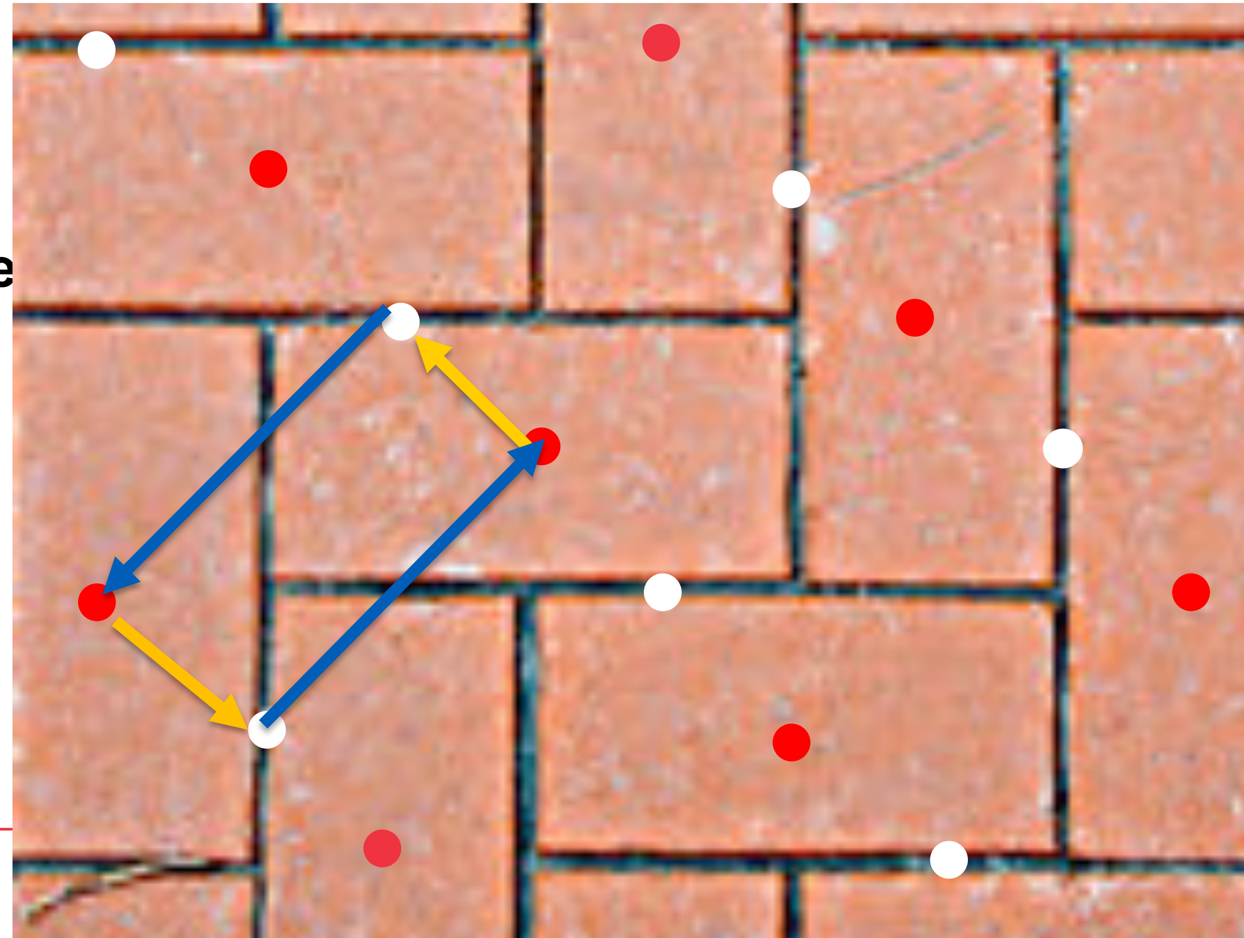
- 2 rotation points
- Mirror images without a reflection line
- => 22x



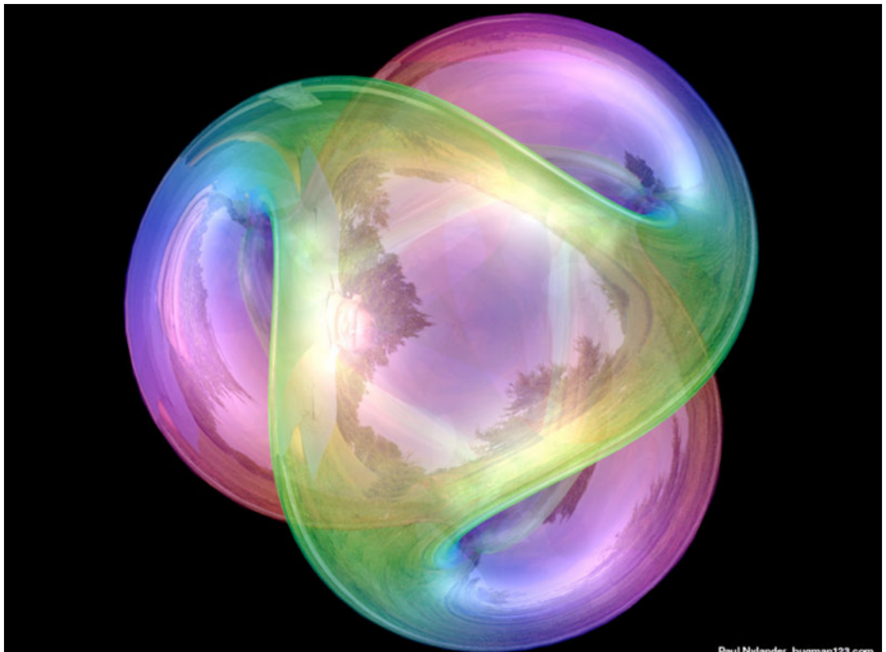


What is the orbifold of 22x symmetry?

What shape do you get when you do the identifications on the boundary ?

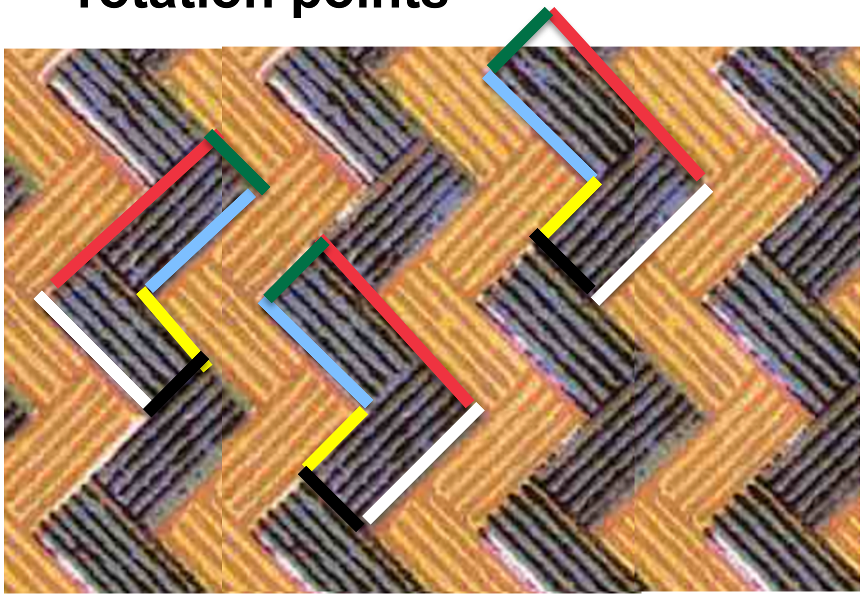


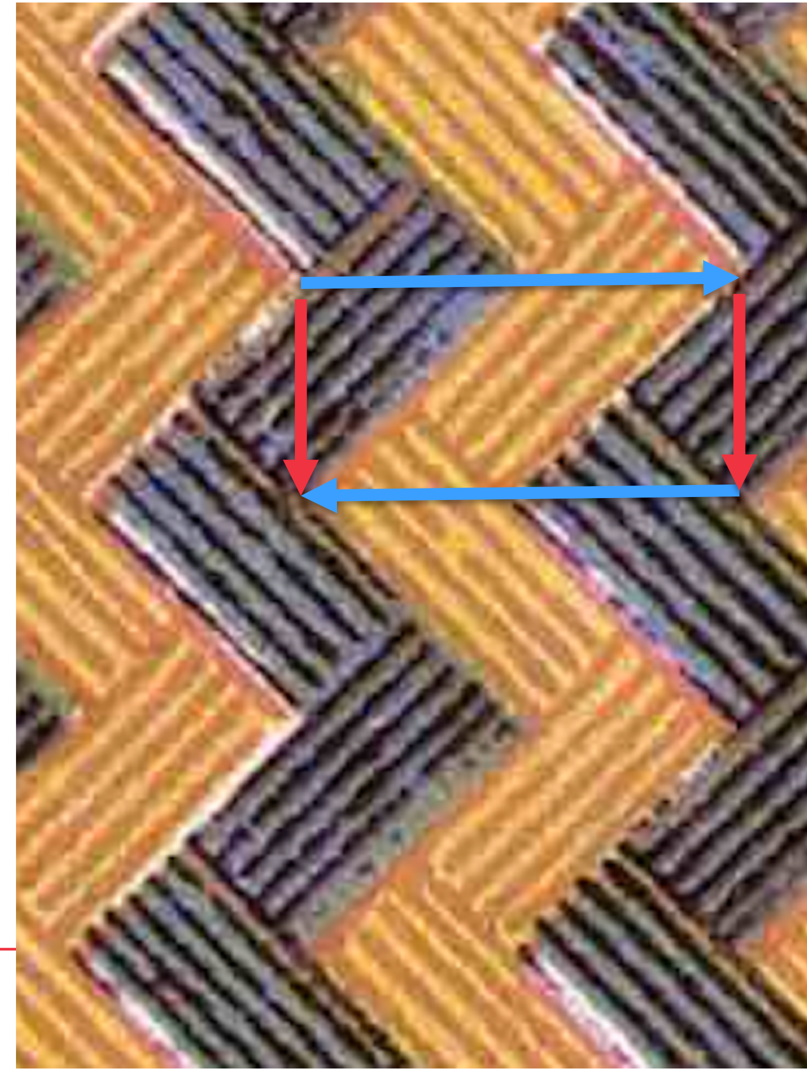
Real projective plane !



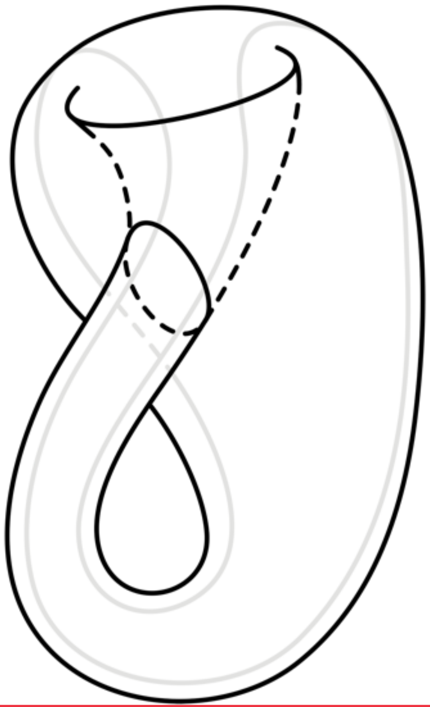
What about xx ?

- **Two miracles (mirror images without reflection lines) no rotation points**

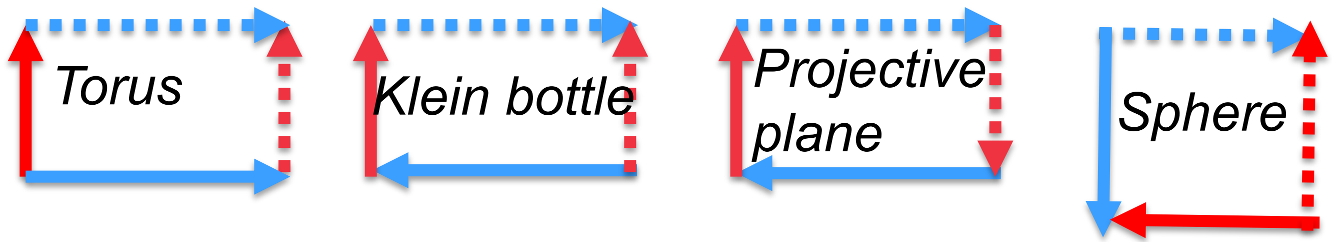




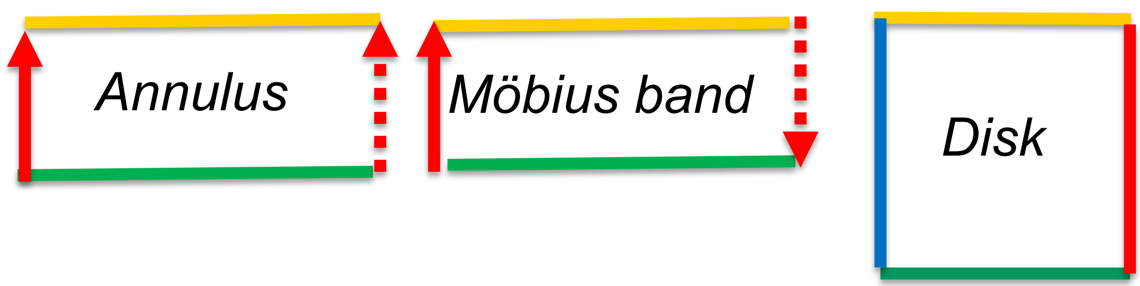
Klein bottle !



Surfaces via identifying boundary components of polyhedrons



(no boundary after gluing)



(non-empty boundary)

Are there other orbifolds coming from planar patterns ?

How many different signatures exist for plane patterns?

Assuming Magic Theorem to hold, this is similar question as asking:

How many different ways can I make change for one euro if I can use only 50, 20, 10 and 5 cents?

- Find all blue types
- Find all red types
- Find all hybrids

Blue types (orientation preserving)

Price for one n-fold rotational point is (n-1)/n <1 => need more than two to cost 2 euros:

- 333, 442, 632
- 2222
- Wonder O

What is the orbifold of the given signatures?

Red types without miracles

Observation: If no miracles x then *AB….N corresponds to ABC…N since

1+(A-1)/2A+(B-1)/2B+….+(N-1)/2N =2 ⇔

(A-1)/A+(B-1)/B+….+(N-1)/N =2

=> Only types *333, *442, *632, *2222

can occur in addition to ** .

What is the orbifold of these?

Hybrids: mixture of blue and red or involve x

Observations :

- switching between **n*** and ***nn** does not change the total cost
- replacing **x** with * -”-
- replacing **final** * with **x** -”-
- **=>**
- **cannot be changed into a hybrid: *632**
- ***442 => 4*2**
- ***333 => 3*3**
- ***2222 => 2*22 => 22* => 22x**
- **** => *x => xx**

Orbifolds of the above?

Conclusions

Only 17 possible signatures = 17 symmetry types for repeating patterns in the plane:

*632	*442	*333	*2222	**
			2*22	*x
	4*2	3*3	22*	xx
			22x	
632	442	333	2222	o

Possible orbifolds for planar patterns

Orientable

Sphere (**632** **442** **333** **2222**)

Torus **o**

Annulus ******

Disk (***632** ***442** ***333** ***2222**
2*22 **4*2** **3*3** **22***)

Non-orientable

Projective plane **22x**

Klein bottle xx

Möbius band *x

Groupwork with textiles

1) Choose the different patterns in your group as instructed by Laura

2) Upload (as a group) to MyCourses by next Tue

3) Group Presentations starting on Tue 29th only 5-10(?)min/group

4) Give criteria/justification (either artistic or mathematical) for your choice.

5) For the repeated patterns, find the signature and orbifold if possible (ignore 'mistakes' and minor details in the prints)

Q: How to benefit from the classification in (flat) surface design in practise?