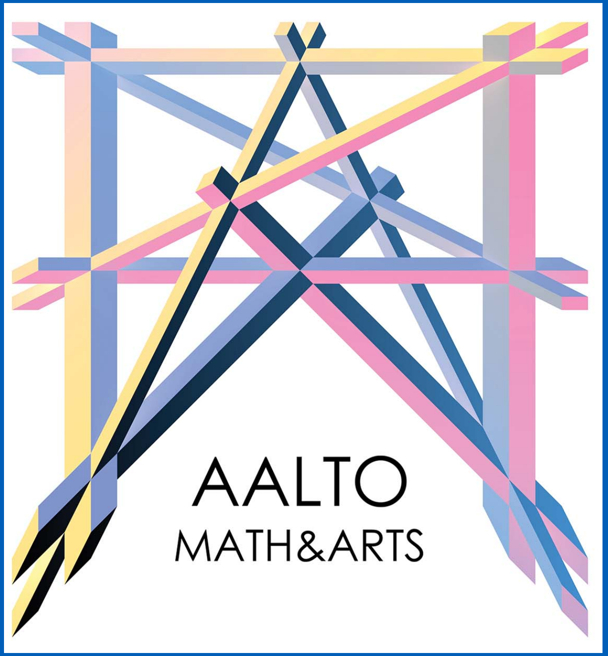
Aalto University

Shapes in Action
Sept 25th

The Magic Theorem



Program schedule for Sept 25th

13:15 Some ideas for Essays

13:45 Spherical symmetry classes

14:00 Break

14:15 Proof of the Magic Theorem

15:00 Break

15:15 Frieze patterns

15:30 Discussion in groups

What is an Essay here?

1) **A practical activity/demonstration completed with pictures and a short report or**
2) **Digital visualization/program providing some additional value to related topics or**
3) **traditional essay in the spirit of the course**

Proposals for topics and a short plan by Tue 6th Oct through MyCourses

DL for submissions in My Courses: Sun 25th Oct

Weight 1/3 of the course

What kind of a activity?

1) **Laura's workshop for 10 fastest:** Registration to directly to laura.isoniemi@aalto.fi by Tue 29th

2) Origami or other crafts (knitting, weaving, crochet, lace, drawing,…) and artistic approaches are highly encouraged

3) Prototypes for future studies

In case you choose this approach, please add one page explanation about the process and several pictures

Digital approaches

- **Programming or use of some existing program**
- **Photography**
- **Sound**
- **Video**
- **VR**
- **Prototypes for future studies**

Short description of the process and used tools must be added also here.

Traditional essay

- Written in a style of scientific writing
- contain clearly all references that are used (literature, web, etc)
- The length of the essay depends on the content, but it should be 10 pages at most (font Times New Roman 12, spacing 1,5) in addition to possible pictures. Note that sometimes it is more informal to use pictures instead of lengthy explanations.
- Background and stage of studies is taken into account in grading

Crafts traditions from different cultures

Finland: Example: https://www.finna.fi/List/831108

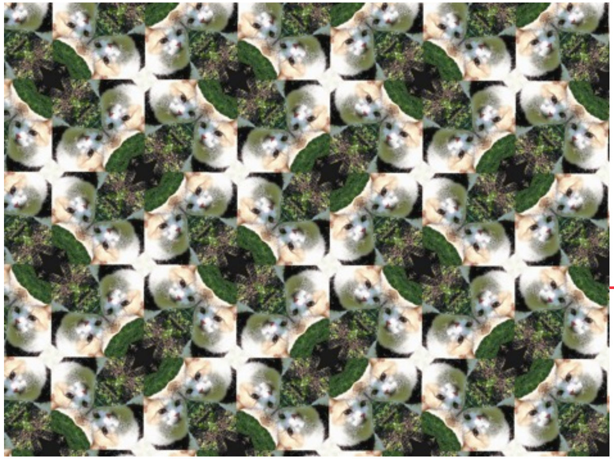
National museum would **love** to have an explanation for broad audience on how to classify symmetries of stuff they have

Philippines: Textiles & Weavings from Philippines, Math Intelligencer, vol 36, 2014

Islamic art

…..

Q: How to benefit from the classification in (flat) surface design in practise?



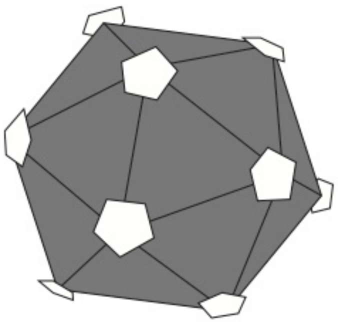
BRIDGES AALTO 2021

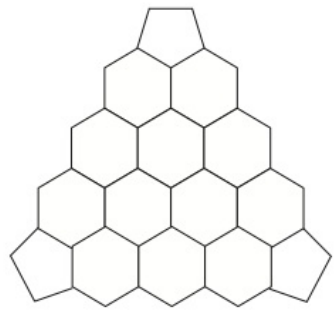
MATHEMATICS / ART / MUSIC / ARCHITECTURE / EDUCATION / CULTURE

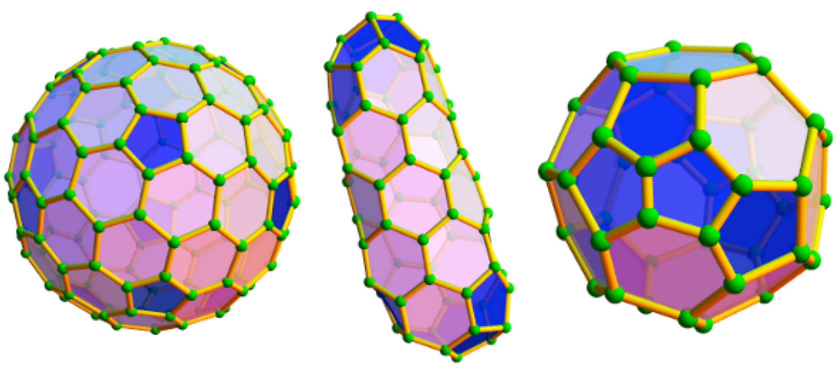
Check http://archive.bridgesmathart.org/ for ideas to essay AND/OR Submissions:

Regular Papers: 1 Feb 2021, Workshop Papers: 1 Mar 2021
Short Papers: 1 Mar 2021
Art Exhibition: 15 Mar, 2021, **Short Films:**19 April, 2021.
Math + Fashion Show:31 Mar, 2021.

Back to Thurston & Conway…..







Every property has its cost (in euros)

Symbol	Price	Symbol	Price
o	2	* or x	1
2	½	2	¼
3	⅔	3	⅓
4	¾	4	⅜
5	⅘	5	⅖
6	⅚	6	5/12
n	(n-1)/n	n	(n-1)/2n

The Magic Theorem for spherical patterns

The signature of a spherical pattern costs exactly 2-2/d euros, where d is the total number of symmetries of the pattern.

Note:

- **ch = 2/d**
- **for the chair d=2, for the table d=4**
- **In the plane case: d=∞ => only *one* Magic Theorem**

Lets produce some objects for analysis via folding …

14 different spherical symmetry classes

*532	*432	*332	*22N	*NN
				N*
		3*2	2*N	
				Nx
532	432	332	22N	NN

Note:

- **N=** 1,2,3… **but** digits 1 are omitted
- 1*=*11=*
- **However:** For example 11 11 = two rotation points of order 11

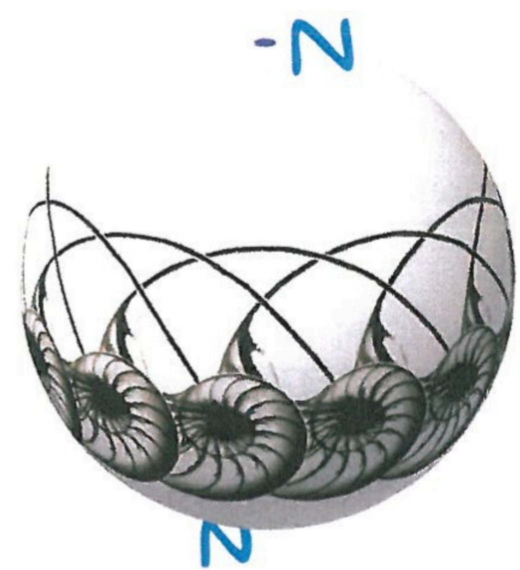
The five ‘true blue’ types (first one)

Total cost = 2-2/d <2 for every **d=** 1, 2, 3, …. =>

- no wonder rings ○
- no more than 3 digits (distinct to 1): (N-1)/N ≥ ½ for all, N=2,3,…
- if three digits, then **at least one** must be 2 (⅔+⅔+⅔=2, (N-1)/N≥⅔ for all N≥3)

Two digit case: MN

(In fact only case M = N occurs)



Case two 2's: 22N (second)

$1+(N-1)/N<2$ for all $N=2,3,4,5,\ldots$

Last 3 of the five ‘true blue’ types

Three digits, one 2:

- one digit must be 3 ($\frac{1}{2}+\frac{3}{4}+\frac{3}{4}=2$)
- the remaining digit must be 3, 4 or 5 ($\frac{1}{2}+\frac{2}{3}+\frac{5}{6}=2$)

$\Rightarrow$ 332, 432, 532

Note: ch(332)=2-($\frac{2}{3}+\frac{2}{3}+\frac{1}{2}$)=$\frac{1}{6}$**=2/12**

ch(432)= 2/24

ch(532)=2/60

The five ‘true red’ types

No ******, ***x**, **xx** signatures, all of type *AB…N

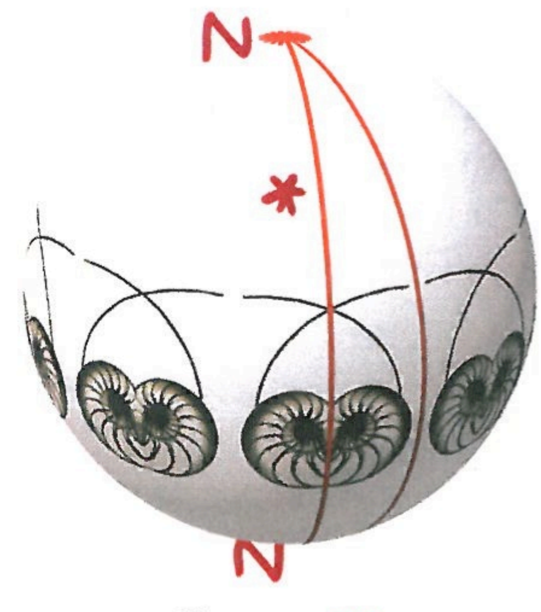
ch(*AB…N)=2-1-((A-1)/2A+…+(N-1)/2N),

ch(AB…N)=2-((A-1)/A+…+(N-1)/N),

=>

ch(*AB…N)=½ch(AB…N)

Note: only *NN is possible with two digits !



*22N

*MN2

*432, *532, *332

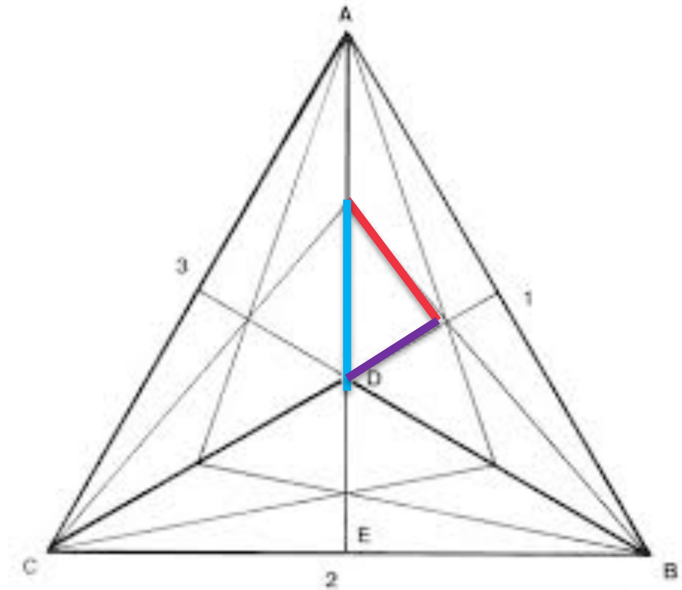
ch(*332) = $2-(1+\frac{1}{4}+\frac{1}{3}+\frac{1}{3})= 1/12$

Compare with orientation reversing symmetries of five platonic solids.

Ex: tetrahedron *332

1+2/6+2/6+1/4= 1+11/12 =2-2/24,

d=24=4x6

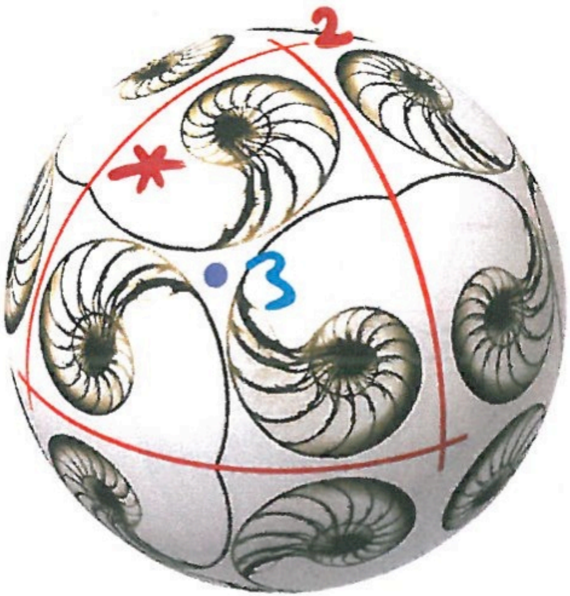


The four Hybrid types

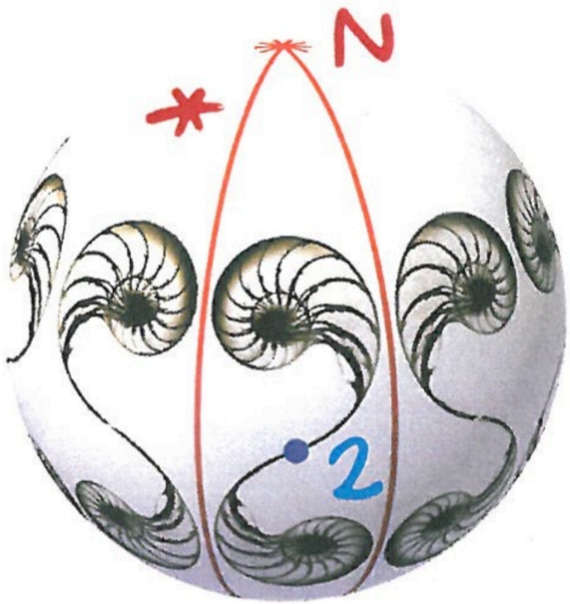
All possible variants (as in the plane case)

- *532
- *432
- *332 -> 3*2
- *22N -> 2*N
- *NN -> N* -> Nx

3*2 and N*



2*N and Nx



An example : A temari ball 2*3


$\mathrm{cost}(2{*}3)=\frac{1}{2}+1+\frac{1}{3}=1+\frac{5}{6}$

$\mathrm{ch}(2{*}3)=2-(1+\frac{5}{6})=\frac{1}{6}=2/12$

12 = total number of symmetries of this temari ball

Why does the Magic Theorem work?

- **Orbifolds** and
- Quantity: **Euler characteristics** = char := V-E+F

are all that matters.

Euler's formula: char = V-E+F= 2-2g for genus g of a (topological) orientable surface

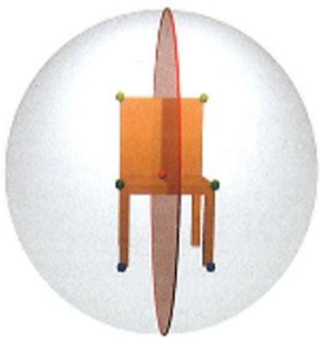
Especially: V-E+F= 2 for sphere (g=0)

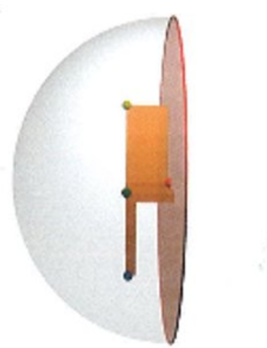
Question: What happens to char under folding along the symmetries ?

Ex:Folding a sphere into a hemisphere

char = V-E+F=2 for a (topological) sphere

What happens to char under folding ?



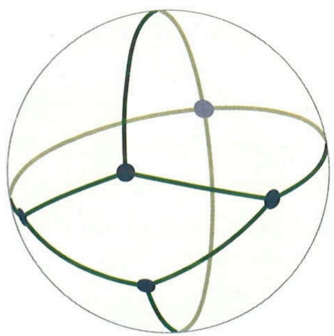


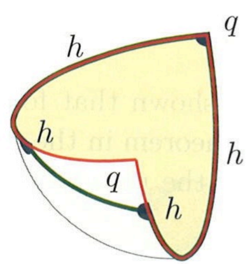
Matching points are fused to form an orbifold.

Claim: char(orbifold)= 2/d, d=number of symmetries

Ex: Pattern with V=5, E=8, F=5 on a sphere (5-8+5=2!)

Signature: *22, price 1+1/4+1/4=3/2 =2- 2/4!



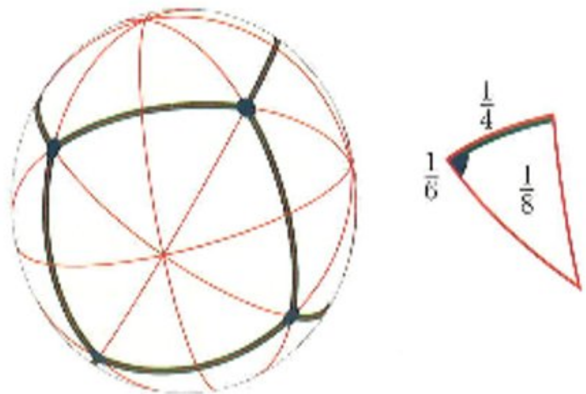


The 4 symmetries (giving rise to 4 identical pieces) of the pattern: identity, 2 reflections, 1 rotation = ‘ the different ways to do nothing’

After folding: V’=1/4+1/2+1/2=5/4, E’=1/2+1/2+1=2=8/4, F’=1+1/4 =5/4
char (quarter sphere)= V’-E’+F’=2/4 = char(sphere) / d

Ex: Folding a cube along reflection lines

Signature *432, price 1+23/24 =2- 1/24 =2-2/48 !



Before: V= 8, E=12, F=6
char(cube**)=8-12+6=2**

After: V'= ⅙, E'=¼, F'=⅛
char(folded cube**)=**
⅙−¼+⅛ =1/24 = 2/48

NOTE: 48 = number of all symmetries of a cube !

General argument for the fact that char (orbifold) = 2/d

Assume: V, **E** and **F** refer to the vertices, edges and faces of the original pattern on the sphere.

Then: under a d-fold symmetry the (weighted) number of vertices of the orbifold is **V'=V/d,** edges **E'=E/d,** faces **F'=F/d.** (Equal amount of vertices, edges and faces are distributed in similar pieces.)

Hence: char(orbifold)= **V'-E'+F'= (V-E+F)/d=2/d**

Note: char(orbifold) **does not depend on the original pattern, but only the number of symmetries it has !**

Why is ch:=change(pattern) :=2-cost(pattern)=char(orbifold) ?

Effect of * on a sphere corresponds (topologically) to punching a hole to the sphere (look at the chair case above)

Suppose * is caused by a k- sided face. Removing that decreases F by 1 and V and E by k/2 => V-E+F is reduced by k/2-k/2+1=1.

Effect of N-fold rotation to char

Claim: Replacing an ordinary point by an N-fold cone point (N) decrease char by (N-1)/N.

Suppose: The point to be folded is a vertex (can always arrange so).

Then: it contributes by 1 to V. After folding it contributes by 1/N

=> **net change to char** is 1-1/N= (N-1)/N

Effect of N-fold reflection to char

Claim: Replacing an ordinary boundary point by an N-fold corner point (N) decrease char by (N-1)/2N

Argumentation:

Contribution of a vertex is ½ (boundary point after *!). After the replacement it will become 1/2N vertex. =>

Net change: ½-1/2N =(N-1)/2N

Signatures related to above operations (all except x)

*532	*432	*332	*22N	*NN
				N*
		3*2	2*N	
				Nx
532	432	332	22N	NN

Introducing holes (-1 effect on char), cone points -(N-1)/N

And corner points –(N-1)/2N gives 13 of the all 14 symmetry types where char = ch

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X corresponds to a *crosscap*

The orbifold of a pattern with signature x (projective space) has char =1. Bringing together antipodal points on the sphere halves the sphere and so halves char .

Note: topologically a crosscap corresponds to adding a *Möbius band* to a hole

Note: We did not prove that these are ALL possibilities…

The Magic theorem for Frieze Patterns as a corollary of the spherical case

Step 1: Roll up an infinite frieze around the equator of a (large enough) sphere so that the pattern is repeated N times.

=> Rotational symmetry of order N within seven spherical symmetries

*22N, *NN, N*, 2*N, Nx, 22N, NN

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EX: N*

Step 2: N->∞, (N-1)/2N->½ and (N-1)/N->1

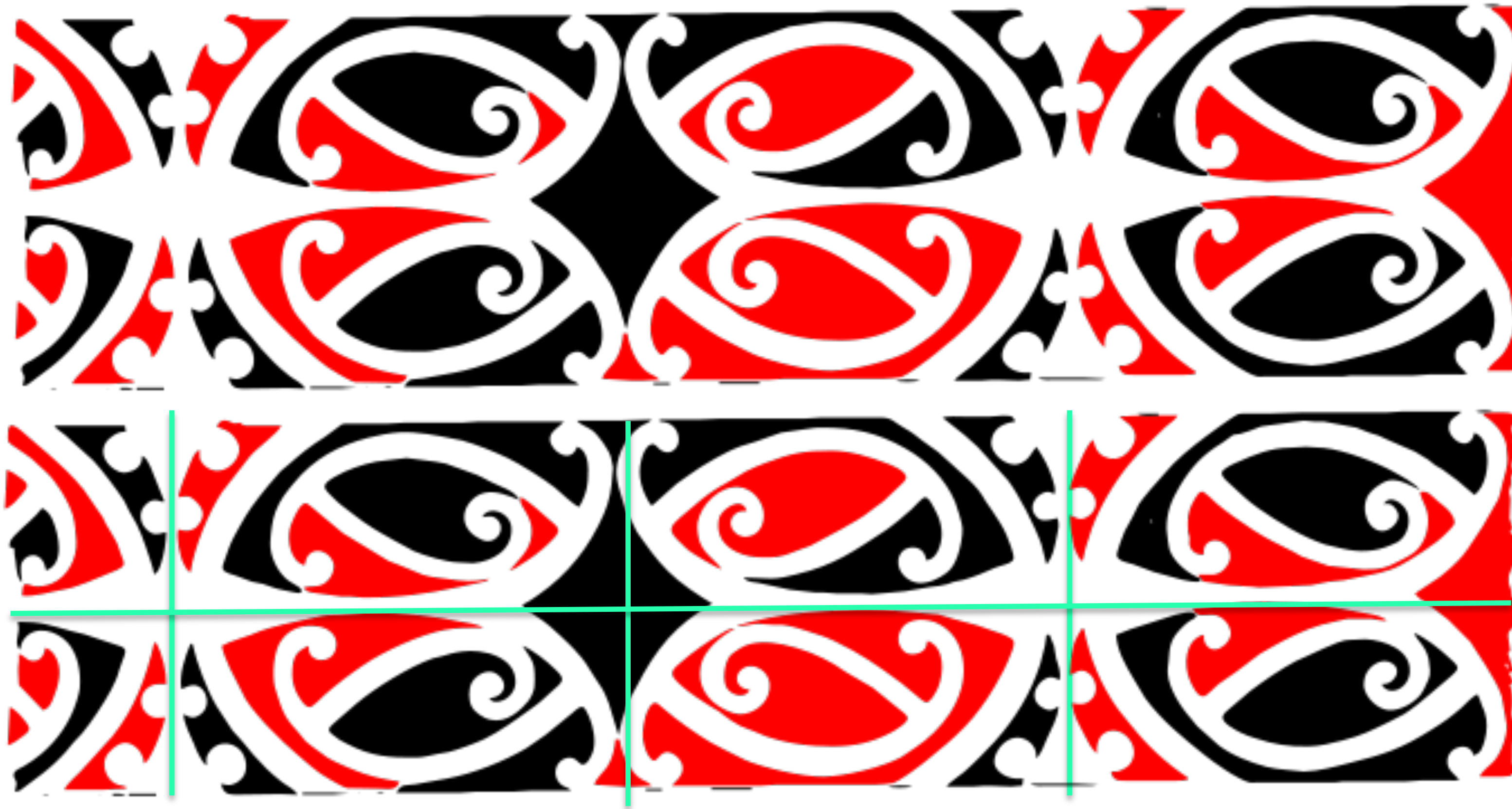
Step 3: Frieze patterns have infinitely many symmetries.

=>Total cost of a frieze pattern is 2 and always contains ∞ s.t. cost(∞)=½, cost(∞)=1

*22N, *NN, N*, 2*N, Nx, 22N, NN give

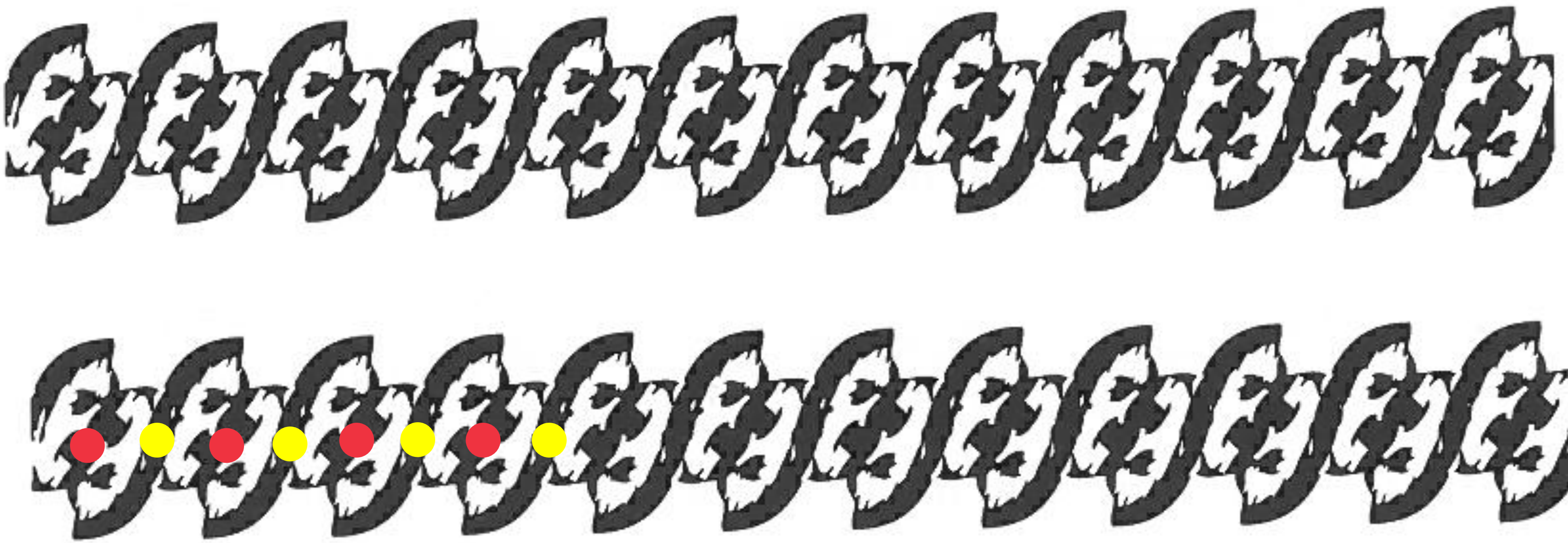
*22∞, *∞∞, ∞*, 2*∞, ∞x, 22∞, ∞∞

EX: *22∞



Ignore colors here!

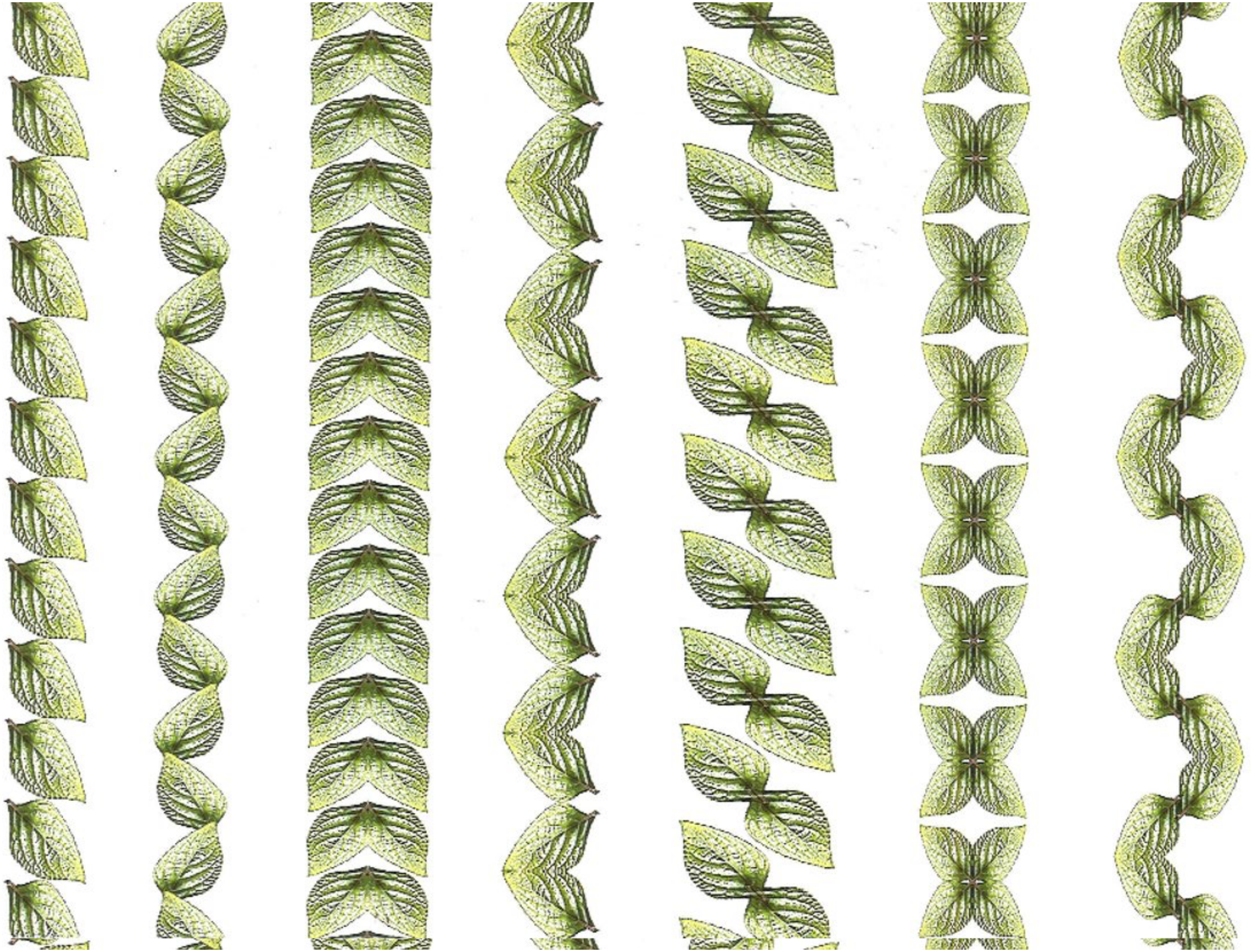
22∞



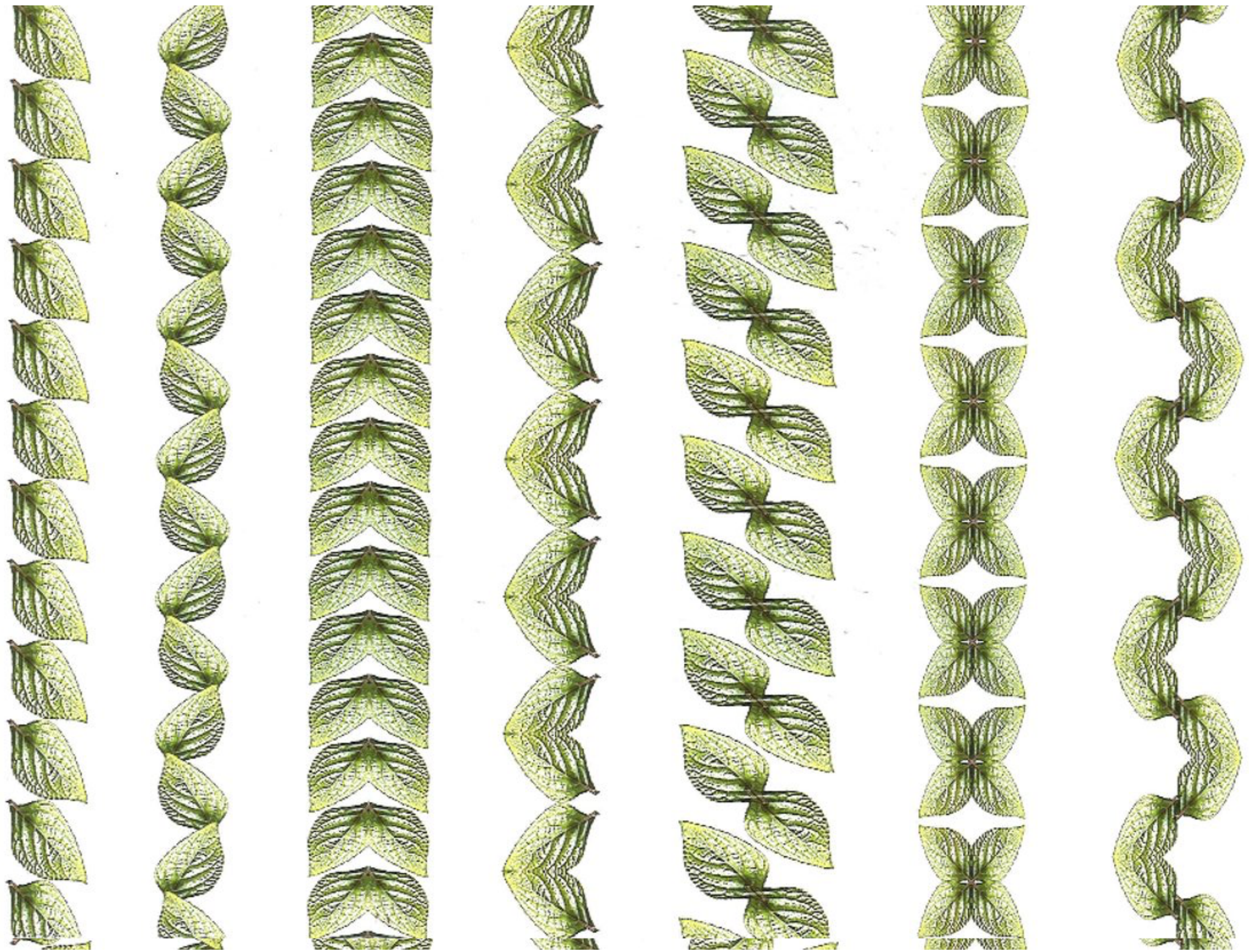
*∞∞

Not **, which is a planar symmetry

What are these symmetries ?



$\infty\infty$ | $\infty\times$ | $\infty *$ | $*\infty\infty$ | 22∞ | $*22\infty$ | $2*\infty$



Next Tue 29th

Be prepared for block printing presentations and feedback from Laura ☺