Taloustieteen matemaattiset menetelmät
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## Problem Set 2: Solutions

1. **Solution** Let $R_1$, $R_2$ and $R_3$ stand for rows 1–3, respectively. Construct the augmented matrix $(A|I)$ and reduce $A$ to its reduced row echelon form by applying Gauss-Jordan elimination to $(A|I)$:

   (1) Add $R1$ to the second row and substract $2R_1$ from the third row

   (2) Substract $3R_2$ from the third row

   (3) Divide the third row by 2

   (4) Add $4R_3$ to the first row and substract $R_3$ from the second row

   (5) Substract $2R_2$ from the first row

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ -1 & -1 & 5 & 0 & 1 & 0 \\ 2 & 7 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(1)} \left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 3 & 5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(2)} \left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & -5 & -3 & 1 \end{array}\right] \xrightarrow{(3)} \left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{(4)} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -9 & -6 & 2 \\ 0 & 1 & 0 & \frac{7}{2} & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{(5)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & -11 & 3 \\ 0 & 1 & 0 & \frac{7}{2} & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

   This is the augmented matrix $(I|A^{-1})$. Therefore $A^{-1} = \begin{bmatrix} -16 & -11 & 3 \\ \frac{7}{2} & \frac{5}{2} & -\frac{1}{2} \\ -\frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$.

2. **Solution** Compute the determinant of the coefficient matrix:

$$D = \begin{vmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ 1 & -2 & -3 \end{vmatrix} = 2 \times \begin{vmatrix} 5 & 2 \\ -2 & -3 \end{vmatrix} + (-3) \times \begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} + (-1) \times \begin{vmatrix} 3 & 5 \\ 1 & -2 \end{vmatrix}$$

$$= 2 \times ((-15) - (-4)) + (-3) \times (-9 - 2) + (-1) \times (-6 - 5)$$

$$= 22 \neq 0, \text{so a unique solution exists.}$$

Compute $B_x$, $B_y$ and $B_z$:

$$B_x = \begin{vmatrix} 1 & 3 & -1 \\ 8 & 5 & 2 \\ -1 & -2 & -3 \end{vmatrix} = 1 \times \begin{vmatrix} 5 & 2 \\ -2 & -3 \end{vmatrix} + (-3) \times \begin{vmatrix} 8 & 2 \\ -1 & -3 \end{vmatrix} + (-1) \times \begin{vmatrix} 8 & 5 \\ -1 & -2 \end{vmatrix} = 66$$

$$B_y = \begin{vmatrix} 2 & 1 & -1 \\ 3 & 8 & 2 \\ 1 & -1 & -3 \end{vmatrix} = 2 \times \begin{vmatrix} 8 & 2 \\ -1 & -3 \end{vmatrix} + (-1) \times \begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} + (-1) \times \begin{vmatrix} 3 & 8 \\ 1 & -1 \end{vmatrix} = -22$$

$$B_z = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 5 & 8 \\ 1 & -2 & -1 \end{vmatrix} = 2 \times \begin{vmatrix} 5 & 8 \\ -2 & -1 \end{vmatrix} + (-3) \times \begin{vmatrix} 3 & 8 \\ 1 & -1 \end{vmatrix} + (1) \times \begin{vmatrix} 3 & 5 \\ 1 & -2 \end{vmatrix} = 44$$

Thus, $x = \frac{B_x}{D} = \frac{66}{22} = 3$, $y = \frac{B_y}{D} = \frac{-22}{22} = -1$, and $z = \frac{B_z}{D} = \frac{44}{22} = 2$.

3. **Solution** A basis of $\mathbb{R}^3$ must contain exactly three vectors. So (a) and (e) are not bases. As for (b), the first and the third vectors are scalar multiples. This implies that the three vectors are linearly dependent, hence they do not constitute a basis. In (c), the determinant of the matrix having the three vectors as columns is equal to zero, so this cannot be a basis. In the remaining case (d), the determinant of the associated matrix is $-8 \neq 0$. Therefore, the three vectors form a basis of $\mathbb{R}^3$.

4. **Solution**

   (a) For any given $\epsilon > 0$, it suffices to take $N > \frac{1}{4\epsilon}$. Now for every $n \geq N$,

$$|x_n - L| = \left|\frac{3n+1}{4n} - \frac{3}{4}\right| = \left|\frac{3n+1-3n}{4n}\right| = \left|\frac{1}{4n}\right| \leq \left|\frac{1}{4N}\right| < \left|\frac{1}{4 \times \frac{1}{4\epsilon}}\right| = \epsilon$$

$$\Longrightarrow \lim_{n \longrightarrow \infty} \frac{3n+1}{4n} = \frac{3}{4}.$$

   (b) Take the sequence $\{2 + \frac{1}{n}\}$. This sequence converges to 2 as $n$ goes to infinity. Now, the sequence $\{(2 + \frac{1}{n})^2\} = \{4 + \frac{1}{n^2} + \frac{4}{n}\}$ converges to 4. But then we have $4 \neq f(2) = 0$. This shows the discontinuity at $x = 2$.

5. **Solution**

(a) $$\frac{\partial f(x,y)}{\partial x} = abx^{b-1}y^c$$

$$\frac{\partial f(x,y)}{\partial y} = acx^b y^{c-1}$$

(b) $$\frac{\partial f(x,y)}{\partial x} = -\frac{a}{1-x}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{b}{y}$$

(c) $$\frac{\partial f(x,y)}{\partial x} = -\frac{ay^d}{b}cx^{-c-1}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{ady^{d-1}}{bx^c}$$

(d) $$\frac{\partial f(x,y,z)}{\partial x} = ae^{ax-by}$$

$$\frac{\partial f(x,y,z)}{\partial y} = -be^{ax-by}$$

$$\frac{\partial f(x,y,z)}{\partial z} = -1$$

(e) $$\frac{\partial f(x,y,z)}{\partial x} = \frac{1}{4}x^{-\frac{1}{2}}\left(x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5z^2\right)^{-\frac{1}{2}}$$

$$\frac{\partial f(x,y,z)}{\partial y} = \frac{1}{6}y^{-\frac{2}{3}}\left(x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5z^2\right)^{-\frac{1}{2}}$$

$$\frac{\partial f(x,y,z)}{\partial z} = 5z\left(x^{\frac{1}{2}} + y^{\frac{1}{3}} + 5z^2\right)^{-\frac{1}{2}}$$