Aalto University

Hyperbolic geometry

Shapes in Action 2nd Oct 2020

Program schedule for Oct 2nd

13:15 Wrap up of Ex this week (Archimedean solids)
& Frieze pattern analysis

14:00 Break

14:15 Some principles of Hyperbolic geometry
Curvature & classification of surfaces

15:00 Break

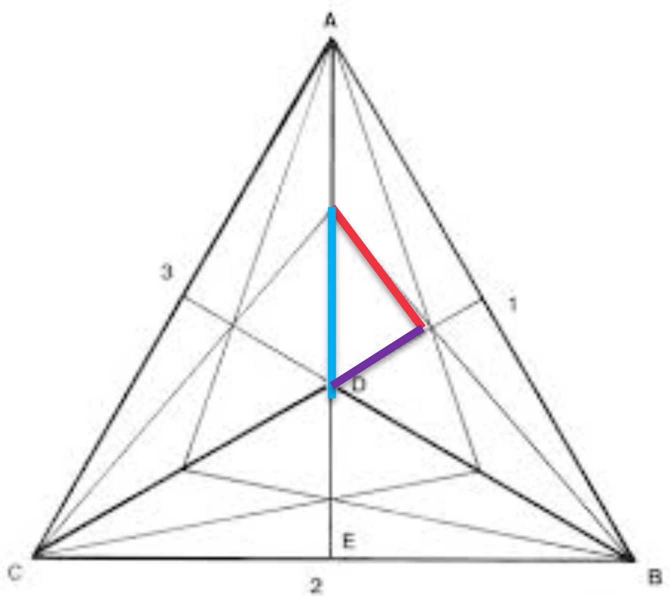
15:15 Magic theorem for hyperbolic tilings ?
What are hyperbolic surfaces ?

Archimedean solids and their symmetries

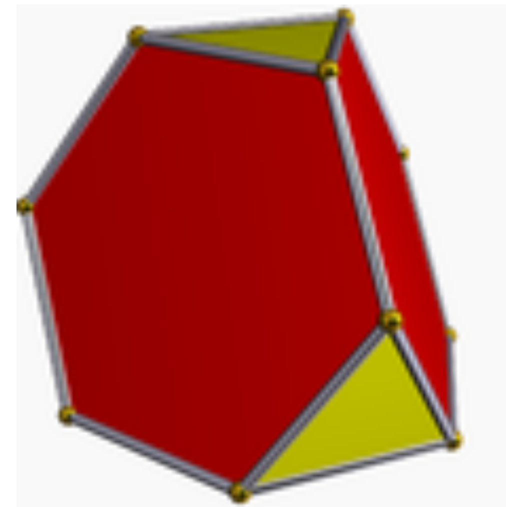
*332 (1)
*432 (5+one 432)
*532 (5+one 532)

families

*332 symmetry

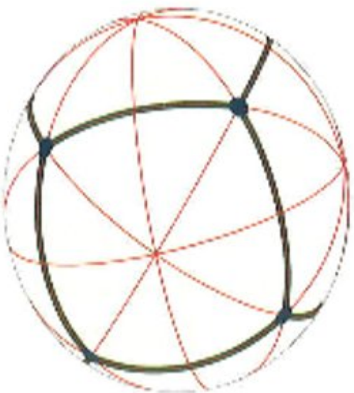
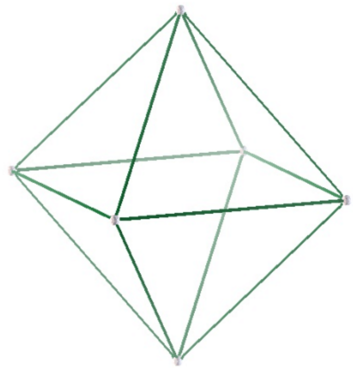


Tetrahedron (Platonic solid)
1+2/6+2/6+1/4= 1+11/12 =2-2/24,
d=24=4x6

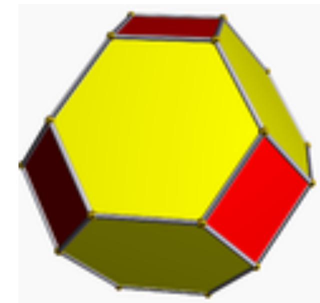


Truncated tetrahedron

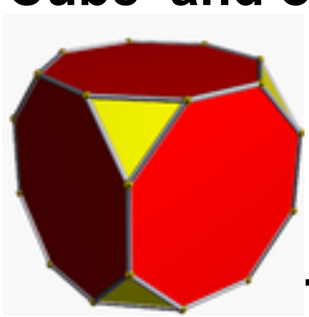
*432 symmetry

Cube and octahedron (Platonic solids)



Truncated octahedron

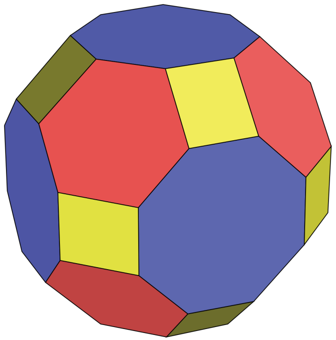


Truncated cube

Cuboctahedron

1+3/8+1/3+1/4=1+23/24 =2- 1/24 =2-2/48, d=6(faces)*8=48 for cube

More *432 symmetry

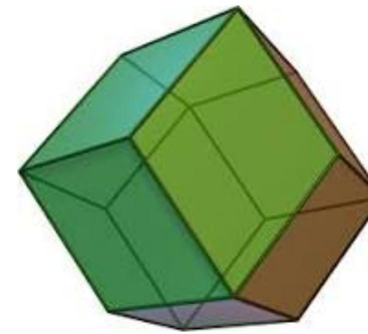


Truncated cuboctahedron

- 12 squares
- 8 hexagons
- 6 octagons

Rhombicuboctahedron
(truncated rhombic dodecahedron)

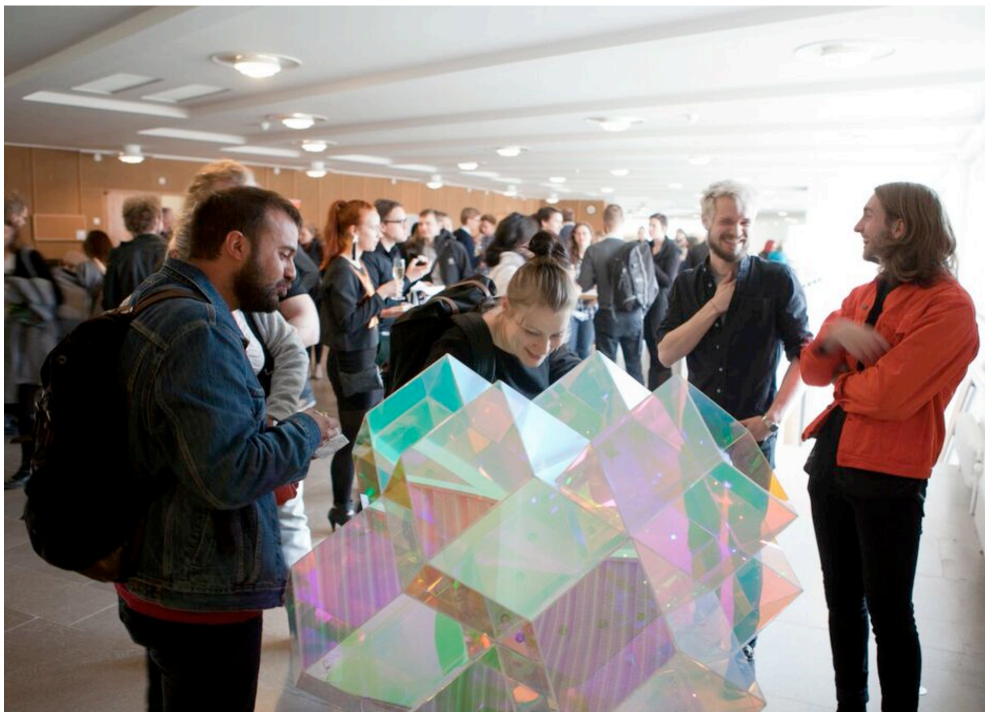
- 8 triangles
- 18 squares



Rhombic dodecahedron
(dual of cuboctahedron)

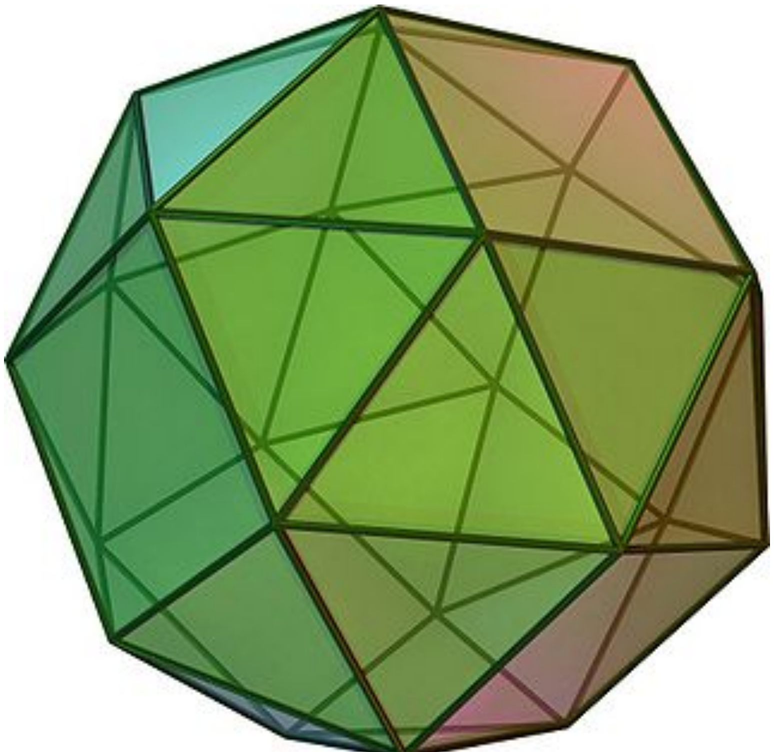
Rhombic dodecacube at Otakaari 1 lobby

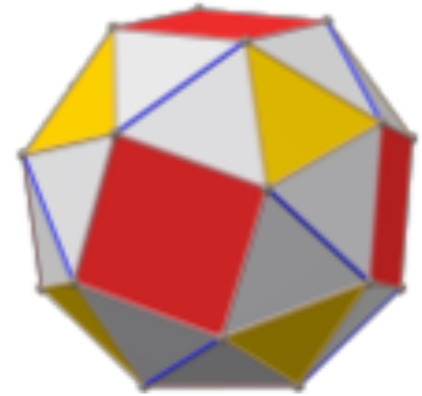
by E.Axelsson (SCI); H. Fred (HU, Computer science), H. Judin (ARTS), V. Livio (SCI) , M. Tuomela (ARTS)



Snub cube 432

From Rhombicuboctahedron

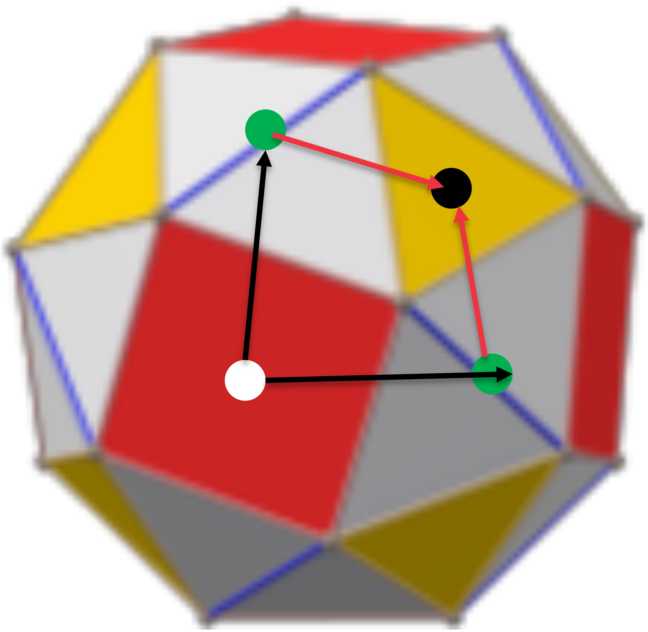




- 6 squares
- 32 triangles

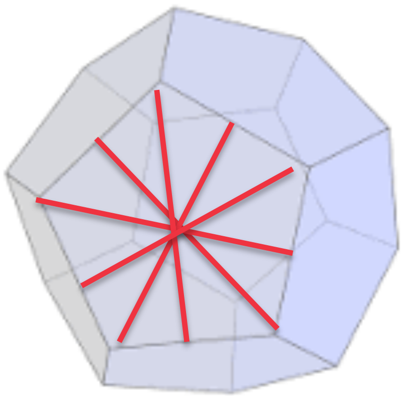
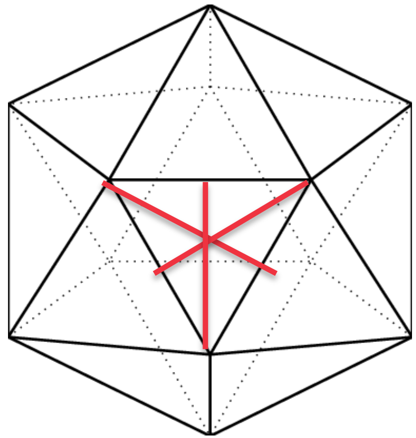
Snub cube 432

$\frac{3}{4}+2/3+1/2=23/12= 2-1/12=2-2/24$



- 6 squares
- 32 triangles

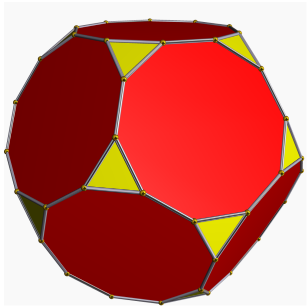
*532 Symmetry

12 (faces)* **10=**120**= 20** (faces)* **6**

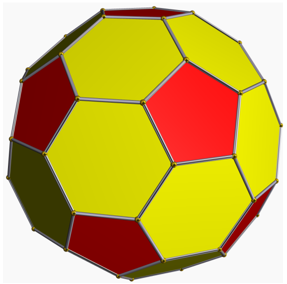
Dodecahedron and Icosahedron
(Platonic solids)

Truncated dodecahedron



- 20 triangles
- 12 decegons

Truncated icosahedron

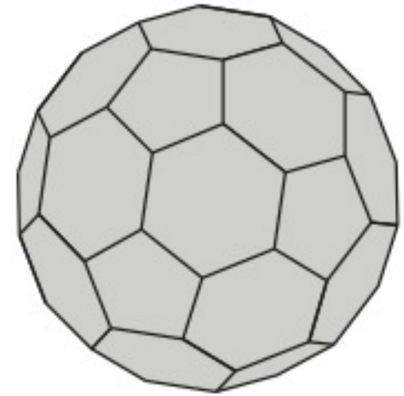


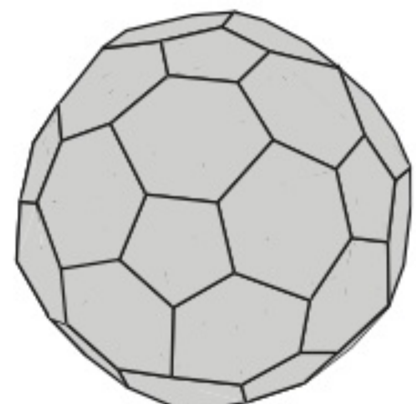
- 12 pentagons
- 20 hexagons

1+2/5+1/3+1/4= 1+59/60 =2- 1/60 =2-2/120

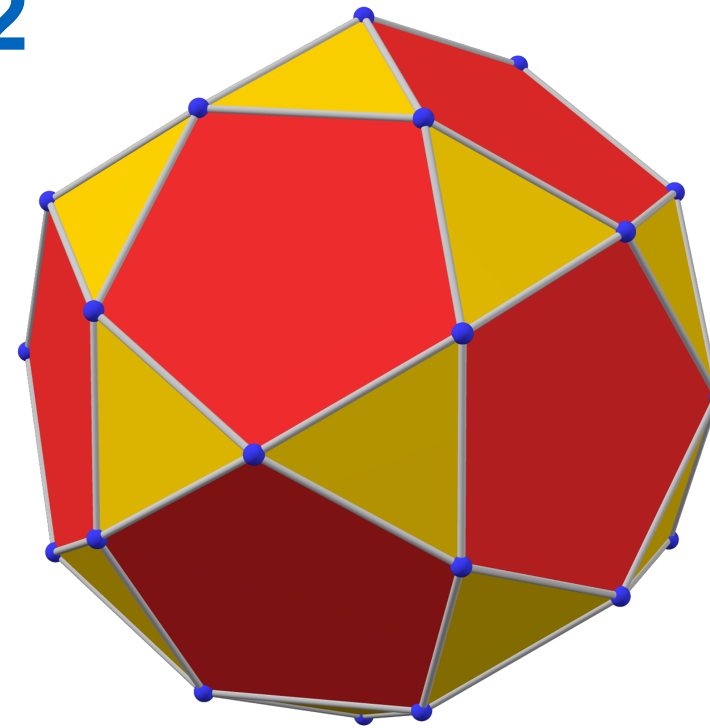
What is the difference between these polyhedra?

- 12 pentagons
- 20 hexagons
- 90 edges, 60 vertices



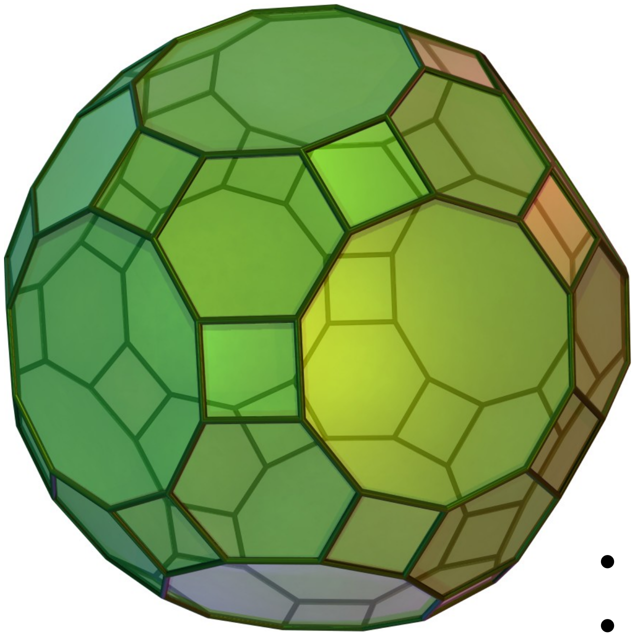


Icosidodecahedron *532

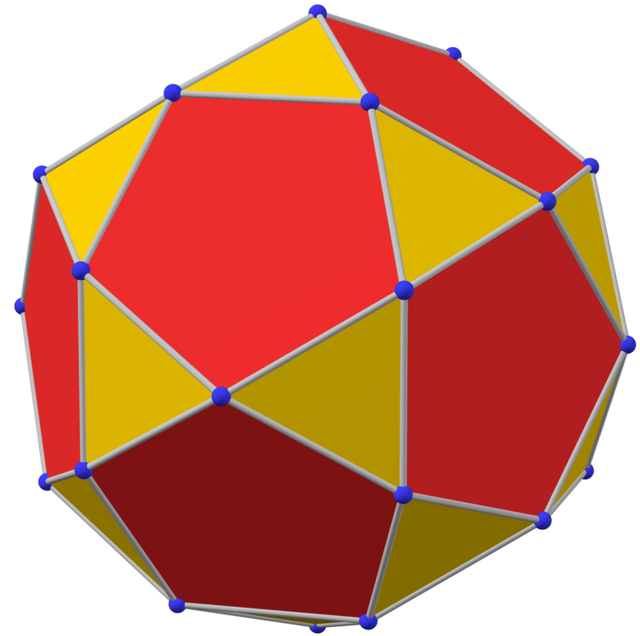


- 20 triangles
- 12 pentagons

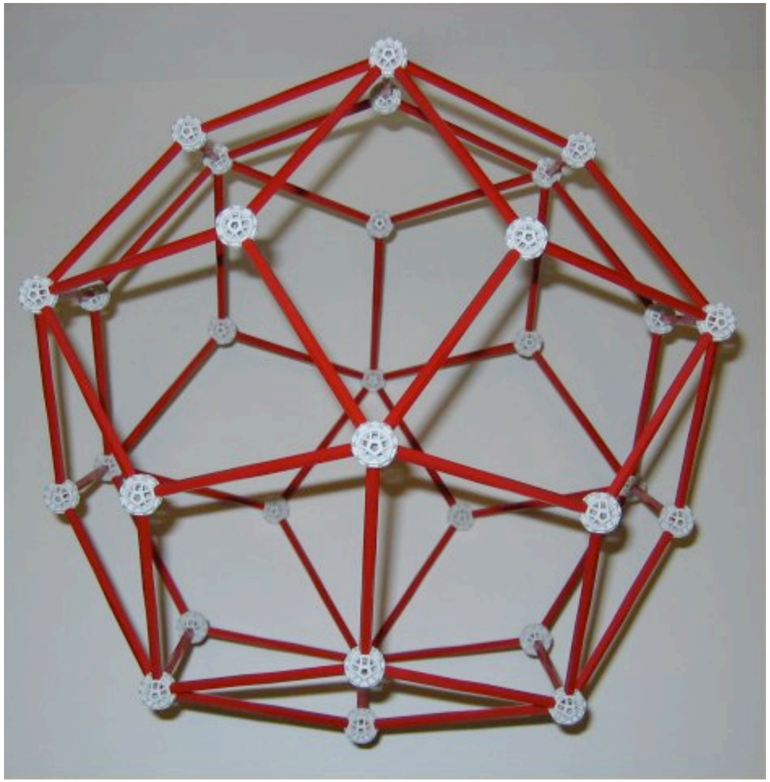
Truncated icosidodecahedron *532

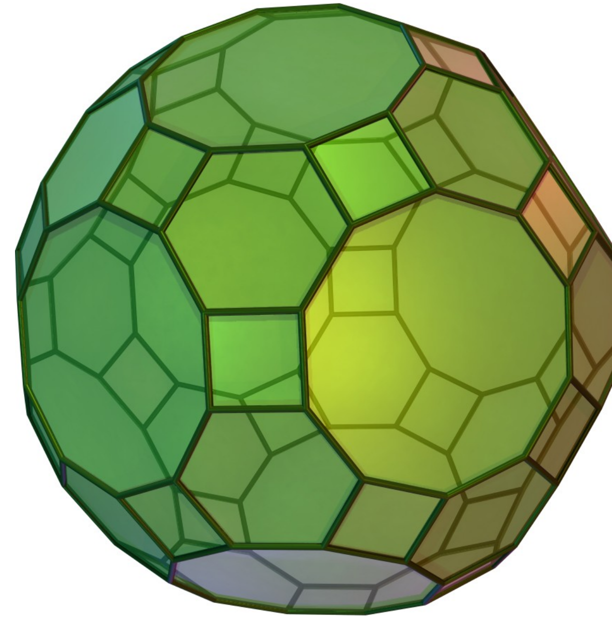


- 30 squares
- 20 hexagons
- 12 decagons



Rhombic triacontahedron

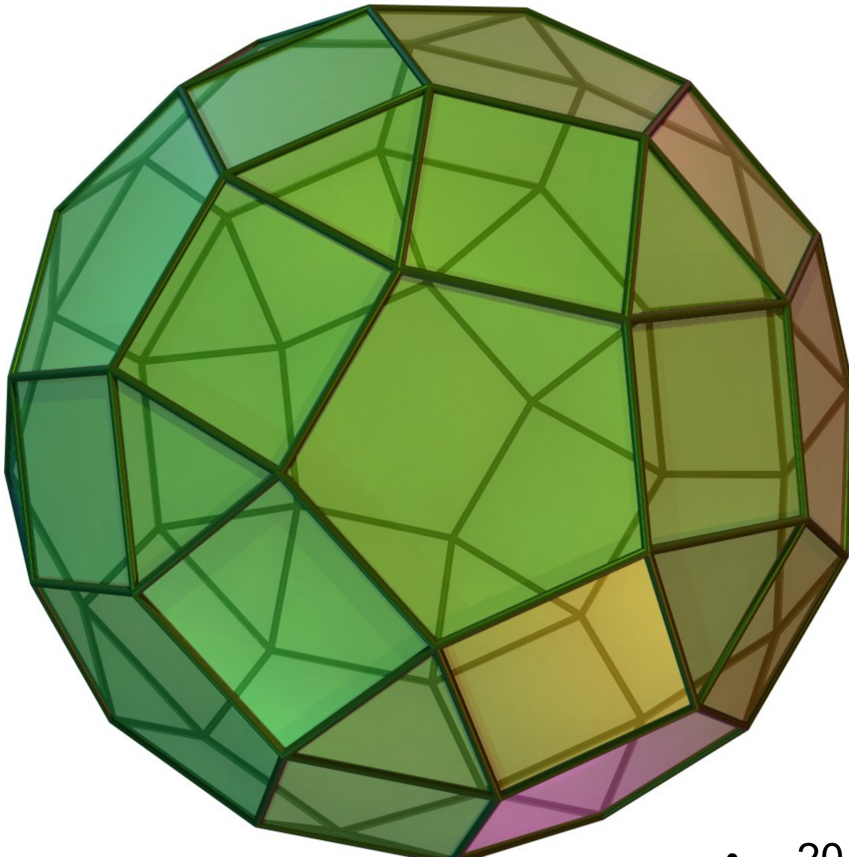




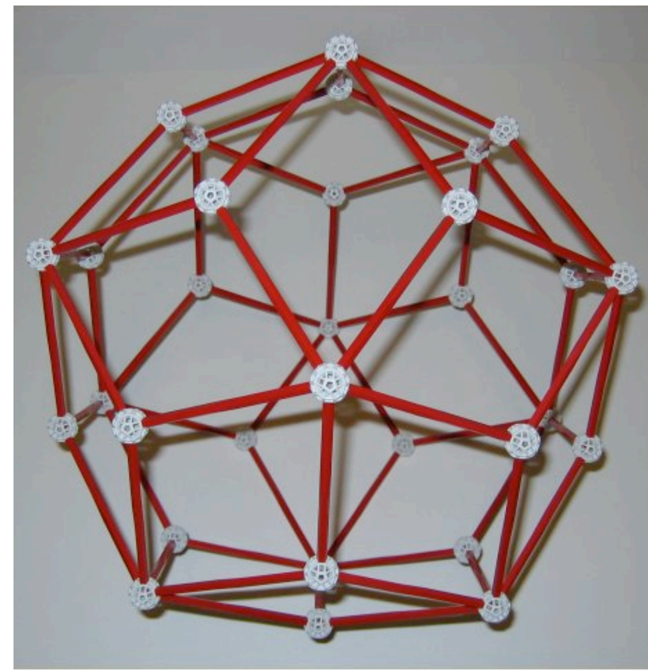
Dual of truncated icosidodecahedron

30 rhombuses

Rhombicosidodecahedron



Truncated rhombic triacontahedron



- 20 triangles
- 30 squares
- 12 pentagons

Rhombicosidodecahedron
design

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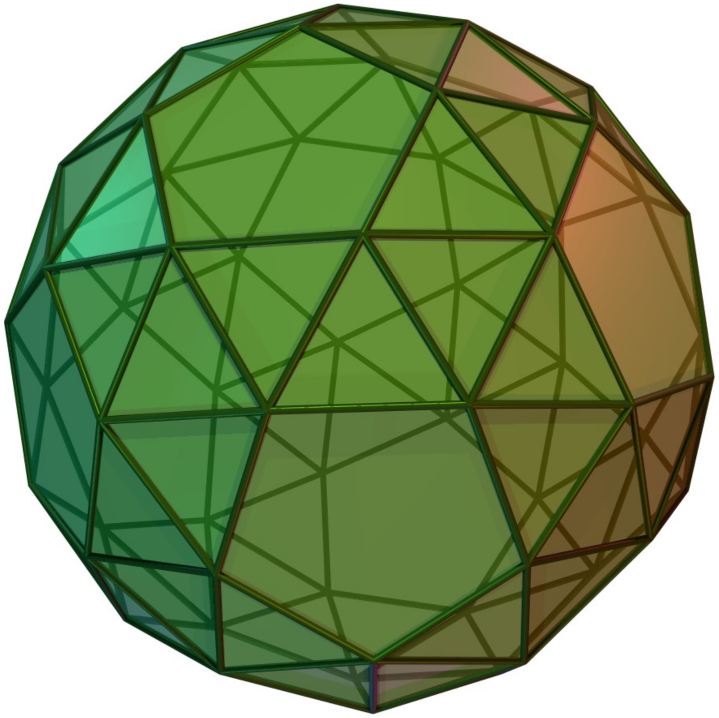
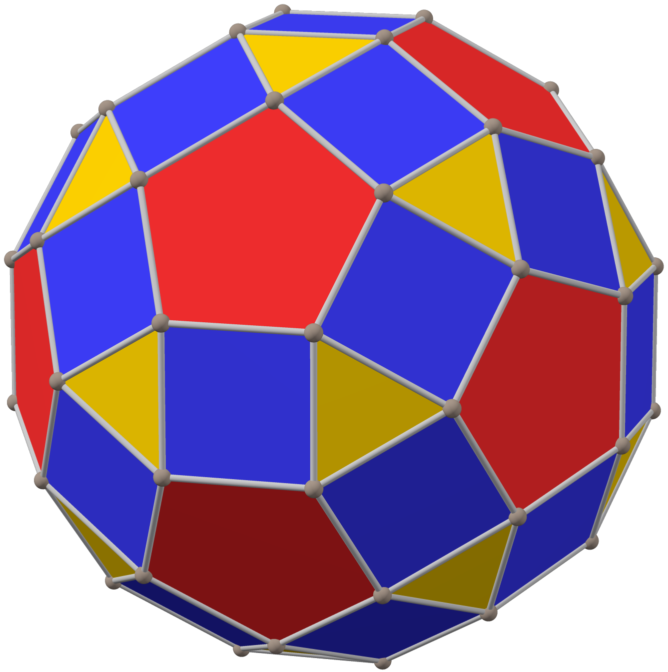
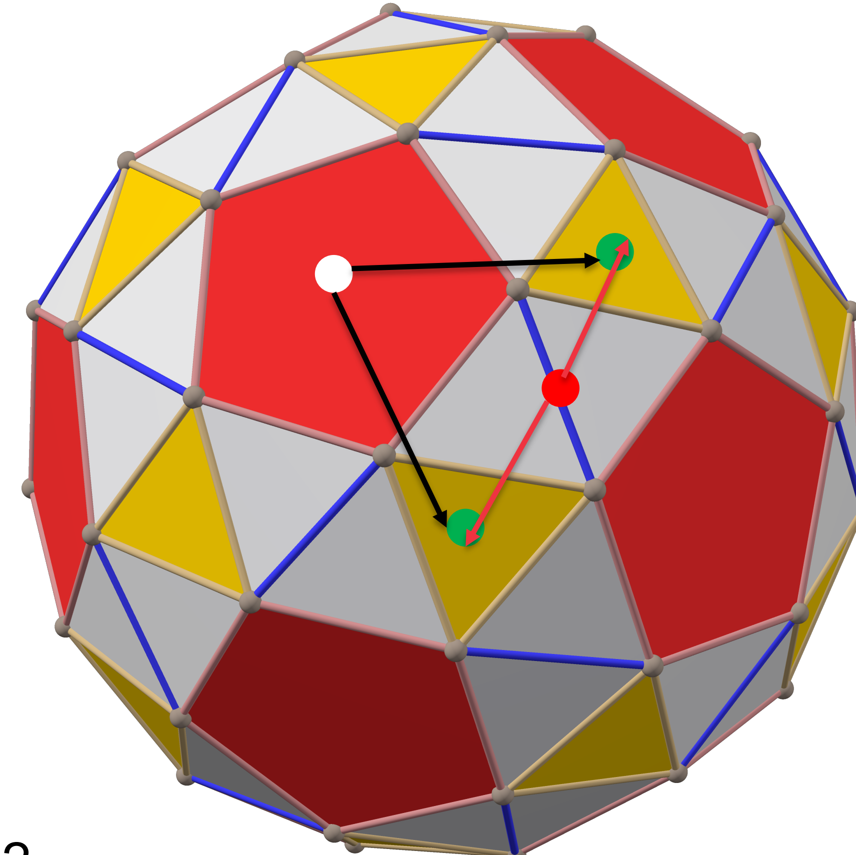
Snub dodecahedron 532



- 12 pentagons
- 80 triangles

Snub dodecahedron 532



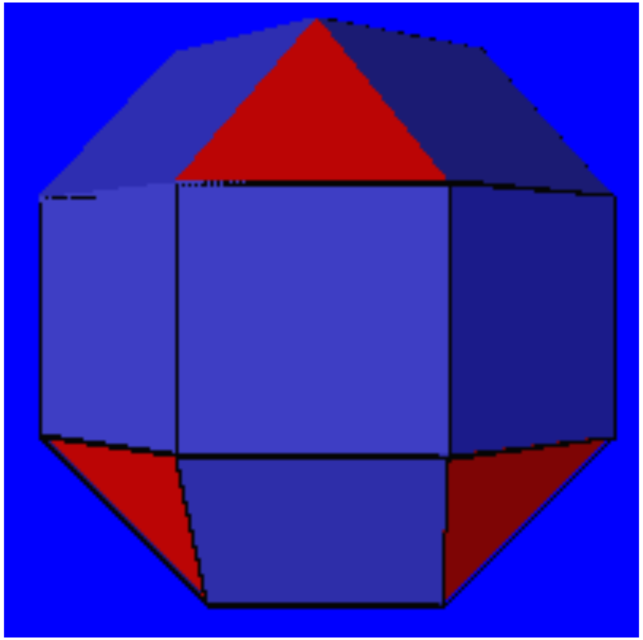


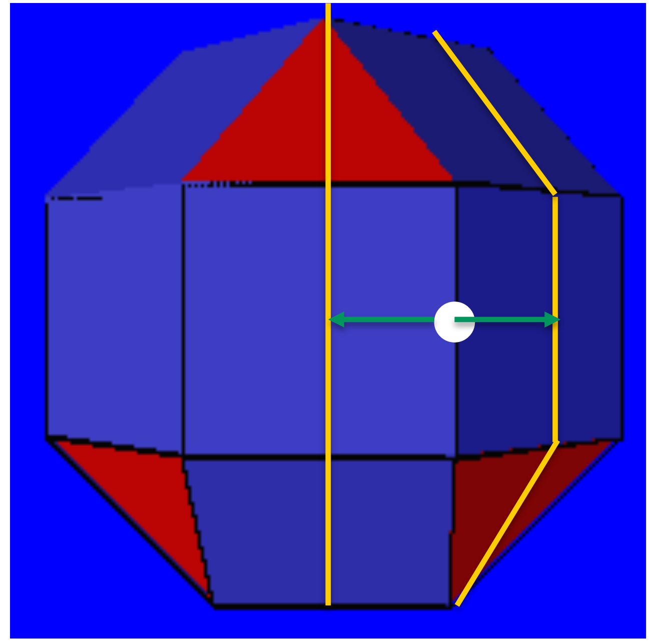
From Rhombicosidodecahedron *532

4/5+2/3+1/2=59/30=2-1/30= 2-2/60

Pseudo rhombicuboctahedron 2*4

½+1+3/8=1+7/8=1-1/8=2-2/16





Compare with the rhombicuboctahedron *432

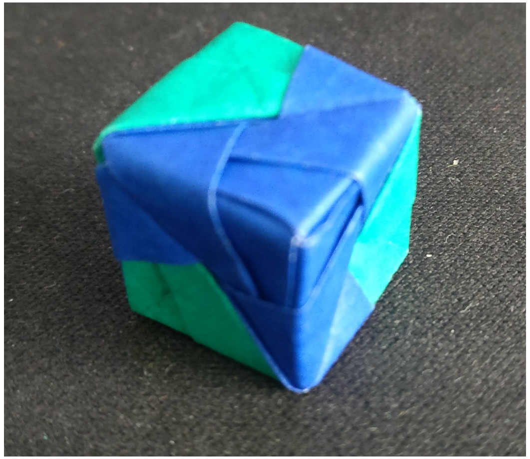
Johnson solid ('the 14th Archimedean solid' not vertex transitive)

Move and turn

By Jouko Koskinen and Jeff Weeks

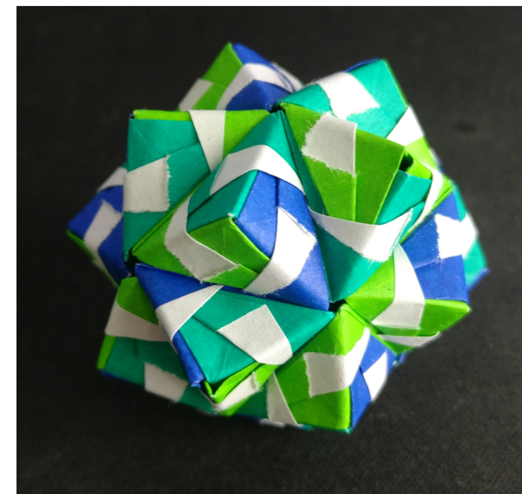
http://www.geometrygames.org/MoveTurn/index.html.en

Beautiful foldings by Elias Seeve



Stellated octahedron

Stellated icosahedron



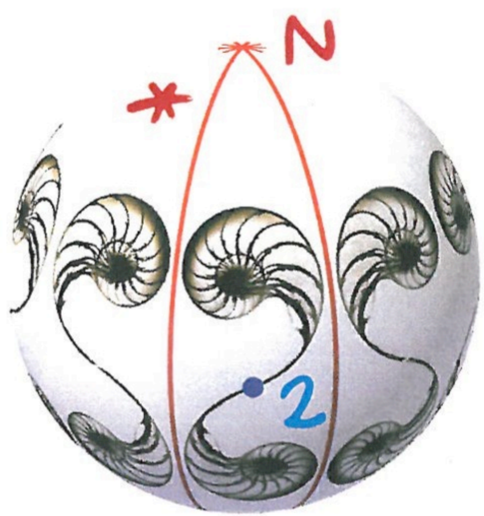
Sonobe units

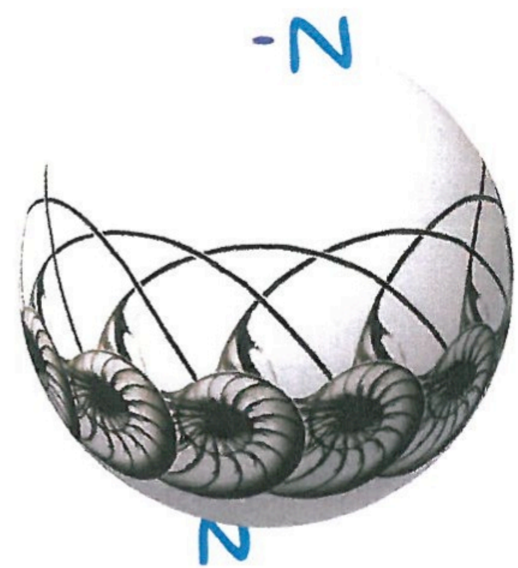
The Magic theorem for Frieze Patterns as a corollary of the spherical case

Step 1: Roll up an infinite frieze around the equator of a (large enough) sphere so that the pattern is repeated N times.

=> Rotational symmetry of order N within seven spherical symmetries

***22N, *NN, N*, 2*N, Nx, 22N, NN**



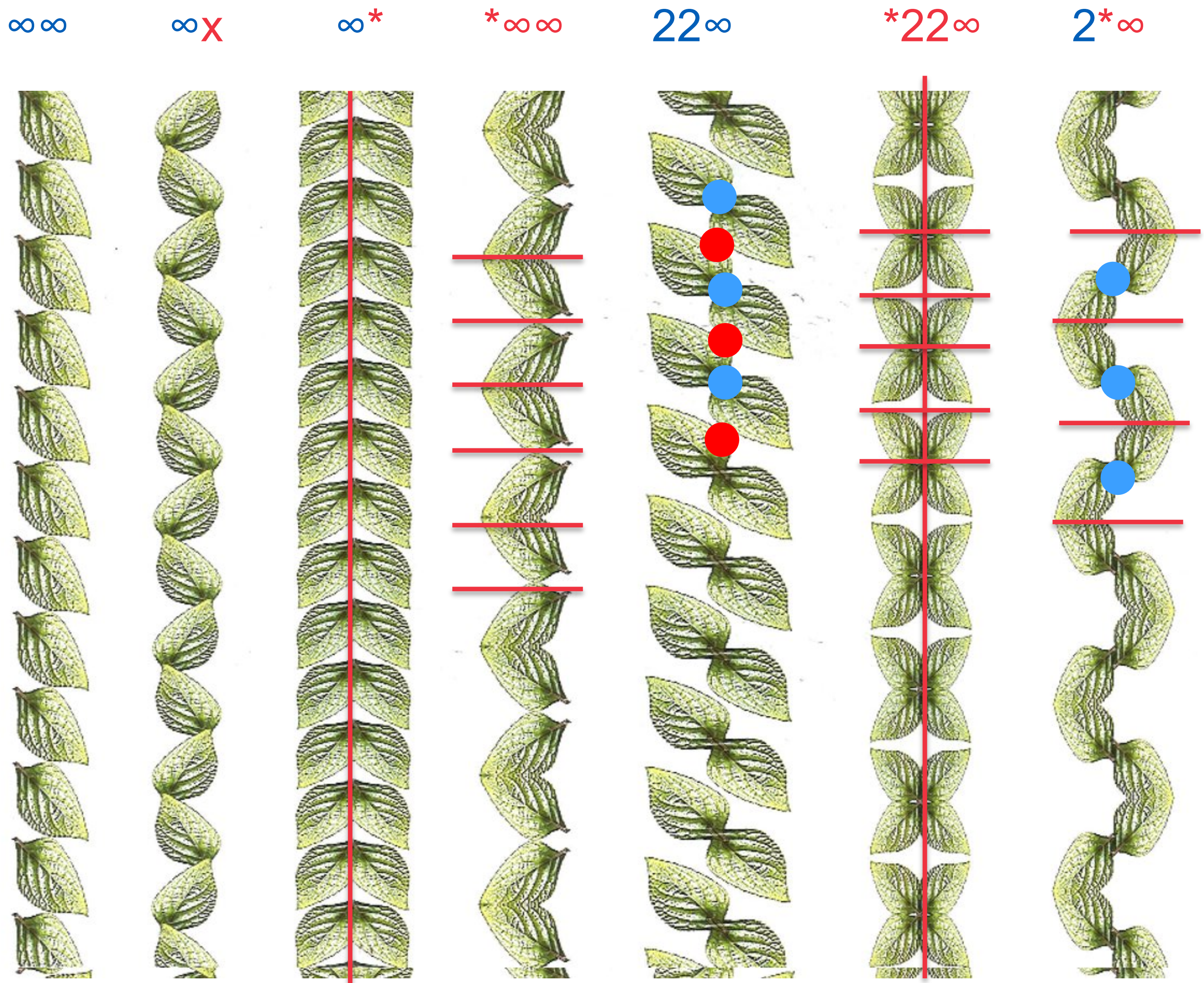


Step 2: N->∞, (N-1)/2N->½ and (N-1)/N->1

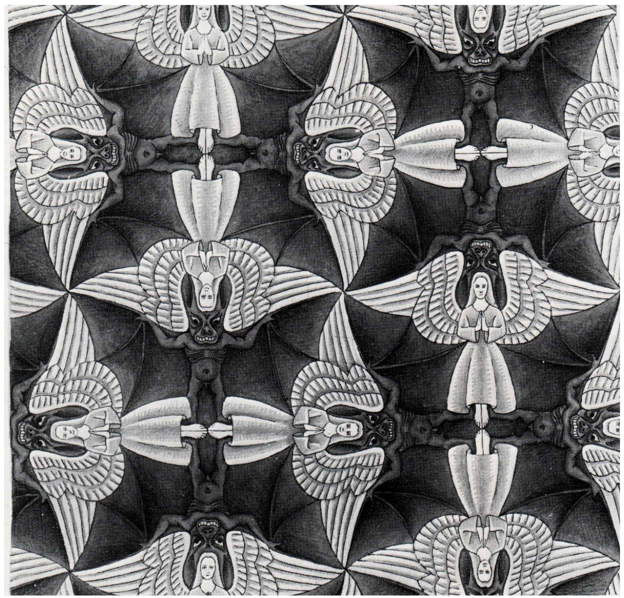
Step 3: Frieze patterns have infinitely many symmetries.
=>Total cost of a frieze pattern is 2 and always contains ∞ s.t. cost(∞)=½, cost(∞)=1

*22N, *NN, N*, 2*N, Nx, 22N, NN give
*22∞, *∞∞, ∞*, 2*∞, ∞x, 22∞, ∞∞

*∞∞

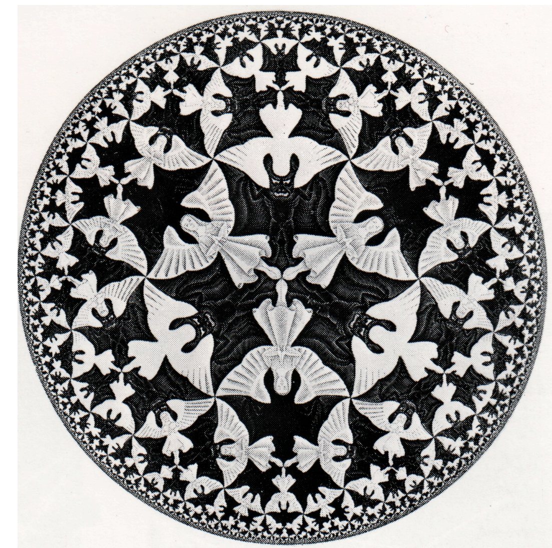

∞∞
∞x
∞*
*∞∞
22∞
*22∞
2*∞

Euclidean (=flat), spherical and hyperbolic models of 2D geometry



K = 0 (17 types)

K>0 (14)

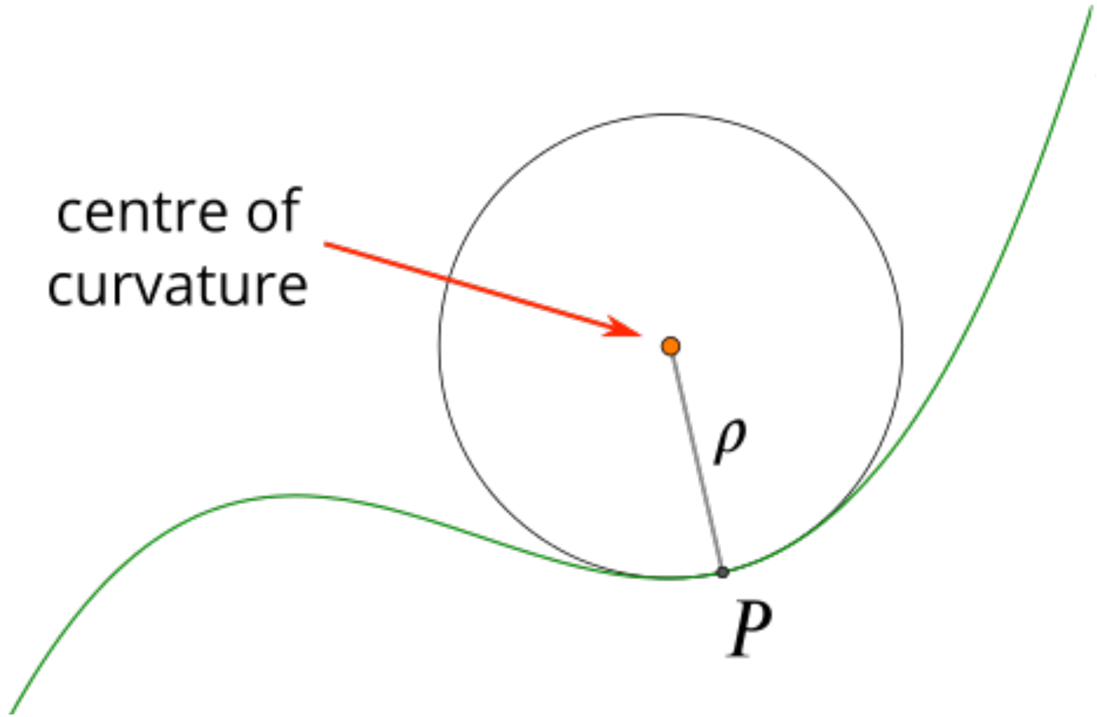


K<0 (∞)

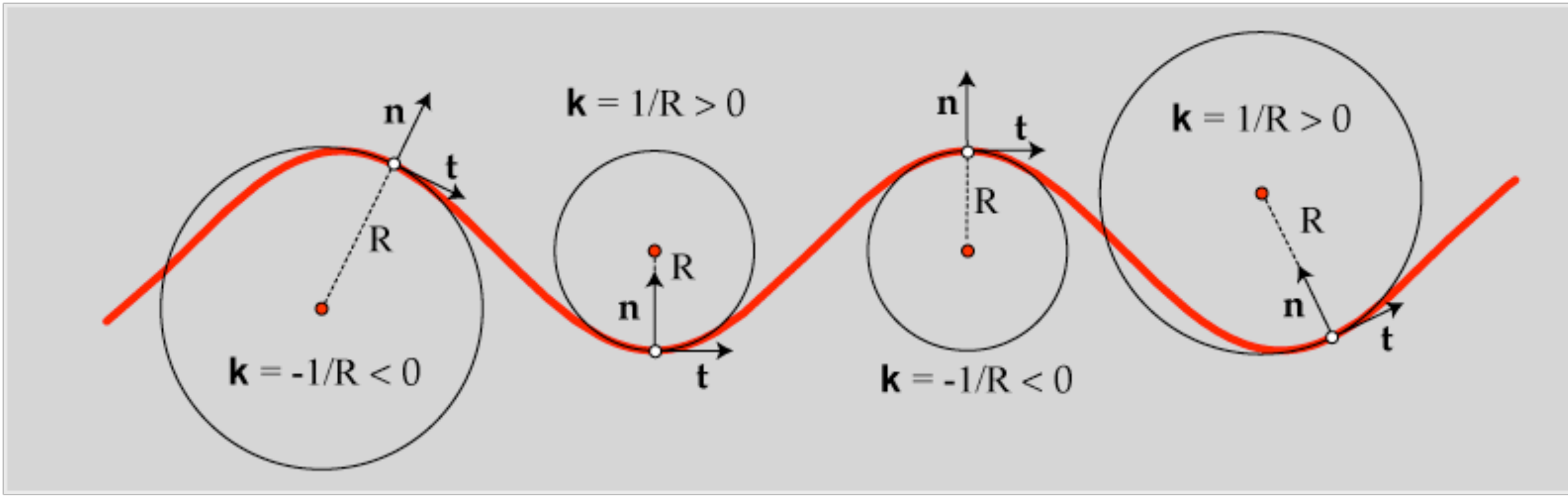
What is curvature ?

Curvature of a smooth planar ***curve*** at point **P** is $\varkappa(P)=1/\boldsymbol{\rho}$

- works also for curves in space or higher dimensions
- points should be approachable with circles
- ***extrinsic*** quantity



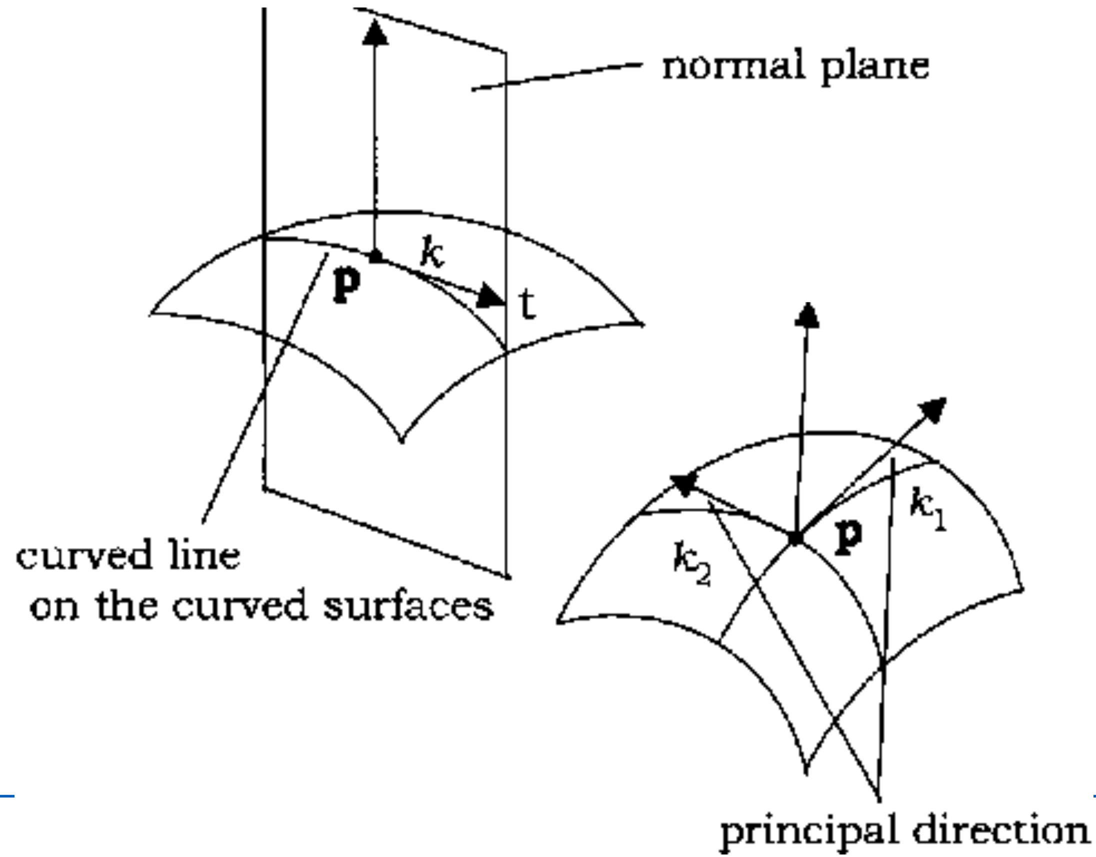
Curvature of a (parametrized planar) curve has a sign



What is curvature of a surface ?

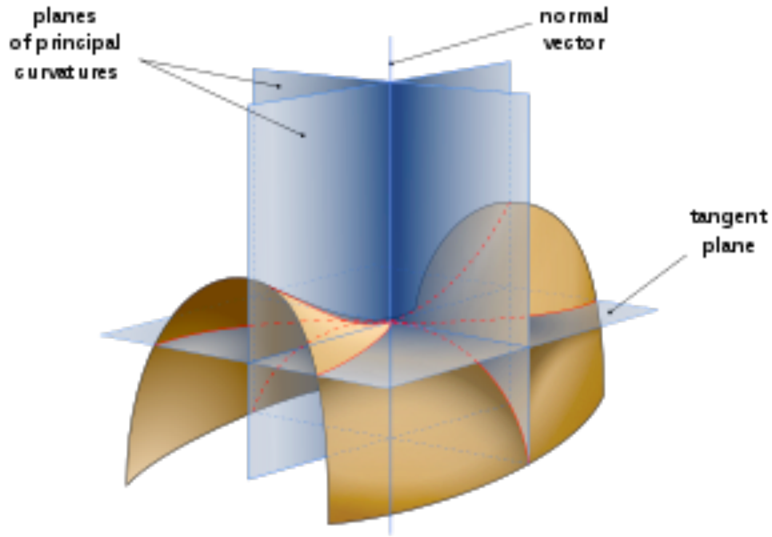
Gauss curvature

$$K(p) = \varkappa_1(p)\,\varkappa_2(p)$$

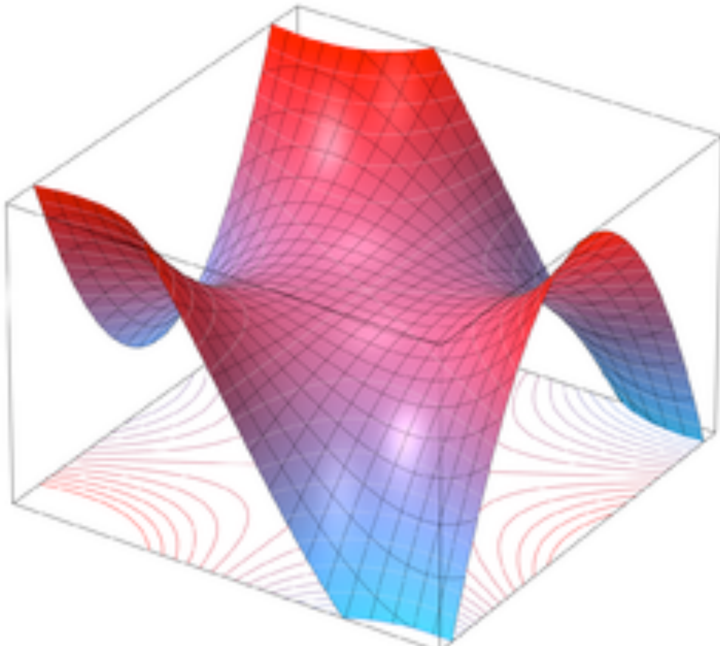


Theorema Egregium (Gauss, 1827)

Curvature K is an *intrinsic* quantity !

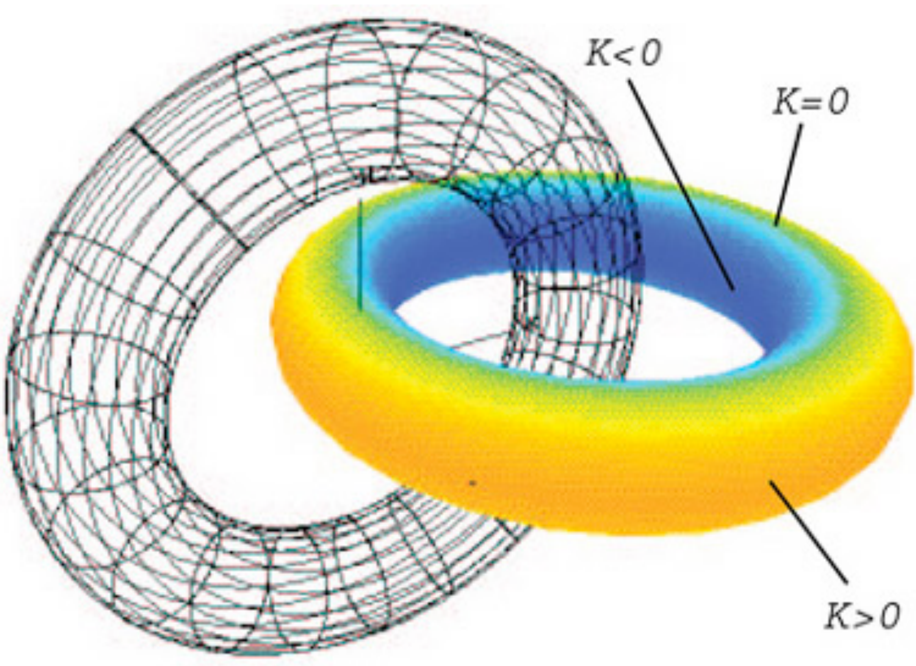


Warning: Monkey saddle



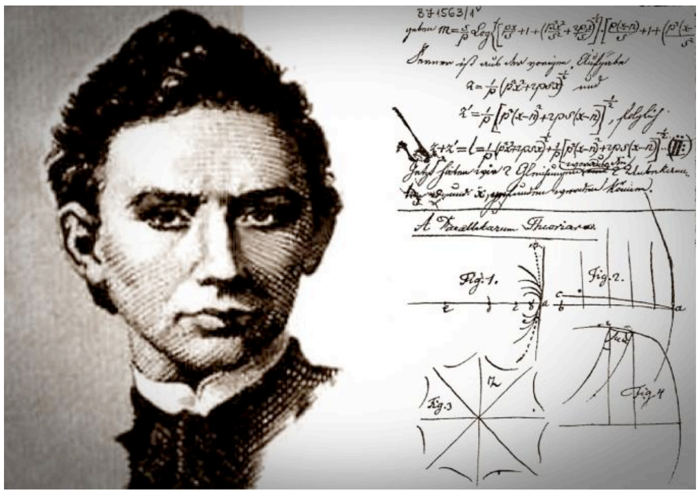
Zero Gaussian curvature at the origin

What are possible constant Gauss curvature geometries for smooth closed surfaces ?



Where did hyperbolic geometry come from ?

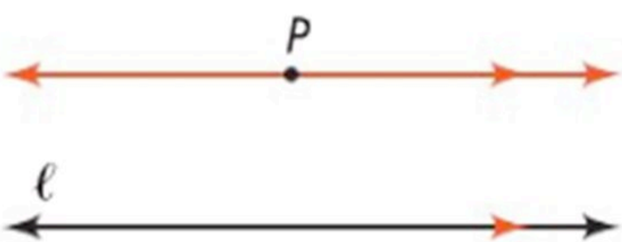
Prehistory: ***Janos Bolyai*** (1801-1860) ***N.I. Lobachevsky*** (1792-1856) independent studies axiomatically (without explicit construction)



Obsession on Euclid's parallel postulate:

Through a point not on a line, there is exactly one line parallel to the given line.

⇒ **There exists an 'Imaginary geometry' violating this postulate (late 1820)!**



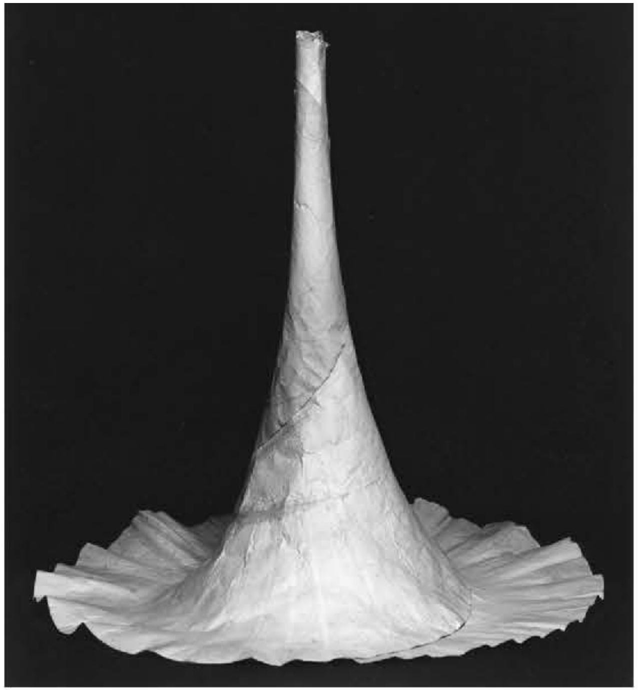
Bernhard Riemann (1826-1866)

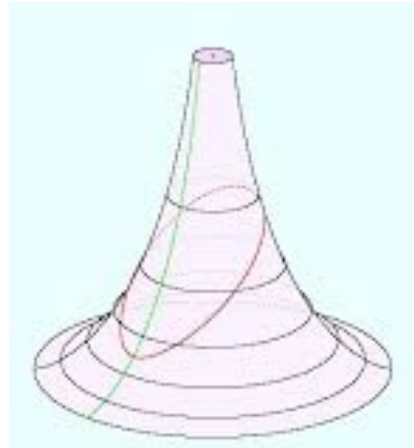
Inaugural lecture (Göttingen, 1854): ‘On the Hypotheses which Lie in the Foundation of Geometry’

Description how hyperbolic geometry would be the *intrinsic* geometry of a surface with constant negative curvature that extend indefinitely in all directions.

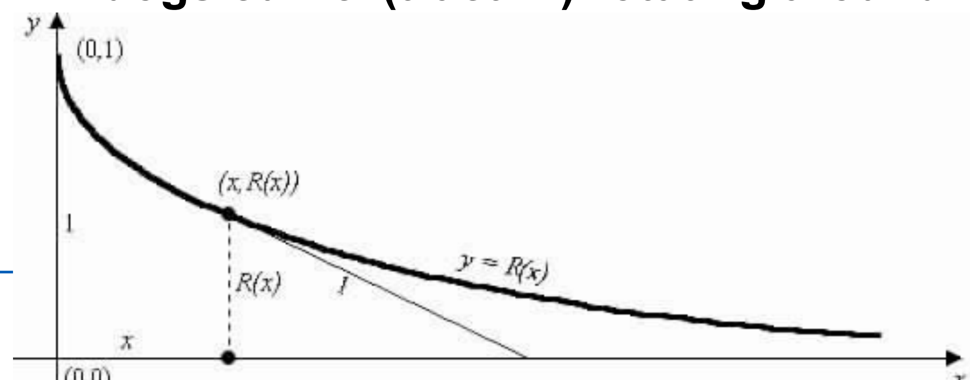
=> Is there a surface in 3-space with constant negative curvature ? (*)

Eugenio Beltrami (1835-1900)





Pseudosphere (1866) : A local model via ‘lazy dogs curve’ (tractrix) rotating around x-axis



Curvature -1

Antonio Candido Capelo, Mario Ferrari, La "cuffia" di Beltrami: storia e descrizione, Bollettino di Storia delle Scienze Matematiche 2 (1982): 233–237.

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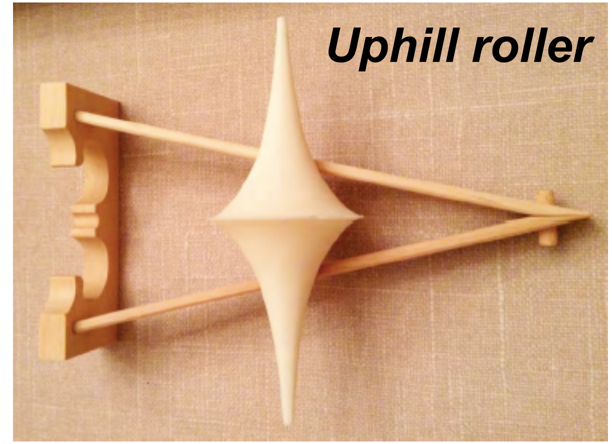
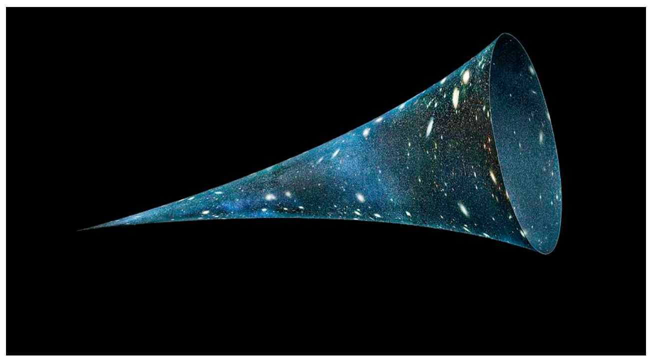
Some appearances of the pseudosphere



P. Voigt 1927: Patent for loudspeaker horn design based on tractrix



Picard's horn universe model
Emile Picard (1856-1941)

Moon flower

David Hilbert (1862-1943)

Answer (1901) to the question (*) posed by Riemann:

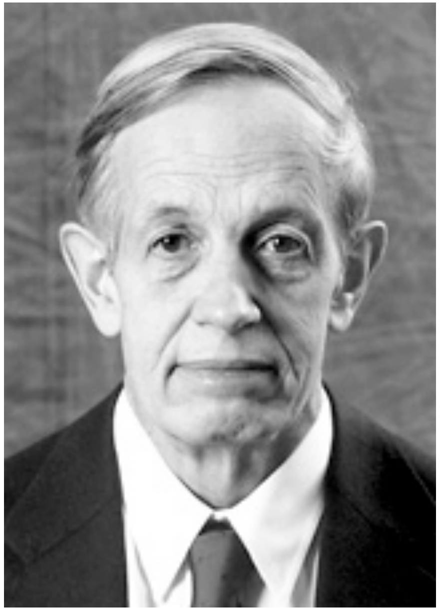
It is not possible to have an equation describe a surface in 3-space that has constant negative curvature and that is extended indefinitely in all directions.

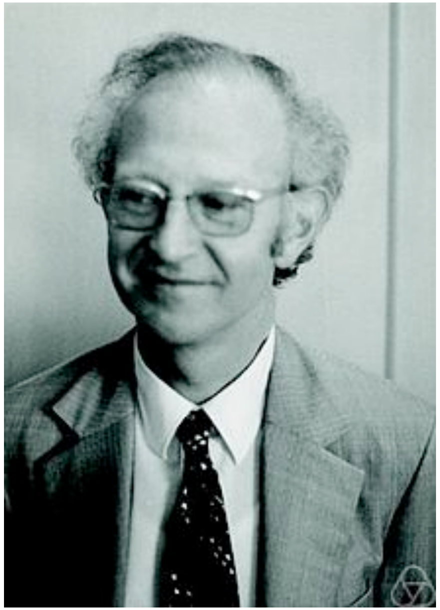
Improvements by Erik Holmgren (1902), Marc Amsler (1955)

Nicolaas Kuiper (1920-1994)
John (‘beautiful mind’) Nash (1928-2015)

Any abstract manifold can be seen as a submanifold in some (enough) high dimensional Euclidean space. (1956, 1966)

⇒ **Holds also for hyperbolic surfaces !**

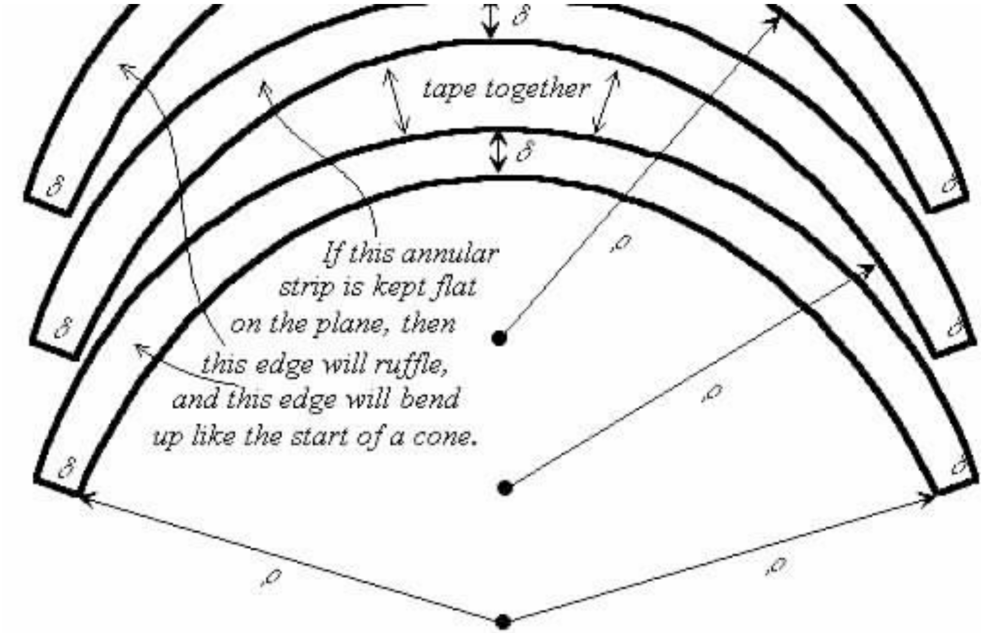




D. Blanusa (1955)

The hyperbolic plane can be embedded smoothly and isometrically into the 6-dimensional Euclidian space.

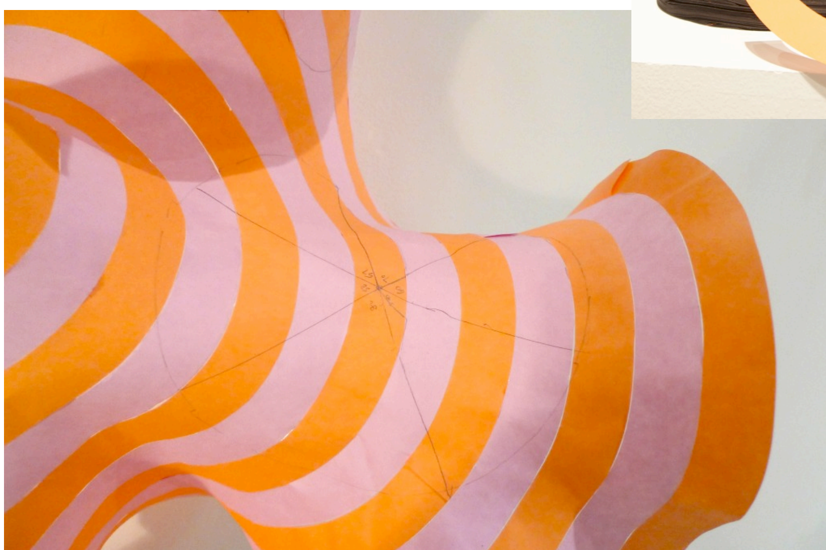
Bill Thurston and his paper annuli to approximate hyperbolic surfaces



ρ = radius of the hyperbolic plane
Curvature - 1/ ρ^2

What I hear I forget,
What I see, I remember,
What I touch, I understand.
- Confius (555-479 CE)

Some outcomes from the workshop at the Institute of Figuring (theiff.org)



Daina Taimina

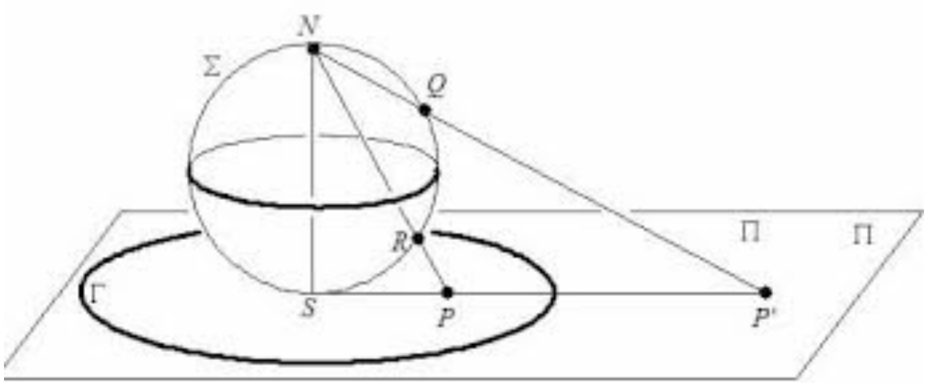
https://www.youtube.com/watch?v=rY8Uo6rSnZc

Study crocheted surfaces

- Try to find curves that realize shortest distances between some points ie *geodesics*
- Try to convince yourself that the parallel axiom does not hold
- can you find the radius of the surface ?

On various ways to map hyperbolic surfaces to Euclidean 3-space

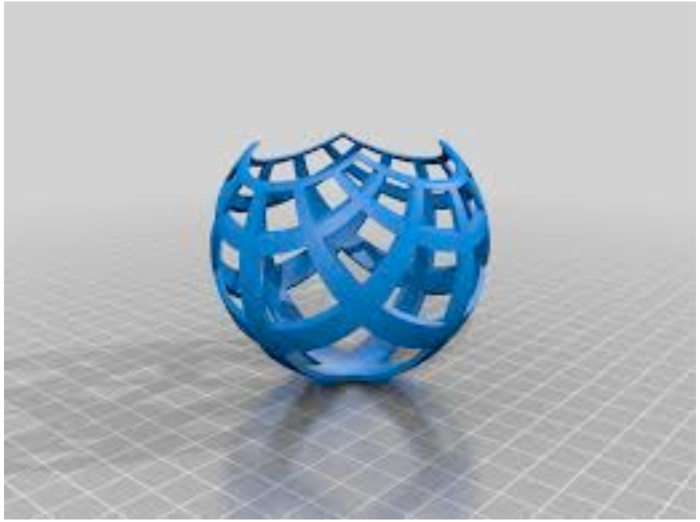
Analogous problem as studying geography of our spherical planet by looking a flat map.

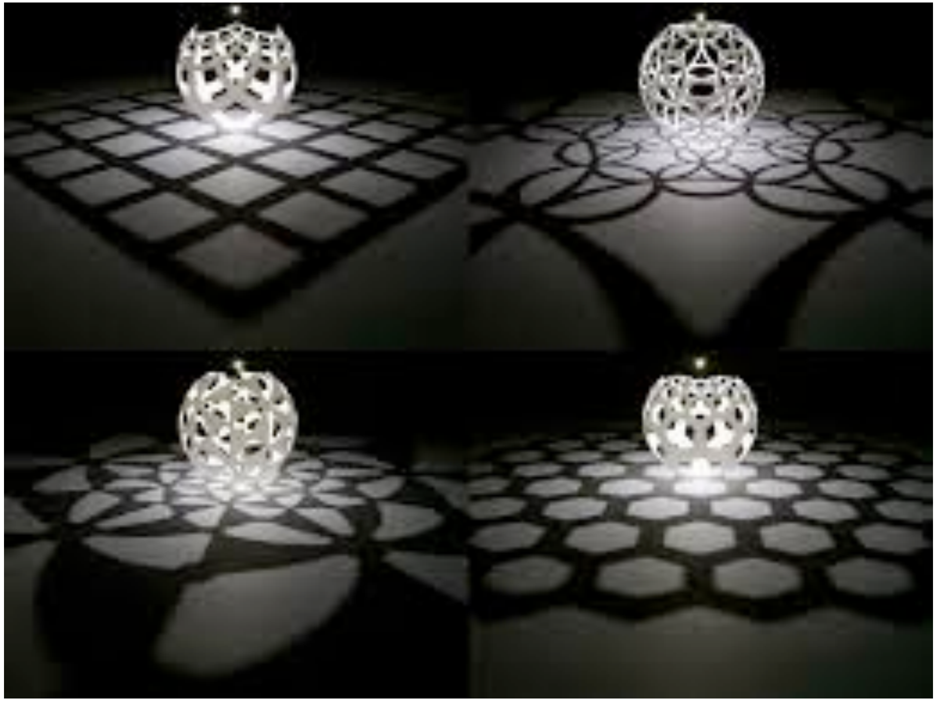


Ex: stereographic projection

Stereographic projection preservers angels !

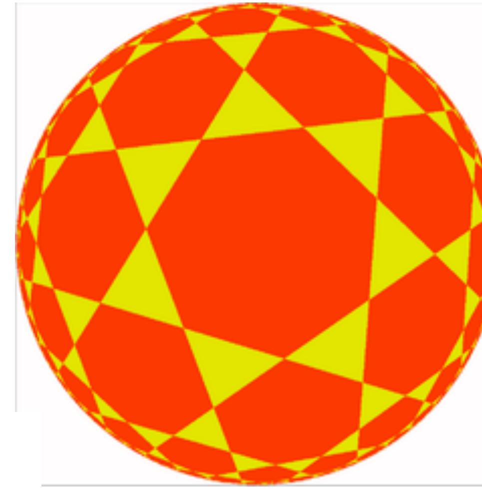
Pictures by Henry Segerman





Can something similar be done for the hyperbolic plane ?

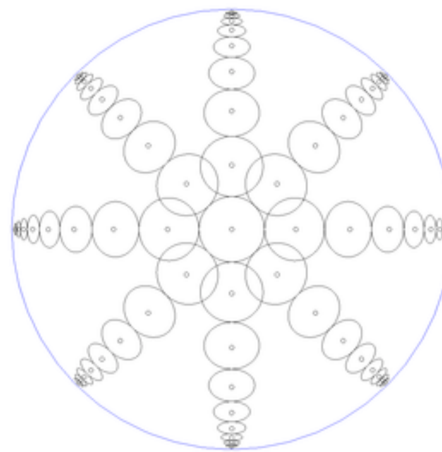
Beltrami-Klein model for the hyperbolic plane (1868, 1871)



Disk model, boundary not included

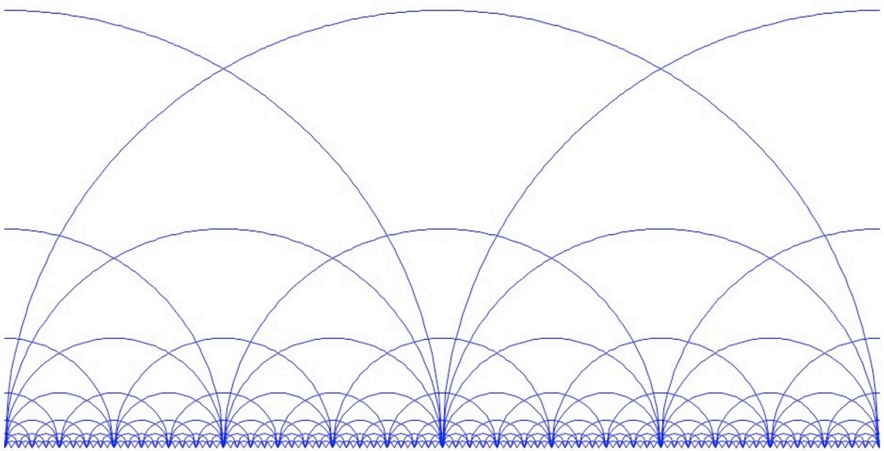
Advantage: shortest distances between points are straight lines

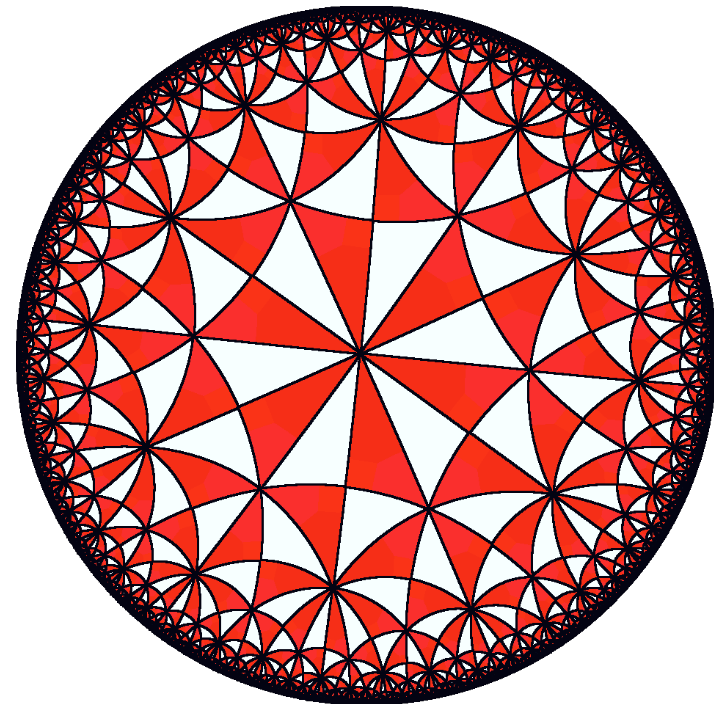
Weakness: Does not preserve angles,
Circles are not circular in general



Henri Poincaré (1854-1912) models

- Preserve angles (conformal model)
- Circular arcs perpendicular to the boundary realise shortest distances between points





Isometries = Möbius maps preserving half plane/unit circle !