Buckling of a straight column

Equilibrium states

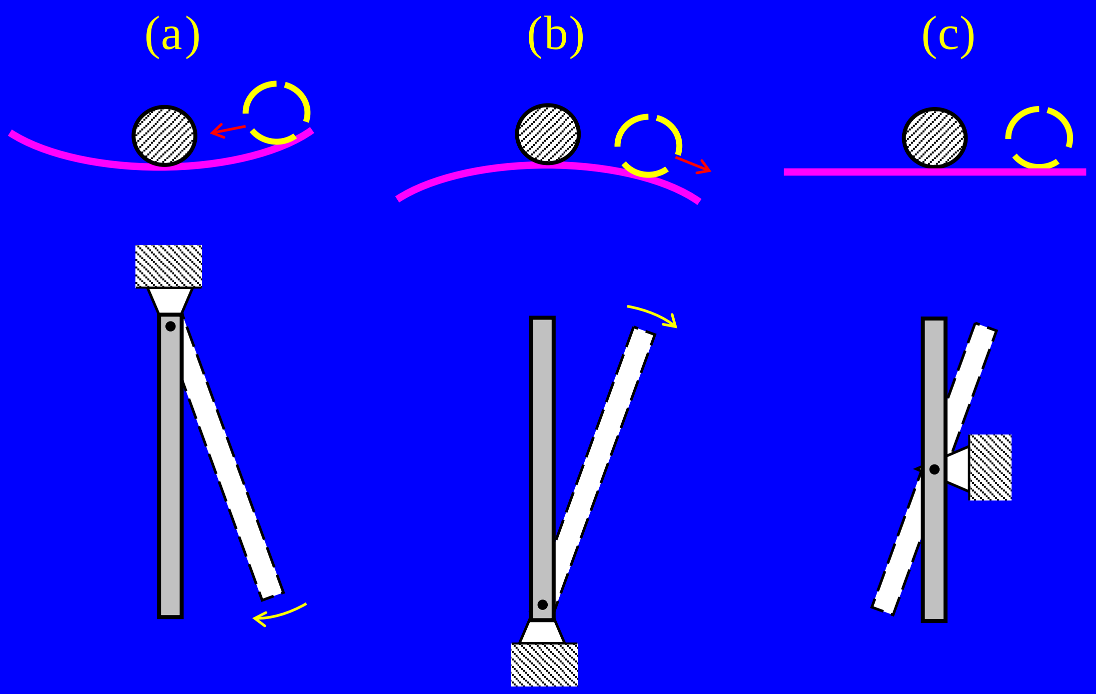
In stability analyses, we make studies about the equilibrium state and its nature.

(a) Equilibrium is stable: When a body is deviated from its equilibrium position, it will return back to its original position.

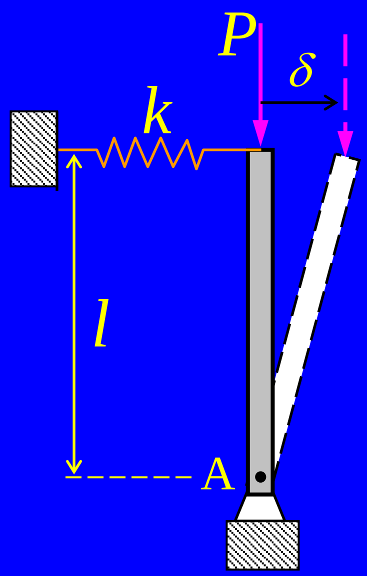
(b) Equilibrium is unstable: When a body is deviated from its equilibrium position, it will move farer out of its original position.

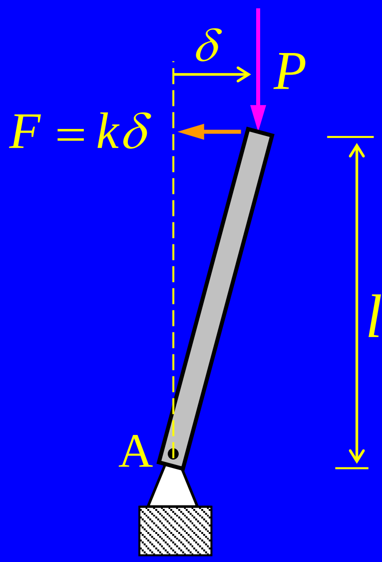
(c) Equilibrium is not well-defined – indifferent: When a body is deviated from its equilibrium position, the deviated position is in equilibrium as well.

Visualization



In a stable configuration, the potential energy takes a minimum value, in unstable configuration maximum value, and in indifferent configuration constant non-zero value.





A rigid beam supported by an elastic spring (linear analysis)

Moment equilibrium about the bottom hinge A:

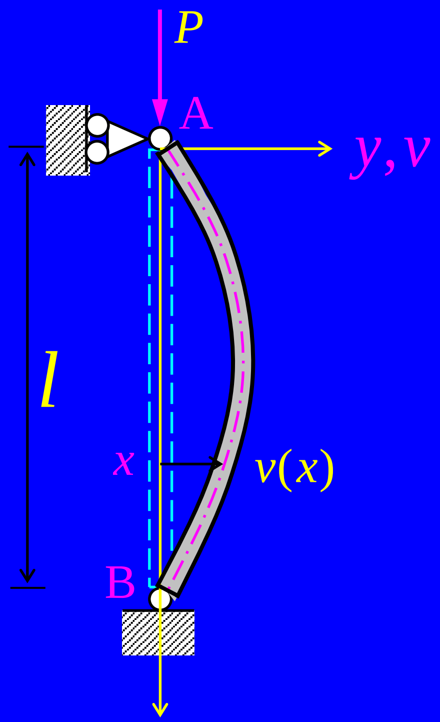
$$M_A = P\delta - Fl = P\delta - k\delta l = (P - kl)\delta .$$

- if $P < kl \Rightarrow M_A < 0$, the structure is in **stable** equilibrium state

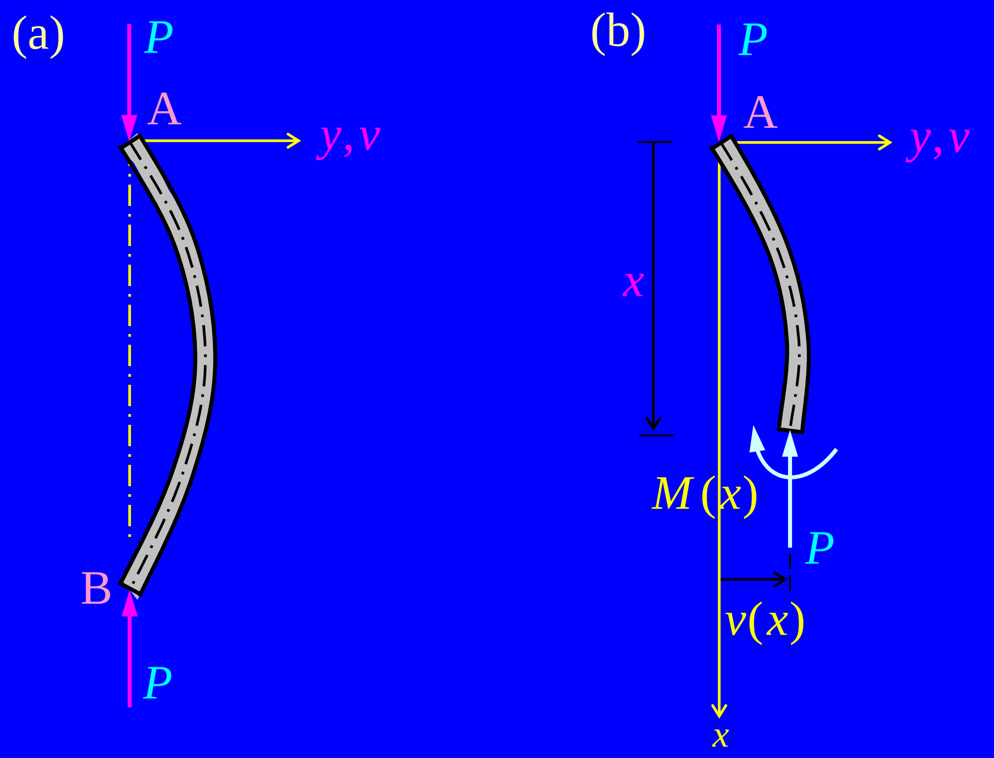
- if $P > kl \Rightarrow M_A > 0$, the structure is in **unstable** equilibrium state
- if $P = kl \Rightarrow M_A = 0$, the equilibrium state is not **well-defined**

$\Rightarrow$ **The critical value** for the load P with respect to felling down is $P_{\mathrm{kr}} = kl$.

Buckling of a straight elastic pin-ended column



It is worth mentioning, that the equilibrium consideration will be performed in the deformed configuration of the column – the problem will be geometrically nonlinear.



Now, we can write up the equilibrium equation

$$X \circlearrowleft: \quad -M(x) + Pv(x) = 0 \;\Rightarrow\; M(x) = Pv(x)$$

By taking into account that bending moment is $M = -EIv''$ or

$$v'' = -\frac{M}{EI}$$

an ordinary differential equation of second order is obtained

$$\Rightarrow \; v'' + k^2 v = 0,$$

where the notation $k = \sqrt{\frac{P}{EI}}$ is adopted.

The solution of this equation is determined by a inserting the solution $v = \exp(rx)$ into the differential equation. This yields the characteristic equation of the problem

$$r^2 + k^2 = 0 \Rightarrow r = \pm ik \;\; (i = \sqrt{-1}) \Rightarrow \;\; v(x) = A\cos kx + B\sin kx$$

The boundary conditions:

$$\left.\begin{aligned} v(0) &\equiv A = 0 \\ v(l) &= A\cos kl + B\sin kl = 0 \end{aligned}\right\} \Rightarrow \; B\sin kl = 0$$

This condition is fulfilled if $B = 0$, which corresponds to the straight configuration and is not of interest. But, the boundary conditions are satisfied also if $\sin kl = 0$, independently of constant B yielding a condition

$$kl = n\pi, \quad n = 0, 1, 2, \ldots$$

$\Longrightarrow$ Taking into account the notation for k we get

$$P_n = \frac{n^2 \pi^2 EI}{l^2}, \quad n = 1, 2, \ldots$$

The buckling mode is thus: $v_n(x) = B \sin \frac{n \pi x}{l}$.

The critical load is the lowest value for *P* with $n = 1$

$$P_{kr} = P_1 = \frac{\pi^2 EI}{l^2}$$

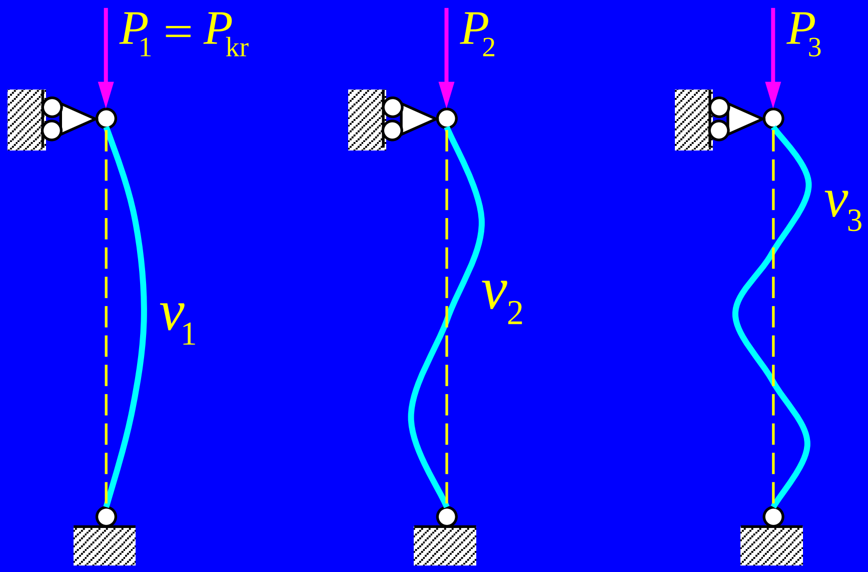
When the compressive load P reaches this value, the structure is in equilibrium in the original configuration. If the value of loading is still increased, the column will buckle immediately.

- phenomenon: **buckling**
- mathematical model: **Euler's theory and equations**

If the buckling mode corresponds to load P_1 is restricted to take place, the next buckling load is P_2.

$$P_2 = 4\frac{\pi^2 EI}{l^2}, \quad P_3 = 9\frac{\pi^2 EI}{l^2}, \cdots$$

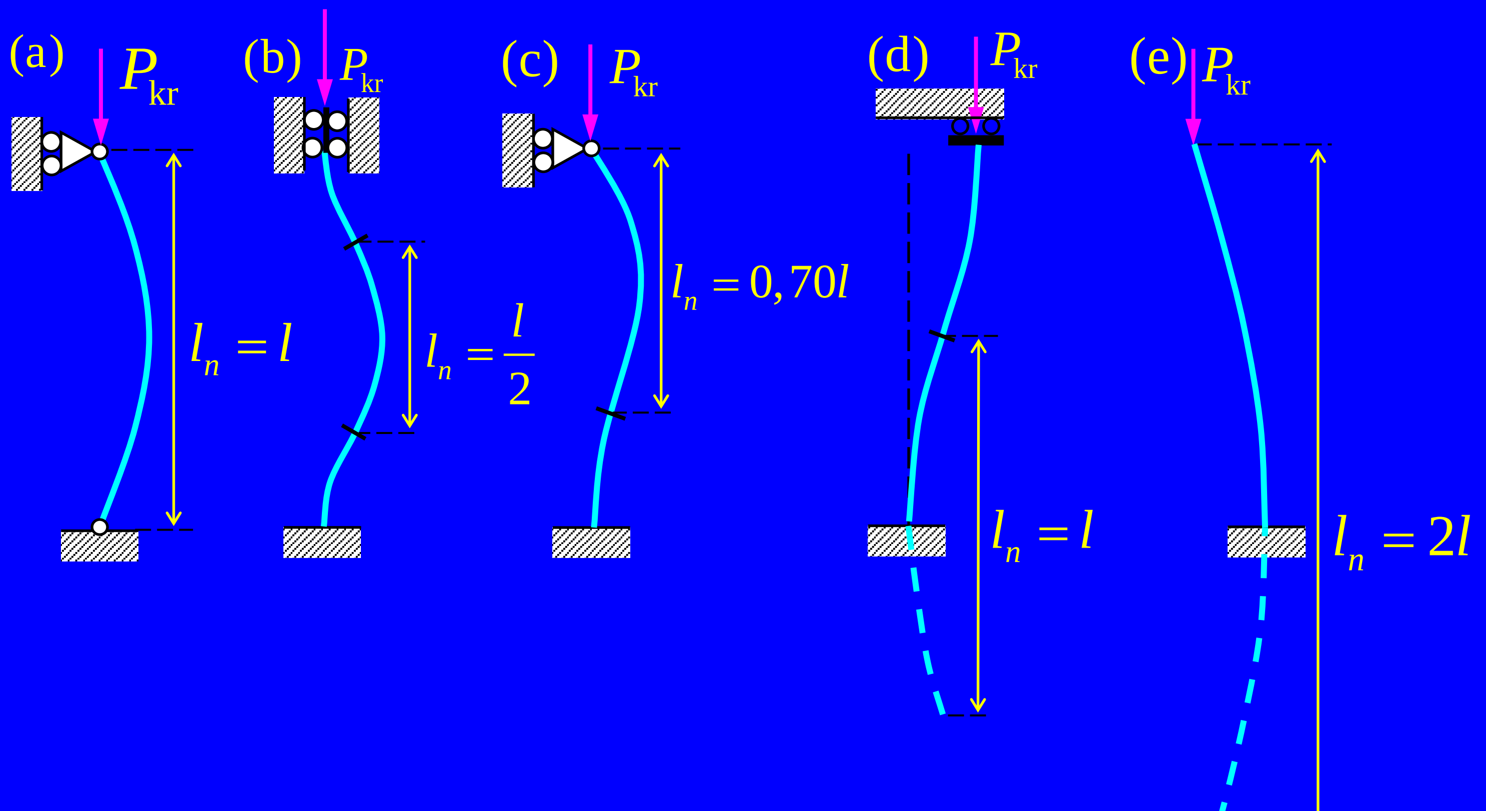
The buckling modes corresponding to various values of buckling loads P_1, P_2 and P_3:



Buckling length:

The buckling length or effective length of a column is the distance between the zero-moment points $M(x) = 0$ (the second derivative of the deflection function $v''(x) = 0$ will disappear) and it is different with various boundary restrictions. Mathematically it is called an inflection point.

The effective length of beams with various supports l_n:



The critical load can be expressed for beams with various support conditions according to so called Eulerian equations for beam buckling as:

$$P_{\mathrm{kr}} = \frac{\pi^2 EI}{l_n^2} \quad \text{or} \quad P_{kr} = \mu \frac{\pi^2 EI}{l^2}$$

where

$$\mu = \left(\frac{l}{l_n}\right)^2$$

is the **effective–length factor** of a beam.

The effective length and effective–length factor with different end conditions

Support	(a)	(b)	(c)	(d)	(d)
l_n	l	$l/2$	$0,70l$	l	$2l$
μ	1	4	2,046	1	$1/4$

Buckling takes always place in the weakest direction of the cross-section, or

$$I = I_{\text{min}} = I_2$$

Buckling stress can be expressed with an equation**:**

$$\sigma_{\text{kr}} = \frac{P_{\text{kr}}}{A} = \frac{\pi^2 E i_{\text{min}}^2}{l_n^2} = \mu \frac{\pi^2 E i_{\text{min}}^2}{l^2},$$

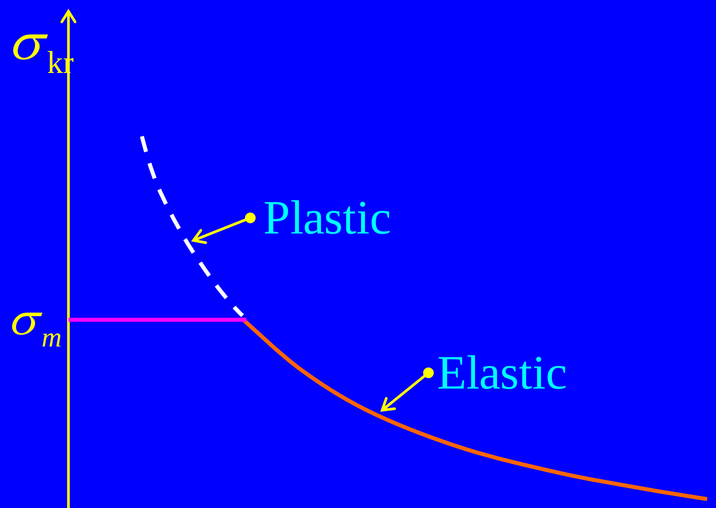
where $$i_{\min} = \sqrt{\frac{I_{\min}}{A}}$$

This can further be given as $$\sigma_{kr} = \frac{\pi^2 E}{\lambda^2}$$

where λ is the **slenderness ratio** of a beam

$$\boxed{\lambda = \frac{l_n}{i_{\min}} = \frac{l}{\sqrt{\mu} \cdot i_{\min}}.}$$

Graphically the same can shown as



A column under an eccentric compressive load

A real rod or column never is absolutely straight or the loading absolutely centric.

- Denote by e the eccentrity of the load

Bending moment is thus:

$M(x) = P[v(x) + e]$.

The differential equation for the deflection takes now the form

$$v'' = -\frac{M}{EI} \Rightarrow v'' + k^2 v = -k^2 e$$

The equation is no more homogenious, and the solution will include the part for the inhomogeneous term in addition

$$v = A\cos kx + B\sin kx - e$$

The boundary conditions are the same, yielding for the constants the values

$$\begin{cases} v(0) \equiv A - e = 0 \\ v(l) \equiv A\cos kl + B\sin kl - e = 0 \end{cases} \Rightarrow \begin{cases} A = e \\ B = e\dfrac{1-\cos kl}{\sin kl} = e\tan\dfrac{kl}{2} \end{cases}$$

Finally, the solution is fully defined

$$v = e(\tan\frac{kl}{2}\sin kx + \cos kx - 1)$$

The maximum value of the deflection is achieved at mid-span

$$v_{max} = v(\frac{l}{2}) = e(\frac{1}{\cos\frac{kl}{2}} - 1) = e(\sec\frac{kl}{2} - 1) = e[\sec(\sqrt{\frac{P}{EI}}\frac{l}{2}) - 1] \Rightarrow$$

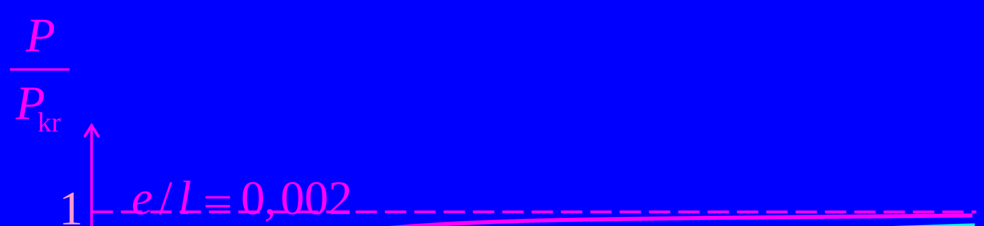
$$v_{\max} = e[\sec(\frac{\pi}{2}\sqrt{\frac{P}{P_{kr}}}) - 1].$$

where $\sec x = 1/\cos x$

the maximum value for the bending moment is

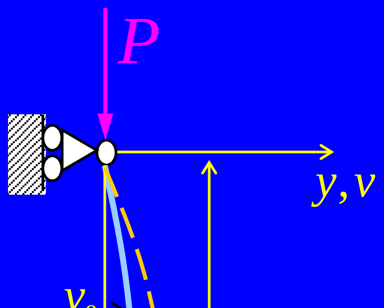
$$M_{\max} = P(v_{\max} + e) = Pe\sec\frac{kl}{2} = Pe\sec(\sqrt{\frac{P}{EI}}\frac{l}{2}) = Pe\sec(\frac{\pi}{2}\sqrt{\frac{P}{P_{\mathrm{kr}}}})$$

Graphically, we can show the dependency between the axial compressive load and the maximum deflection by taking into account the eccentrity of the load



$$\frac{P}{P_{kr}} = \left(\frac{2}{\pi}\arccos\frac{e/l}{v_{\max}/l + e/l}\right)^2$$

A column with an initial deflection in compression



The initial deflection

$$v_0(x) = \delta_0 \sin\frac{\pi x}{l}$$

δ_0 value at mid-span

The curvature will be consisted of the initial part, and the one due to the compression.

$$\kappa = \kappa_M + \kappa_0 = \frac{M}{EI} + \kappa_0 \quad \Rightarrow \quad -v'' = \frac{M}{EI} - v_0'' \quad \Rightarrow \quad v'' + \frac{M}{EI} = v_0'',$$

where

$$M = Pv(x), \quad v_0''(x) = -\delta_0 \frac{\pi^2}{l^2} \sin \frac{\pi x}{l}$$

The differential equation is now inhogeneous

$$v'' + k^2 v = -\delta_0 \frac{\pi^2}{l^2} \sin \frac{\pi x}{l}$$

The particular solution is of form

$v_y(x) = C \sin \frac{\pi x}{l}$ Inserting this into the differential equation gives an equation

$$[(-\frac{\pi^2}{l^2} + k^2)C + \delta_0 \frac{\pi^2}{l^2}] \sin \frac{\pi x}{l} = 0$$

The constant C will take the value

$$\Rightarrow\ C = \frac{\delta_0}{1-\frac{k^2 l^2}{\pi^2}} = \frac{\delta_0}{1-\frac{Pl^2}{\pi^2 EI}} = \frac{\delta_0}{1-\frac{P}{P_{kr}}}\ \Rightarrow$$

And the particular solution

$$v_y(x) = \frac{\delta_0}{1-\frac{P}{P_{kr}}}\sin\frac{\pi x}{l}$$

The general solution is finally

$$v(x) = A\cos kx + B\sin kx + \frac{\delta_0}{1-\frac{P}{P_{kr}}}\sin\frac{\pi x}{l}$$

Boundary conditions

$$\left.\begin{array}{l} v(0) \equiv A = 0 \\ v(l) = A\cos kl + B\sin kl = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} A = 0 \\ B = 0 \end{array}\right.$$

The deflection is reduced to the particular solution only

$$v(x) = \frac{\delta_0}{\frac{P}{P_{kr}} - 1} \sin \frac{\pi x}{l}$$

Bending moment is obtained by differentiation

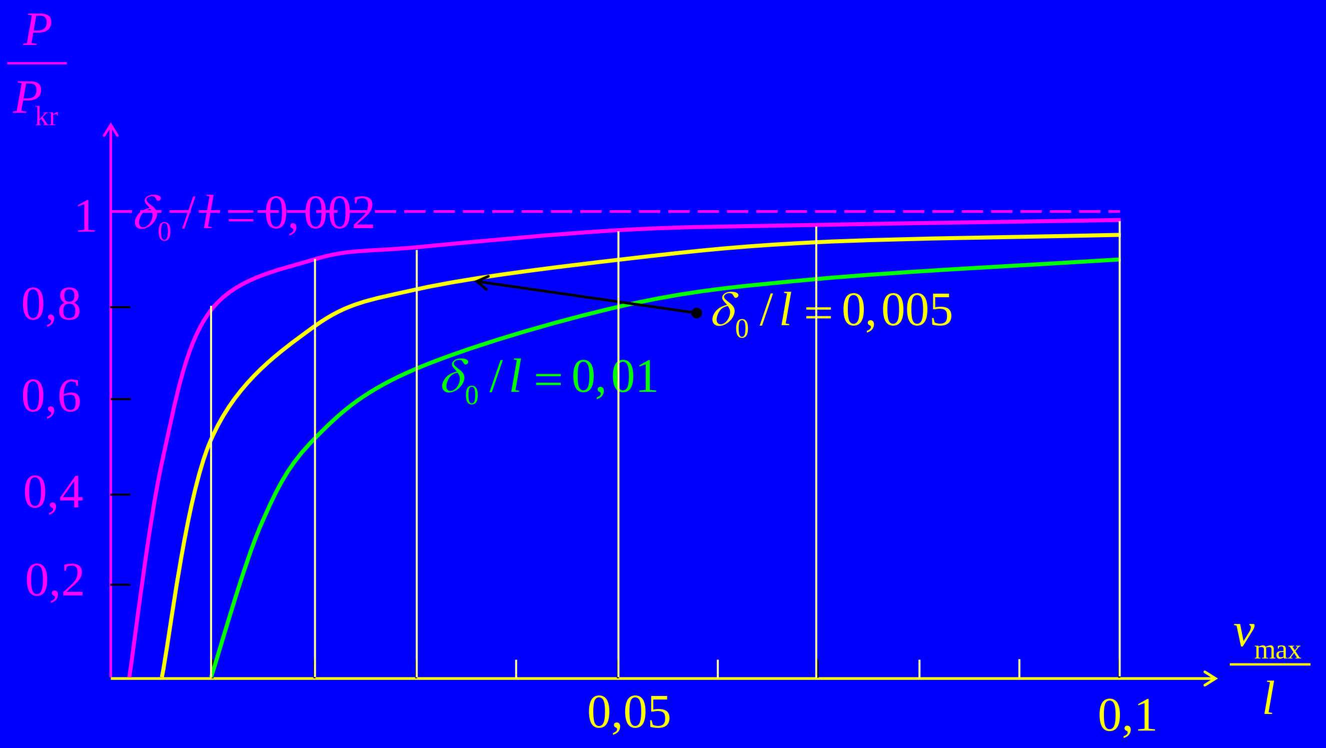
$$M(x) = -EI(v'' - v_0'') = \frac{P\delta_0}{1 - \frac{P}{P_{kr}}} \sin \frac{\pi x}{l}$$

Maximum deflection and bending moment take place at mid-span

$$\underline{\underline{v_{\max} = \frac{\delta_0}{1 - \frac{P}{P_{kr}}}}} \quad \text{and} \quad \underline{\underline{M_{\max} = \frac{P\delta_0}{1 - \frac{P}{P_{kr}}}}}.$$

Graphically, the dependency between the compressive load and the maximum deflection is shown by taking into account the initial deflection.

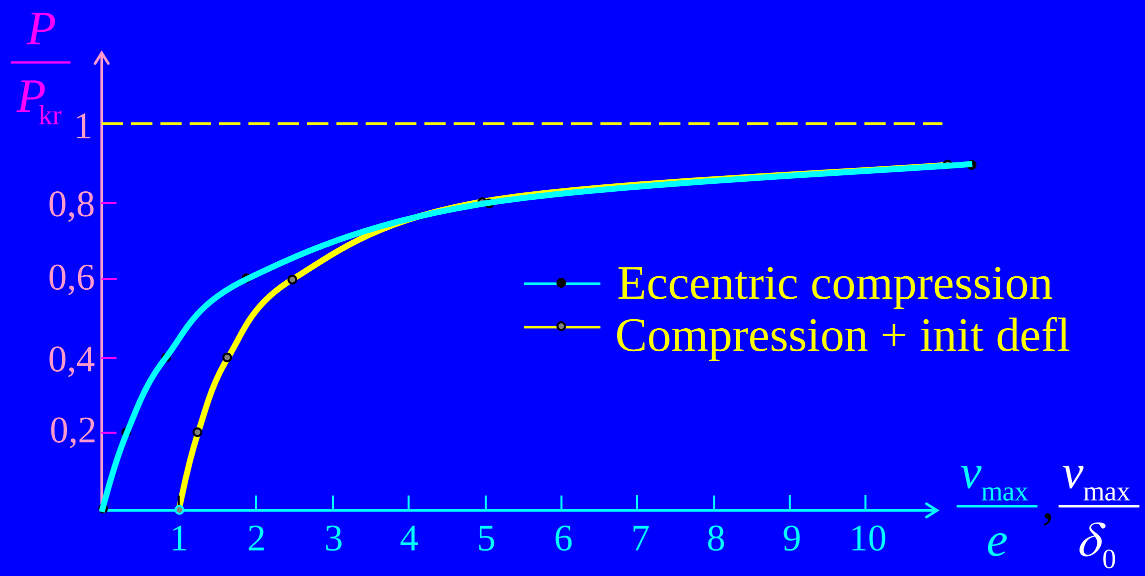
$$\frac{P}{P_{kr}} = 1 - \frac{\delta_0 / l}{v_{max} / l}$$



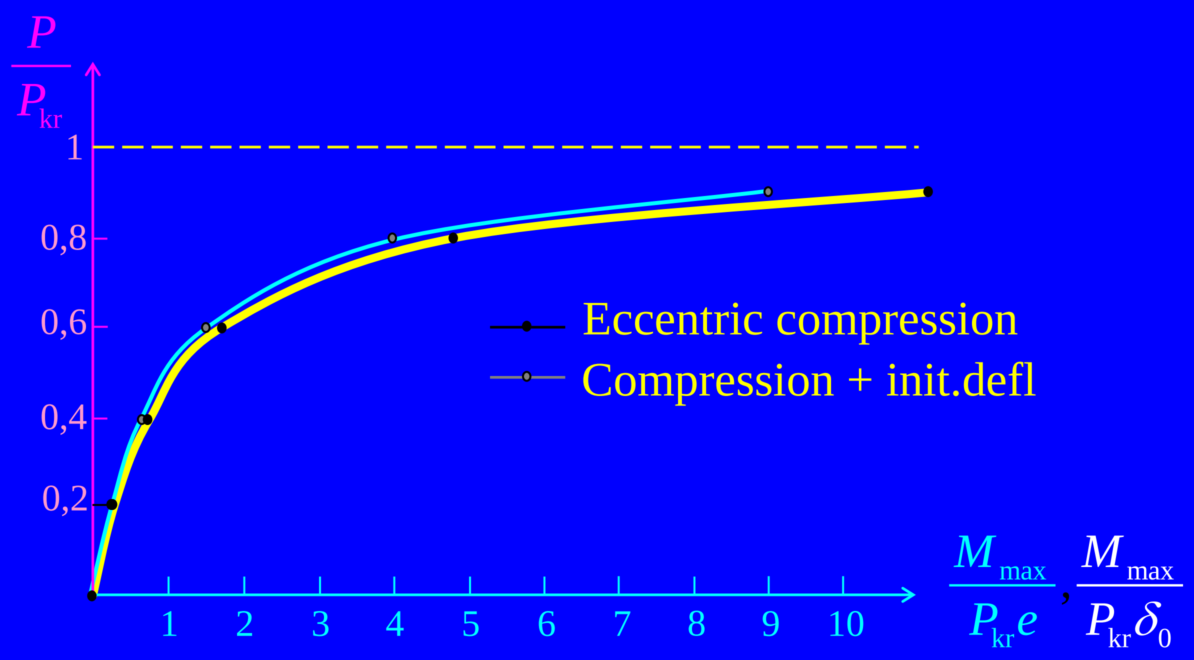
Comparison between eccentrically compressed column with and

without initial deflection

The dependency between compressive load and maximum deflection



Correspondingly, the dependency between compressive load and maximum bending moment



The secant formula

We consider the maximum normal stress in a column under combined compression and bending due to an eccentric compressive load.
The maximum normal stress appears at mid-span

$$\sigma_{\max} = \frac{P}{A} + \frac{M_{\max} a}{I}$$

$$= \frac{P}{A}[1 + \frac{ea}{i^2}\sec(\sqrt{\frac{P}{EI}}\frac{l}{2})] = \frac{P}{A}[1 + \frac{ea}{i^2}\sec(\frac{\lambda}{2}\sqrt{\frac{P}{EA}})]$$

- a is the distance from the axis of the column on the cross-section plane
- $i = \sqrt{\frac{I}{A}}$ is the radius of gyration
- $\lambda = \frac{l_n}{i}$ is the effective slenderness ratio

The secant formula is given in the form

$$\frac{P}{A} = \frac{\sigma_{\max}}{1 + \frac{ea}{i^2}\sec(\frac{\lambda}{2}\sqrt{\frac{P}{EA}})}.$$