Statistical Signal Processing. Homework set #1
due November 9, 2020

1. a) Let $Y_1, \ldots, Y_n$ be iid observations from distribution with density function

$$f(y|\theta) = \theta y^{\theta-1}, \quad 0 < y < 1, \quad \theta > 0.$$

Find the MLE of θ.

b) Let $Y_1, \ldots, Y_n$ be iid observations from a location-shifted exponential distribution with density function

$$f(y|\alpha, \lambda) = \frac{1}{\lambda} exp\left(\frac{-(y-\alpha)}{\lambda}\right), \quad y \geq \alpha, \quad \alpha \in \mathbb{R}, \quad \lambda > 0.$$

Find the MLE of (α,λ).

c) Let $Y_1, \ldots, Y_n$ be iid observations from a zero-inflated Poisson distribution with density function given below. Find the MLE of θ.

$$f(y|\theta, \lambda) = \begin{cases} \theta + (1-\theta)e^{-\lambda}, & y = 0 \\ (1-\theta)\frac{e^{-\lambda}\lambda^y}{y!}, & y = 1, 2, 3, \ldots \end{cases}$$

2. Suppose that a random variable x is distributed according to the Lévi density function with parameter c

$$f(x|c) = \sqrt{\frac{c}{2\pi}}\, x^{-\frac{3}{2}} \exp\left(-\frac{c}{2x}\right), \quad x \geq 0.$$

a) Given N statistically independent measurements $x_1, \ldots x_N$, find the maximum likelihood estimate of the parameter c.

b) Find the Cramér-Rao lower bound on the variance of any unbiased estimator of c.

3. Let $y_1, y_2, \ldots, y_N$ be i.i.d. samples from a continuous distribution given by:

$$f(y, \theta) = \frac{y^2}{2\theta^3} e^{-\frac{y}{\theta}}, \quad y \geq 0, \quad \theta > 0 \tag{1}$$

a) Find the maximum likelihood estimate for the parameter θ, $\hat{\theta}_{ML}$

b) Show that $\hat{\theta}_{ML}$ is unbiased for θ and find its variance

c) Determine the Cramer-Rao lower bound on the variance of the unbiased estimators of θ. Is $\hat{\theta}_{ML}$ efficient? Explain why.

d) Prove that $T(\mathbf{y}) = \sum_{n=1}^{N} y_n$ is a sufficient statistic for θ

4. Suppose that given 256 measurements $y(n)$ ($n = 0, ..., 255$) we need to estimate the amplitude $\theta = [\alpha_0\ \alpha_1\ \alpha_2]^T$ of a sinusoidal in additive Gaussian noise with $\sigma^2 = 0.81$:

$$y(n) = \alpha_0 + \alpha_1 cos(2\pi f_0 n) + \alpha_2 sin(2\pi f_0 n) + v(n),$$

where $f_0 = \frac{1}{16}$ is known. Furthermore, 10% of the original observations are randomly replaced by outliers that have the maximum signal value (8.0). Desired value of α_0 is 0. Pick desired values of $\alpha_1 \in [2.0, 3.0]$ and $\alpha_2 \in [0.5, 1.0]$ randomly.

Estimate α_0, α_1 and α_2 using M-estimation and so-called Andrew's sine ψ-function given as follows:

$$\psi(\tilde{y}) = \begin{cases} sin(\tilde{y}/a), & |\tilde{y}| \leq a\pi \\ 0, & |\tilde{y}| > a\pi \end{cases}$$

and the value of the tuning parameter a is set such that samples yielding measurement residuals larger that 3σ are rejected completely (have no influence). Use the IRLS method described in the lecture notes.

In your solution show plots of the desired signal, the noisy signal, the weighting function, the estimated signal using both M-estimation and Maximum likelihood (LS in this case) estimation, as well as the desired values and estimated values of the parameters using M-estimation and LS-estimation. Enclose your matlab code as well.

5. Expectation-Maximization Algorithm The EM algorithm simultaneously segments and fits data generated from multiple parametric models. We consider the measured data $y(n)$ generated by the two linear models:

$$\begin{aligned} y(n) &= a_1 x(n) + b_1 + v_1(n) \\ y(n) &= a_2 x(n) + b_2 + v_2(n) \end{aligned} \tag{2}$$

where a_1, b_1 and a_2, b_2 are the model parameters. The noise terms $v_1(n)$ and $v_2(n)$ are assumed to be Gaussian and zero mean. Generate the data corresponding to the two models, considering $N = 64$ data points produced by each model. Afterwards, assume that you have only the noisy measurements available. Then, estimate the model parameters from the noisy data by using the EM Algorithm. The noise variance for the two models is known $\sigma_1 = \sigma_2 = \sigma = 0.1$.

Requirements:

a) Write the analytical derivation for the EM estimation (E-step, M-step)

b) By using Matlab, simulate the data fitting of the two linear models. Plot the data points together with the estimated models (the two lines in (x, y) plane) after every iteration.

c) Enclose your Matlab code as well

Hints:

- EM Algorithm does not assign directly each data point to one of the models. EM assigns to each data point, a probability of belonging to one of the models (likelihood).
- E-step: Assume random model parameters in the beginning. Calculate the likelihood of each data point belonging to each model. For this, you consider the residual error of each point n for each model k, $r_k(n) = a_k x(n) + b_k - y(n), \quad k = 1, 2$. The likelihood of each data point is $w_k(n) = P(a_k, b_k | r_k(n))$.
- M-step: Take the likelihood of each data point belonging to each model and re-estimate the model parameters using Weighted Least Squares. For this, you need to build a weighted quadratic error function. (The weight for the squared error of each point is the likelihood of that point).