Exercises

Because the Desargues theorem implies its converse, another way to show that the Desargues theorem fails in the Moulton plane is to show that its converse fails. This plan is easily implemented with the help of Figure 6.16. (Moulton himself used this figure when he introduced the Moulton plane in 1902.)

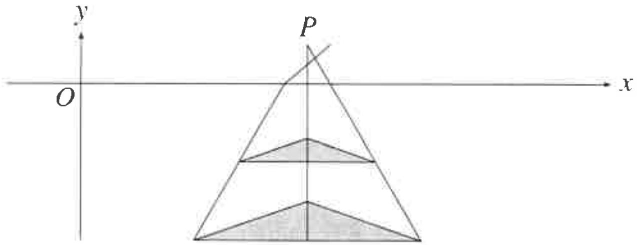

Figure 6.16: The converse Desargues theorem fails in the Moulton plane

6.3.1 Explain why Figure 6.16 shows the failure of the converse Desargues theorem in the Moulton plane.

6.3.2 Formulate a converse to the little Desargues theorem, and show that it follows from the little Desargues theorem.

6.3.3 Show that the converse little Desargues theorem implies a "little scissors theorem" in which the quadrilaterals have their vertices on parallel lines.

6.3.4 Design a figure that directly shows the failure of the little scissors theorem in the Moulton plane.

6.4 Projective arithmetic

If we choose any two lines in a projective plane as the x- and y-axes, we can add and multiply any points on the x-axis by certain constructions. The constructions resemble constructions of Euclidean geometry, but they use straightedge only, so they make sense in projective geometry. To keep them simple, we use lines we call "parallel," but this merely means lines meeting on a designated "line at infinity." The real difficulty is that the construction of $a+b$, for example, is different from the construction of $b+a$, so it is a "coincidence" if $a+b=b+a$. Similarly, it is a "coincidence" if $ab=ba$, or if any other law of algebra holds. Fortunately, we can show that the required coincidences actually occur, because they are implied by certain geometric coincidences, namely, the Pappus and Desargues theorems.

Addition

To construct the sum $a+b$ of points a and b on the x-axis, we take any line $\mathscr{L}$ parallel to the x-axis and construct the lines shown in Figure 6.17:

1. A line from a to the point where $\mathscr{L}$ meets the y-axis.
2. A line from b parallel to the y-axis.
3. A parallel to the first line through the intersection of the second line and $\mathscr{L}$.

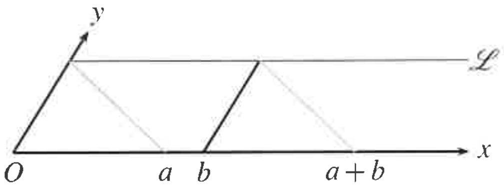

Figure 6.17: Construction of the sum

This construction is similar in spirit to the construction of the sum in Section 1.1. There we "copied a length" by moving it from one place to another by a compass. The spirit of the compass remains in the projective construction: the black line and the gray line form a "compass" that "copies" the length Oa to the point b.

We need the line $\mathscr{L}$ to construct $a+b$, but we get the same point $a+b$ from any other line $\mathscr{L}'$ parallel to the x-axis. This coincidence follows from the little Desargues theorem as shown in Figure 6.18.

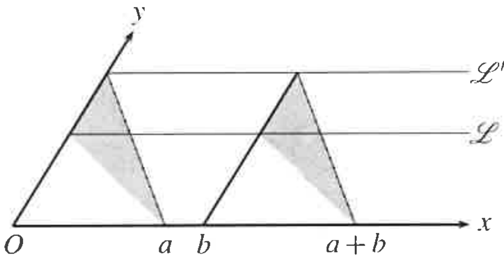

Figure 6.18: Why the sum is independent of the choice of $\mathscr{L}$

The black sides of the solid triangles are parallel by construction, as are the gray sides, one of which ends at the point $a+b$ constructed from $\mathscr{L}$. Then it follows from the little Desargues theorem that the dotted sides are also parallel, and one of them ends at the point $a+b$ constructed from $\mathscr{L}'$. Hence, the same point $a+b$ is constructed from both $\mathscr{L}$ and $\mathscr{L}'$.

Multiplication

To construct the product ab of two points a and b on the x-axis, we first need to choose a point $\neq O$ on the x-axis to be 1. We also choose a point $\neq O$ to be the 1 on the y-axis. The point ab is constructed by drawing the black and gray lines from 1 and a on the x-axis to 1 on the y-axis, and then drawing their parallels as shown in Figure 6.19. This construction is the projective version of "multiplication by a" done in Section 1.4.

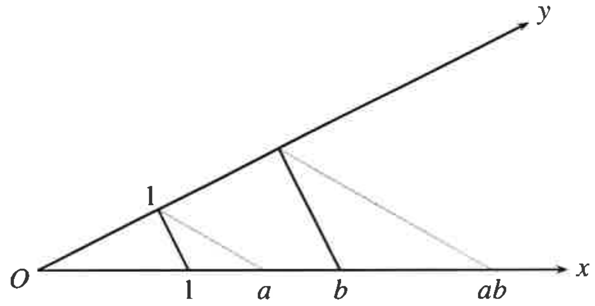

Figure 6.19: Construction of the product ab

Choosing the 1 on the x-axis means choosing a unit of length on the x-axis, so the position of ab definitely depends on it. For example, $ab = b$ if $a = 1$ but $ab \neq b$ if $a \neq 1$. However, the position of ab does not depend on the choice of 1 on the y-axis, as the scissors theorem shows (Figure 6.20).

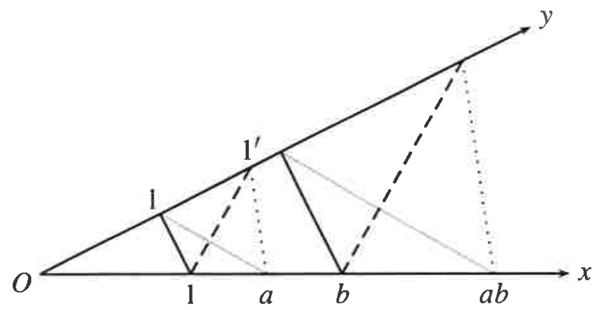

Figure 6.20: Why the product is independent of the 1 on the y-axis

If we choose $1'$ instead of 1 to construct ab, the path from b to ab follows the dashed and the dotted line instead of the black and the gray line. But it ends in the same place, because the dotted line to ab is parallel to the dotted line to a, by the scissors theorem.

Interchangeability of the axes

Once we have chosen points called 1 on both the x- and y-axes, it is natural to let each point a on the x-axis correspond to the point on the y-axis obtained by drawing the line through a parallel to the line through the points 1 on both axes (Figure 6.21).

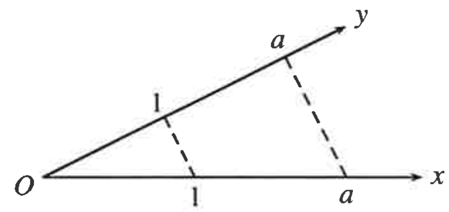

Figure 6.21: Corresponding points

It is also natural to define sum and product on the y-axis by constructions like those on the x-axis. But then the question arises: Do the y-axis sum and product correspond to the x-axis sum and product?

To show that *sums correspond*, we need to construct $a+b$ on the x-axis, and then show that the corresponding point $a+b$ on the y-axis is the y-axis sum of the y-axis a and b. Figure 6.22 shows how this construction is done.

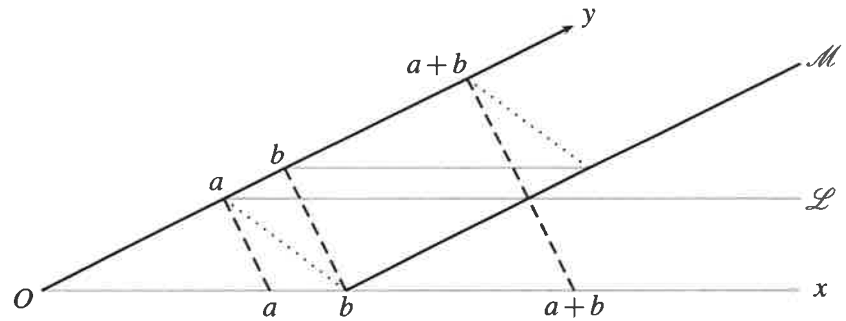

Figure 6.22: Corresponding sums

We construct $a+b$ on the x-axis using the line $\mathscr{L}$ through a on the y-axis. That is, draw the line $\mathscr{M}$ through b on the x-axis and parallel to the y-axis, and then draw the line (dashed) from the intersection of $\mathscr{M}$ and $\mathscr{L}$ parallel to the line from a on the x-axis to the intersection of $\mathscr{L}$ and the y-axis. This dashed line meets the x-axis at $a+b$, and (because it is parallel to the line from a to a) it also meets the y-axis at $a+b$.

Now we construct $a+b$ on the y-axis using the line $\mathscr{M}$ (as on the x-axis, the sum does not depend on line chosen, as long as it is parallel to the y-axis). That is, draw the line through b on the y-axis parallel to the x-axis, and then draw the line (dotted) parallel to the line from a on the y-axis to b on the x-axis (because this b is the intersection of the x-axis with $\mathscr{M}$).

But then, as is clear from Figure 6.22, we have a Pappus configuration of gray, dashed, and dotted lines between the y-axis and $\mathscr{M}$, hence the dotted line (leading to the y-axis sum) and the dashed line (leading to the point corresponding to the x-axis sum) end at the same point, as required. □

To show that *products correspond*, we use the scissors theorem from Section 6.3. Figure 6.23 shows the corresponding points 1, a, b, and ab on both axes. The gray lines construct ab on the x-axis, and the dotted lines construct ab on the y-axis.

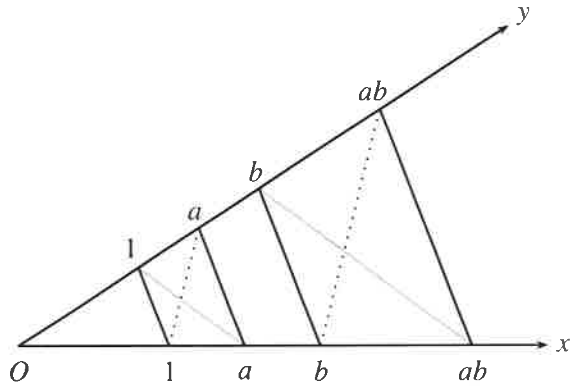

Figure 6.23: Construction of the product ab on both axes

It follows from the scissors theorem that the dotted line on the right ends at the same point as the black line from ab on the x-axis parallel to the lines connecting the corresponding points a and the corresponding points b. Hence, the product of a and b on the y-axis (at the end of the dotted line) is indeed the point corresponding to ab on the x-axis. □

Exercises

These definitions of sum and product lead immediately to some of the simpler laws of algebra, namely, those concerned with the behavior of 0 and 1. The complete list of algebraic laws is given in the Section 6.5.

6.4.1 Show that $a+O=a$ for any a, so O functions as the zero on the x-axis.

6.4.2 Show that, for any a, there is a point b that serves as $-a$; that is, $a+b=O$. (Warning: Do not be tempted to use measurement to find b. Work backward from $O=a+b$, reversing the construction of the sum.)

6.4.3 Show that $a1=a$ for any a.

6.4.4 Show that, for any $a \neq O$, there is a b that serves as a^{-1}; that is, $ab=1$. (Again, do the construction of the product in reverse.)

You will notice that we have not attempted to define sums or products involving the point at infinity ∞ on the x-axis.

6.4.5 What happens when we try to construct $a+\infty$?

6.4.6 What is $-\infty$?

You should find that the answers to Exercises 6.4.5 and 6.4.6 are incompatible with ordinary arithmetic. This is why we do not include ∞ among the points we add and multiply.

6.5 The field axioms

In calculating with numbers, and particularly in calculating with symbols ("algebra"), we assume several things: that there are particular numbers 0 and 1; that each number a has a *additive inverse*, $-a$; that each number $a \neq 0$ has a *reciprocal*, a^{-1}; and that the following *field axioms* hold. (We introduced these in the discussion of vector spaces in Section 4.8.)

$$\begin{aligned}
a+b&=b+a, & ab&=ba && \text{(commutative laws)}\\
a+(b+c)&=(a+b)+c, & a(bc)&=(ab)c && \text{(associative laws)}\\
a+0&=a, & a1&=a && \text{(identity laws)}\\
a+(-a)&=0, & aa^{-1}&=1 && \text{(inverse laws)}\\
& & a(b+c)&=ab+ac && \text{(distributive law)}
\end{aligned}$$

We generally use these laws unconsciously. They are used so often, and they are so obviously true of numbers, that we do not notice them. But for the projective sum and product of points, they are *not* obviously true. It is not even clear that $a+b=b+a$, because the construction of $a+b$ is different from the construction of $b+a$. It is truly a *coincidence* that $a+b=b+a$ in projective geometry, the result of a geometric coincidence of the type discussed in Section 6.2.