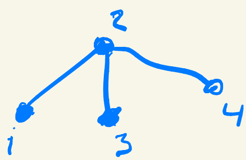
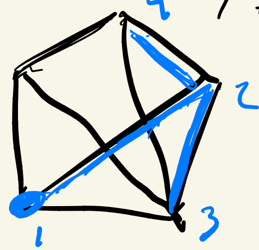
Missing lemmas & graph polynomials

Missing lemmas about embedding trees

Lm: A graph with $\delta(G) \geq m-1$ contains all trees with m nodes as a subgraph.

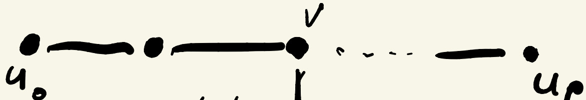
Proof: Embed arbitrarily, keeping the embedded graph connected



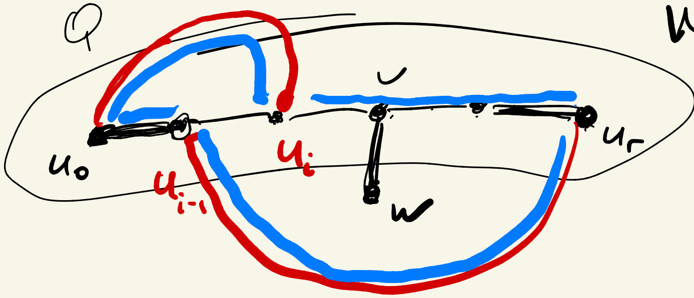


Lm: A connected graph on $\geq m$ nodes with min degree $> \frac{m}{2}$ contains a path on m nodes as a subgraph.

Pf: Assume Q longest path in G,
$|Q| < m$.
$= r+1$



v has a n'bour outside Q.



WTS there is a path containing all vertices of Q and $w \in N(v) \setminus Q$.

Enough to show there is a cycle whose vertex set is Q.

Q longest path $\Rightarrow$ $N(u_0) \subseteq Q$
$N(u_r) \subseteq Q$

Pigeonhole principle:

$$\{\underset{\{1\ldots r-2\}}{\underset{\cap}{i}} : u_0 u_{i+1} \in E\} \cap \{\underset{\{1\ldots r-2\}}{\underset{\cap}{i}} : u_r u_i \in E\}$$

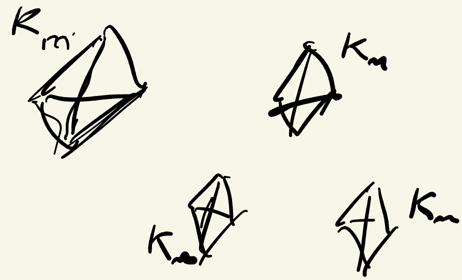
There is a "blue" cycle

$u_0 u_i Q u_r u_{i-1} Q u_0$.

Cut this cycle to a path with v as an endpoint and extend the path to w.

Longer path. ↯ ☒

Notice: Not true if we do not assume connected:



Ramsey numbers of graphs:

Recall $R(n,m)$ = smallest N s.t any 2-partition of $E(K_N)$ has a blue K_n or a red K_m

$R(G,H)$ = smallest N s.t if I 2-colour the edges of K_N I always get a blue G or a red H.

Note
$R(G,H) \leq R(|G|, |H|)$.

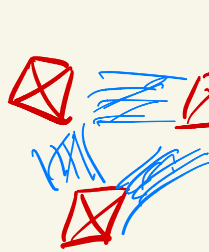
Thm: $R(K_n, T) = (m-1)(n-1)+1$

T: m vertex tree

regardless of the tree.

Proof:

$\geq$:



cliques of size $m-1$

blue subgraph is Turan graph $\underbrace{K_{m-1,\dots,m-1}}_{n-1 \text{ parts}}$

so has no K_n.

$\leq$: Proof by induction on n.

(vacuous when $n=1$
trivial when $n=2$)

Assume blue-red colouring of $E(K_{(m-1)(n-1)+1})$. If red subgraph had min degree $\geq m-1$, then red T by lemma.

So assume some node x has $\geq (m-1)(n-1)-(m-1)$ blue n'bours
$= (m-1)(n-2)$.

$\geq (n-2)(m-1)+1$

By induction hypothesis, among $N_{blue}(x)$ is either

$\underbrace{\text{blue } K_{n-1}}$ or red T.

so blue K_n in

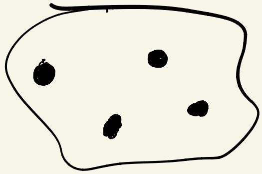
$\{x\} \cup N_{blue}(x)$.

$\blacksquare$

Chromatic polynomial

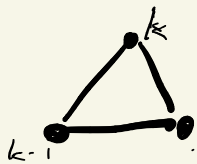
Def: $\chi_G(k) = \#\{k\text{-colourings of } G\}$

Ex:



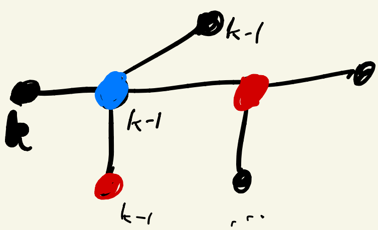
$\chi_{\overline{K_n}} = k^n$

$\chi_e = k(k-1)$



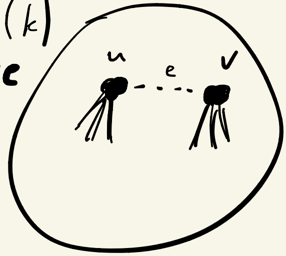
$\chi_{K_n} = \boxed{k(k-1)\cdot\ldots\cdot(k-n+1)}$

$= (k)_n = \frac{k!}{(k-n)!}$ if $n<k$



$\chi_T = k(k-1)^{n-1}$

($n = \#$ nodes)

$\chi_{G-e}^{(k)}$ 

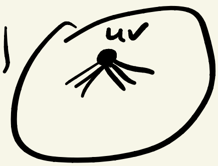
$\|$

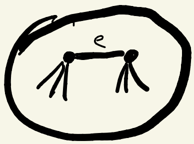
#colourings of $G-e$ where u,v get same colour

$+$

——"—— different colours

$\|$

$\chi_{G/e}^{(k)}$ 

$+\chi_G^{(k)}$ 

for any k.

- $\chi_{\bar{K}_n}(k) = k^n$ degree n degree $n-1$.
- $\chi_G(k) = \boxed{\chi_{G-e}(k)} - \boxed{\chi_{G/e}(k)}$ for any edge $e \in E(G)$.

so by induction over # edges

- χ_G polynomial of degree $n = |G|$.
  - leading coefficient 1.

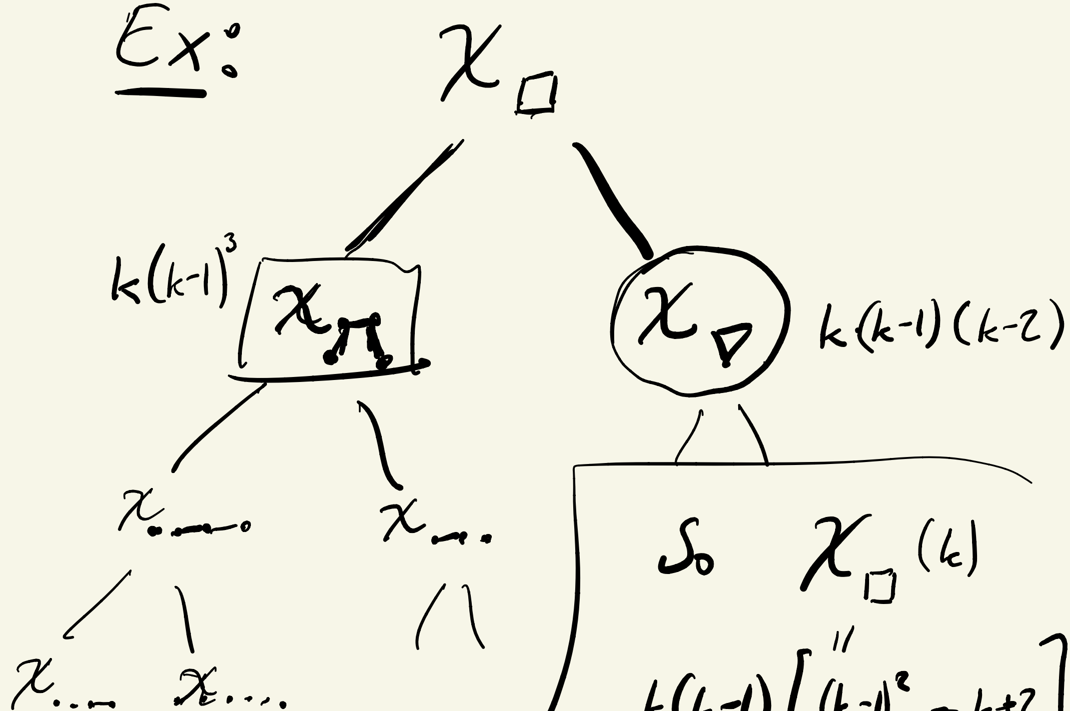
- integer coefficients
- alternating coefficients:

$$\chi_G(k) = k^n - a_1 k^{n-1} + a_2 k^{n-2} - \dots$$

where $a_i \in \mathbb{N}$

$a_n = 0$

<u>Ex:</u>



So $\chi_{\square}(k)$

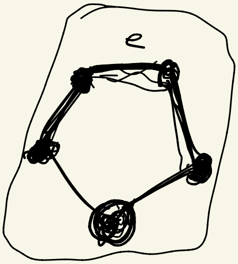
$$= k(k-1)\left[(k-1)^2 - k + 2\right]$$

$$= k(k-1)\left[k^2 - 3k + 3\right]$$

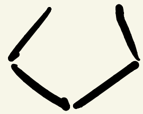
Ex $\chi_{C_n}(k) = \chi_{P_{n-1}}(k) - \chi_{C_{n-1}}(k)$

$= k(k-1)^{n-1} - \chi_{C_{n-1}}(k)$

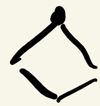
C_n



$C_n - e$



C_n / e

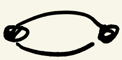


Base case

$\chi_{C_3} = \chi_{K_3}$

$= k(k-1)(k-2)$

$\chi_{C_2}(k)$



$= k(k-1)$

$\chi_{C_1}(k)$

0

Induction $\Rightarrow$

$$\chi_{C_n}(k) = (k-1)^n + (k-1)(-1)^n$$

$$\boxed{\chi_G(k)} = \#k\text{-colourings of } G$$

if $k \in \mathbb{N}$.

has n roots (with multiplicity) in $\mathbb{C}$.

$$\chi_G(0) = \chi_G(1) = \dots = \chi_G(\underbrace{\chi(G)}_{\text{chrom number}}-1) = 0$$

What are the other $n - \chi(G)$ roots?

Answer: $A = \{\text{algebraic numbers in } \mathbb{C}\}$

There is a real number r s.t. the set of possible roots of chromatic polynomials of graphs is

$r \in (\frac{32}{27}, \varphi^2)$

$$A \setminus (-\infty, r) \cup \{0, 1\}$$

$\varphi =$ golden ratio.

$$\bar{\chi}_G(k) \qquad \chi_G(k) = k^n - a_1 k^{n-1} + a_2 k^{n-2} - \dots \pm a_{n-1} k$$

$$:= (-1)^n \chi_G(-k) = k^n + a_1 k^{n-1} + a_2 k^{n-2} + \dots + a_{n-1} k$$

$$\begin{array}{ccc} \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\subseteq$} \\ \chi_G(k) & & \mathbb{N} \quad \text{if } k \in \mathbb{N} \end{array}$$

<u>**Thm:**</u> $\bar{\chi}_G(k) = \#(\rho, \gamma)$ where

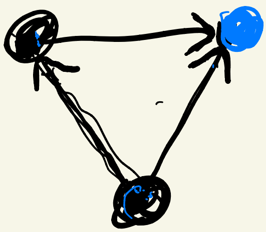
- ρ acyclic orientation of G
- $\gamma : V(G) \to \{1 \dots k\}$ such that

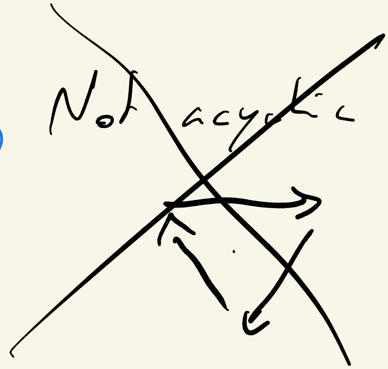
$$u \xrightarrow{\rho} v \implies \gamma(u) \geq \gamma(v)$$

"ρ and γ are weakly compatible"

"ρ and γ strongly compatible" if γ is also a proper graph colouring.

Acyclic orientation





Thm:

$(-1)^n \chi_G(-k)$ is the number of pairs (ρ, γ) where

- ρ acyclic orientation of G
- $\gamma : V(G) \to \{1 \ldots k\}$

"weakly compatible" s.t. $u \xrightarrow{\rho} v \Rightarrow \gamma(u) \leq \gamma(v)$.

Pf: $\bar{\chi}_G = (-1)^n \chi_G(-k)$

ψ_G = numbers of weakly compatible pairs (ρ, γ).

Recall $\chi_G(x) = \chi_{G-e}(x) - \chi_{G/e}(x)$ if e ordinary edge

$\chi_G(x) = k^n$ if G has no edges

$\chi_G(x) = 0$ if G has loop

So $\bar{\chi}_G(k) = k^n$ if G has no edges

$\bar{\chi}_G(k) = 0$ if G has loops

$\bar{\chi}_G(k) = \bar{\chi}_{G-e}(k) + \bar{\chi}_{G/e}(k)$ if e ordinary edge

Enough to show

no edges to orient, every map $V \to \{1 \ldots k\}$ ok

$\psi_{\bar{K}_n}(k) = k^n$

$\psi_G(k) = 0$ if G has loops

no acyclic orientation

$\psi_G(k) = \psi_{G-e}(k) + \psi_{G/e}(k)$ if e ordinary edge!

$e = uv$

(ρ, γ)-pairs with $\gamma(u) \neq \gamma(v)$

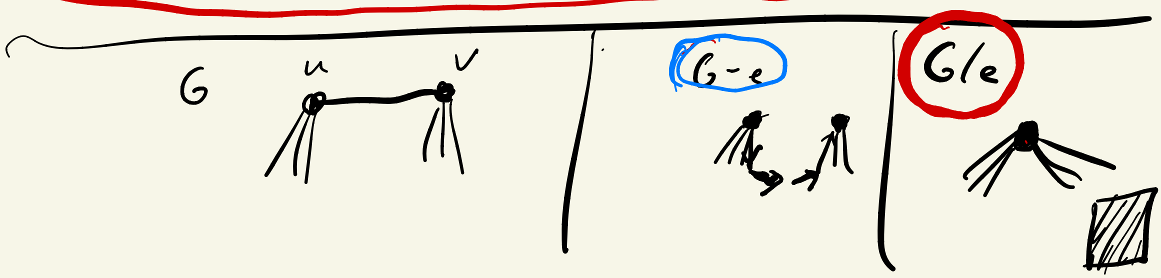
+

(ρ, γ)-pairs with $\gamma(u) = \gamma(v)$ $\quad u \xrightarrow{\rho} v$

$= \psi_{G-e}$

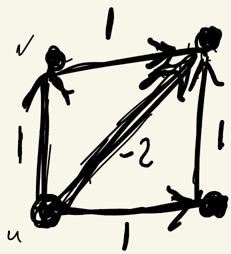
+

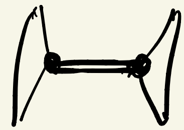
(ρ, γ)-pairs with $\gamma(u) = \gamma(v)$ $\quad u \xleftarrow{\rho} v$ $\quad = \psi_{G/e}$



A $\mathbb{Z}_n$ Flow on directed graph G is a map $f: E(G) \to \mathbb{Z}_n$ s.t.

$$\sum_{e\in N_-(v)} f(e) = \sum_{e\in N_+(v)} f(e)$$

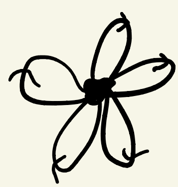




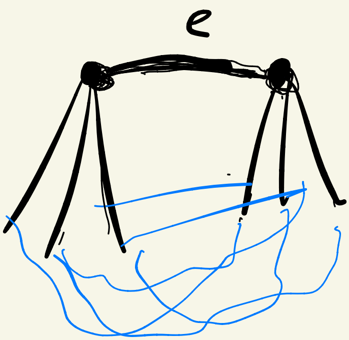
$$\#\left\{\begin{matrix}\text{nowhere zero } \mathbb{Z}_n\text{-flows} \\ \text{on } G\end{matrix}\right\} = \varphi_G(\mathbb{Z}_n)$$

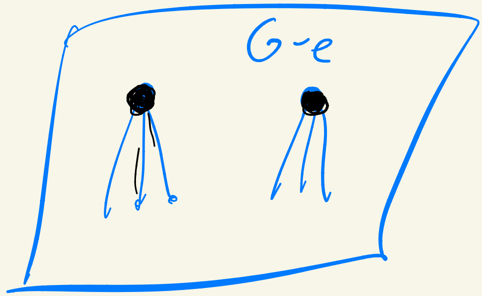
Note: φ_G does not depend on the orientation of the graph.

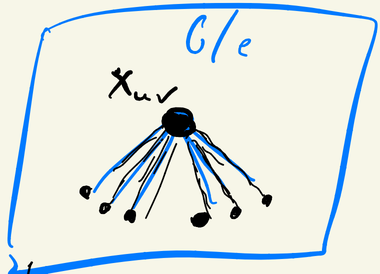
- If G has a bridge, then $\varphi_G(\mathbb{Z}_n) = 0$.
- If G has r loops then $\varphi_G(\mathbb{Z}_n) = (n-1)^r$



- If e ordinary (no loop, no bridge) $\varphi_G(\mathbb{Z}_n) = -\varphi_{G-e}(\mathbb{Z}_n) + \boxed{\varphi_{G/e}(\mathbb{Z}_n)}$







A nowhere zero flow on G/e gives a flow on G (by assigning the appropriate value to $f(e)$) and an almost-flow on $G-e$.

This flow on G is nowhere zero

$\iff$ the "flow" on $G-e$ was **not** conserved at u & v, i.e the induced almost-flow on $G-e$ is a flow.

so $$\varphi_G(\mathbb{Z}_n) = -\varphi_{G-e}(\mathbb{Z}_n) + \varphi_{G/e}(\mathbb{Z}_n).$$

$\square$

So we get a graph polynomial

$$\varphi_G(n) := \varphi_G(\mathbb{Z}_n) \qquad \text{in } n.$$

of degree $\xi(G) = |E| - |V| + |K|$

set of connected components.

leading term $n^{\xi(G)}$

alternating integer coefficients

$$\varphi_G = \varphi_{G/e} - \varphi_{G-e}$$

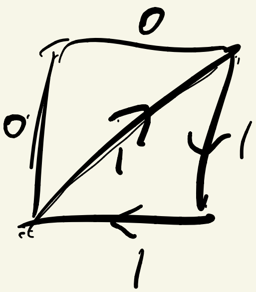
So again

$$(-1)^{\xi(G)} \varphi_G(-n) \geq \varphi_G(n) \qquad \text{if } n \in \mathbb{N}.$$

What does $(-1)^{\xi(G)} \varphi_G(-n)$ count?

Thm: $(-1)^{\xi(G)} \varphi_G(-n)$ is the number of pairs (f, ρ) where $f : E(G) \to \mathbb{Z}_n$ flow and ρ is a totally cyclic orientation of $G/_{\mathrm{supp} f}$

G



$G/\operatorname{supp} f$

A totally cyclic orientation is one where every edge is included in a directed cycle.

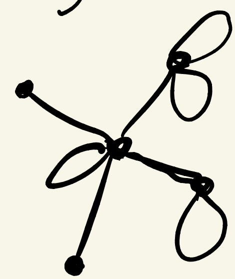
In particular, $|\varphi_G(-1)|$ counts the number of totally cyclic orientations of G

$|\chi_G(-1)|$ counts the number of acyclic orientations of G.

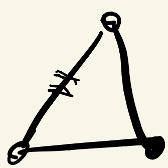
Tutte polynomial

Def: $T_G(x,y)$ is

- $x^b y^\ell$ if G has b bridges and ℓ loops and no other edges.
- $T_{G-e}(x,y) + T_{G/e}(x,y)$ if e is an ordinary edge



Ex: $G = K_3$.



$$T_\Delta = T + T$$

$$= x^2 + T + T$$

$$= x^2 + y + x$$

Thm: $$T_G(x,y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{n(A)}$$

where $r(A) = |V(G)| - K(G[A])$

"rank"

connected components

$n(A) = |A| - r(A)$

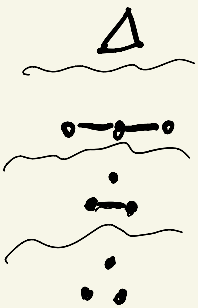
"nullity"

$r(\Delta) = 2$

Ex: $T_{\Delta} =$

$x^2 + x + y$

(?) $\|$



$$\left(\begin{array}{c} (x-1)^0 (y-1)^1 \\ + \\ 3\,(x-1)^0 \cdot (y-1)^0 \\ + \\ 3(x-1)^1 (y-1)^0 \\ + \\ (x-1)^2 \cdot (y-1)^0 \end{array} \right) = \boxed{\begin{array}{c} y-1 \\ + \\ 3 \\ + \\ 3(x-1) \\ + \\ (x-1)^2 \end{array}}$$

Thm: $\varphi_G(x) = (-1)^{|E|-|V|+\kappa(G)} T_G(0, 1-x)$

$$\chi_G(x) = (-x)^{-\kappa(G)} T_G(x-1, 0)$$

$T_G(2,1) = \#$ forests

$T_G(1,1) = \#$ spanning forests

$T_G(1,2) = \#$ spanning subgraphs

$T_G(-2,0) = \#$ acyclic orientation

$\vdots$