Exercises 1.

Convexity and seminorms

Exercise 1.1. Let $A : V \to W$ be linear. Show that $\mathrm{Ker}(A)$ is a vector subspace of V and that $\mathrm{Ran}(A)$ is a vector subspace of W. Let $[u] := u + \mathrm{Ker}(A)$ for $u \in V$. Prove that $\widetilde{A}[u] := Au$ gives a well-defined linear bijection

$$\widetilde{A} : V/\mathrm{Ker}(A) \to \mathrm{Ran}(A).$$

Exercise 1.2. Let $s : X \to \mathbb{R}$ a be seminorm, and for $r > 0$ define the r-strip by

$$S_r := \{u \in X : \ s(u) < r\}. \tag{1.1}$$

Prove that $S_r \subset X$ is
(a) **a**bsorbing, (for all $u \in X$, $u \in tS_r$ whenever $t > 0$ is large enough),
(b) **b**alanced ($\lambda u \in S_r$ whenever $|\lambda| \leq 1$ and $u \in S_r$),
(c) **c**onvex ($tx + (1-t)y \in S_r$ whenever $0 < t < 1$ and $x, y \in S_r$).

Exercise 1.3. Let $S \subset X$ be an **a**bsorbing **b**alanced **c**onvex subset of a vector space X. Show that the *Minkowski functional* $s : X \to \mathbb{R}$ is a seminorm, where

$$s(u) := \inf \{r > 0 : \ u/r \in S\}.$$

Exercise 1.4. Let s be a seminorm on a $\mathbb{K}$-vector space X, and define

$$u \sim v \quad \overset{\text{definition}}{\Longleftrightarrow} \quad s(u - v) = 0.$$

Let $[u] := \{v \in X : \ u \sim v\}$. Prove the following claims:
(a) $\sim$ is an equivalence relation on X.
(b) $L := X/s^{-1}(0) = \{[u] : \ u \in X\}$ is a normed space with

$$[u] + [v] := [u + v], \quad \lambda[u] := [\lambda u], \quad \text{quotient norm} \quad [u] \mapsto s(u).$$