Hyperbolic geometry & Fractals

Shapes in Action 13th Oct 2020

Program schedule for Oct 13th

15:15 Hyperbolic geometry (cont.)

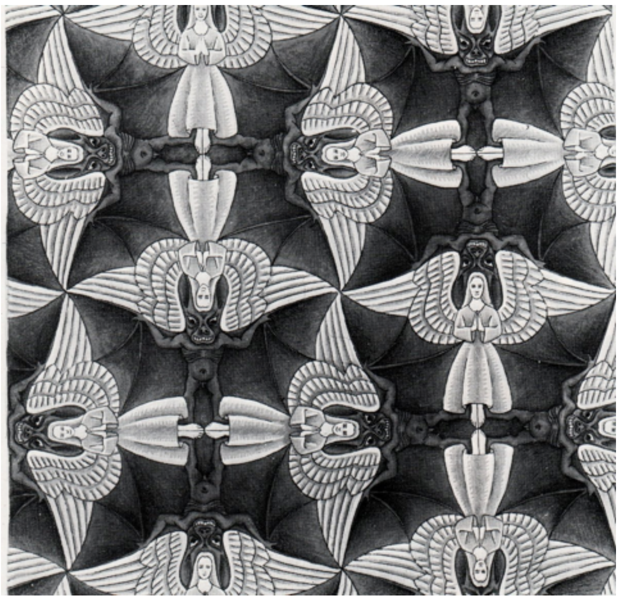
16:00 Break

16:15 Some ideas about fractals

17:00 Break

17:15 A folding activity

Euclidean (=flat), spherical and hyperbolic models of 2D geometry



K = 0 (17 types)

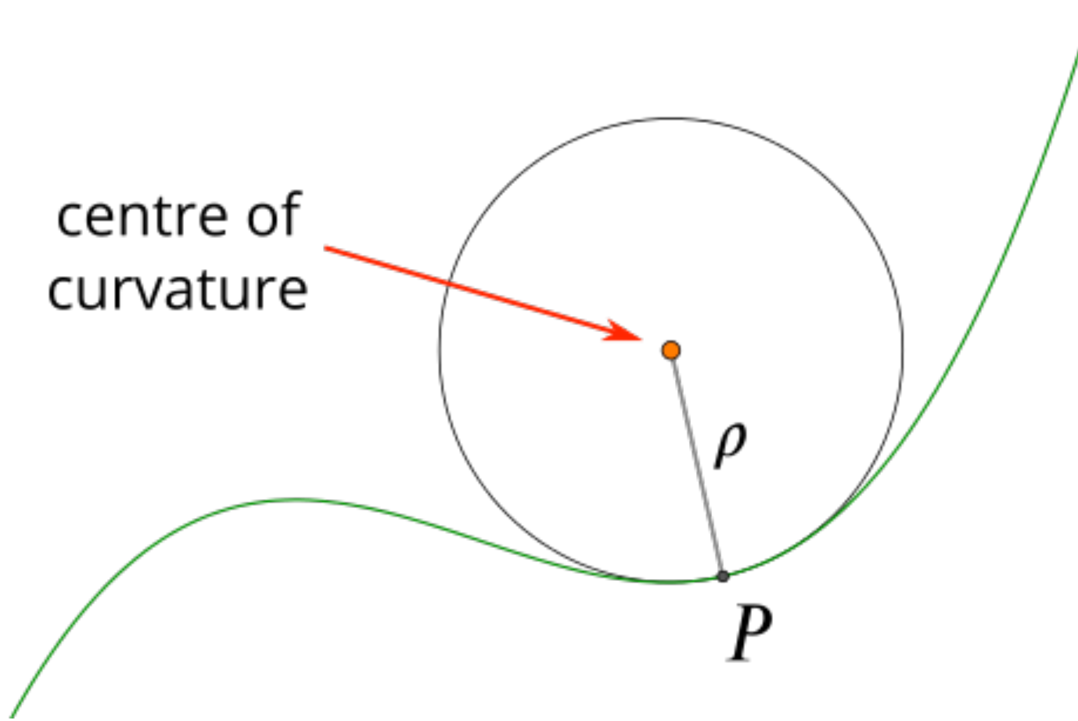
K>0 (14)

K<0 (∞)

What is curvature ?

Curvature of a smooth planar ***curve*** at point **P** is $\varkappa(P)=1/\boldsymbol{\rho}$

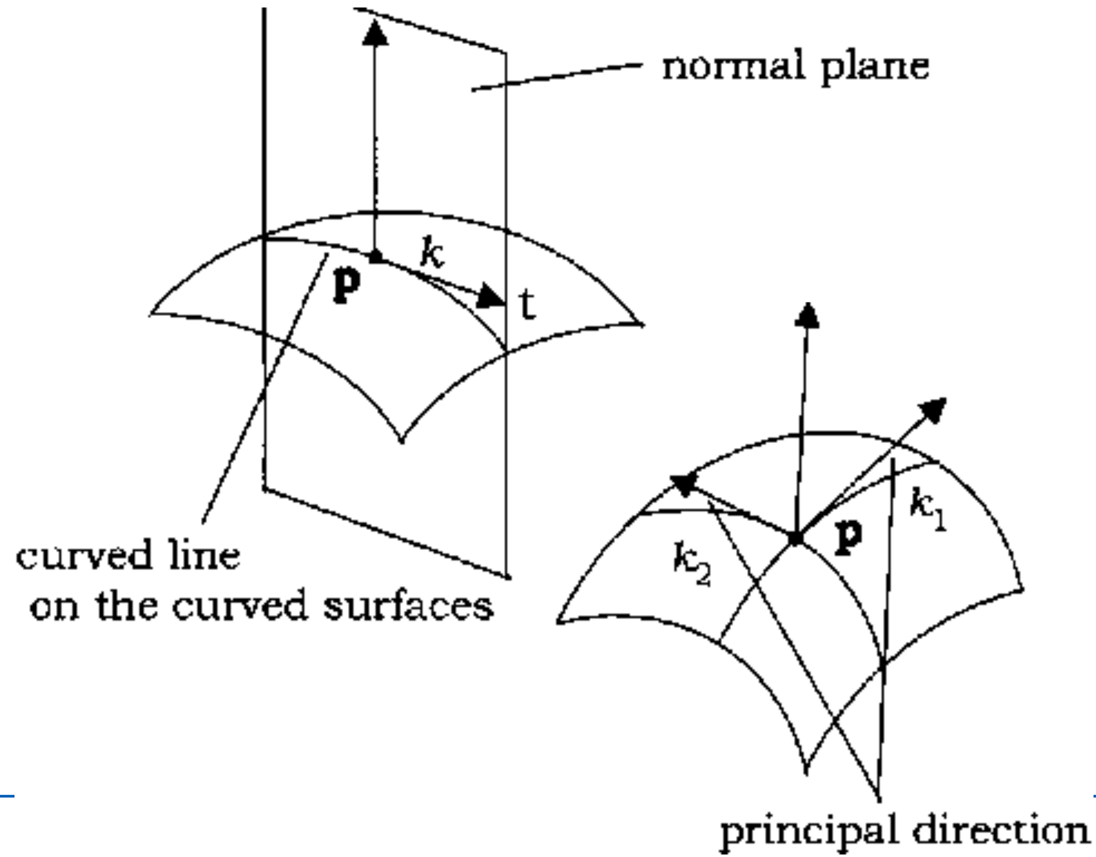
- works also for curves in space or higher dimensions
- points should be approachable with circles
- ***extrinsic*** quantity



What is curvature of a surface ?

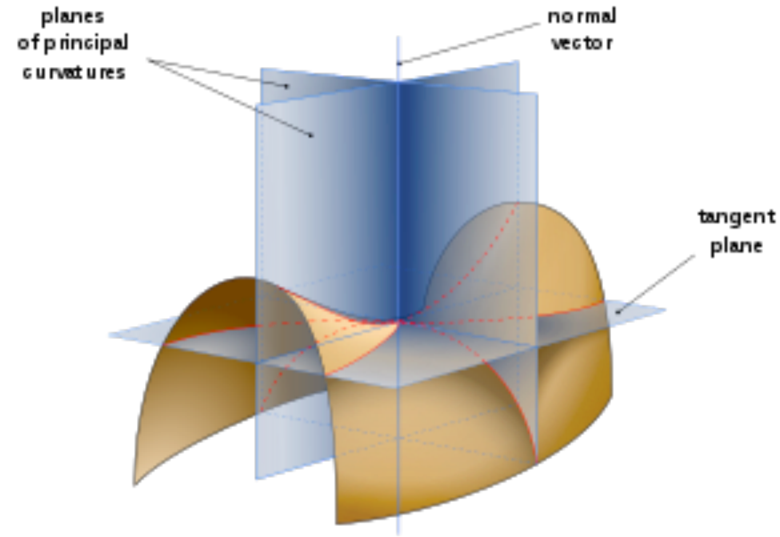
Gauss curvature

$K(p) = \varkappa_1(p)\,\varkappa_2(p)$

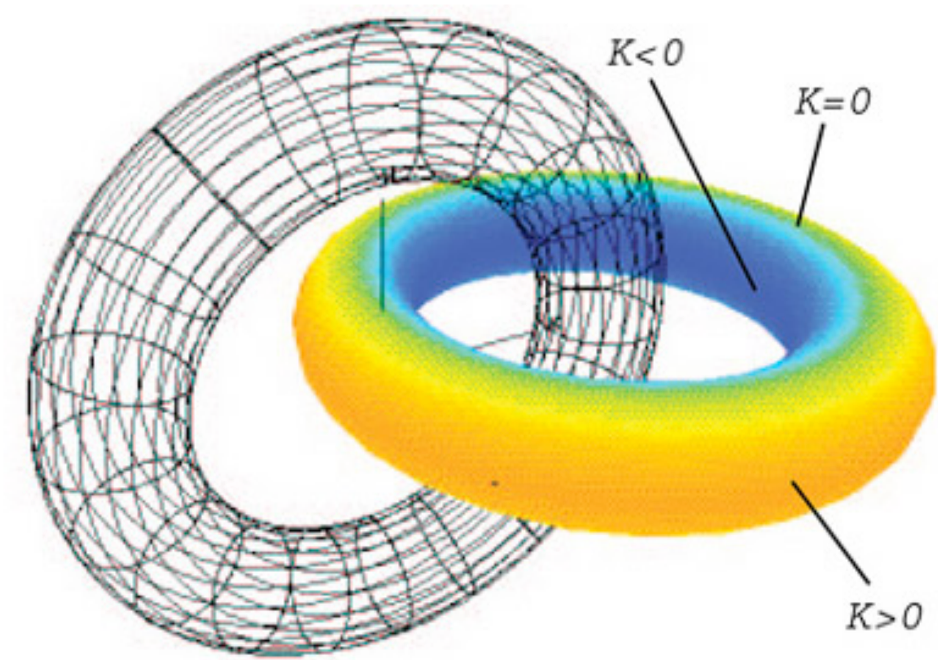


Theorema Egregium (Gauss, 1827)

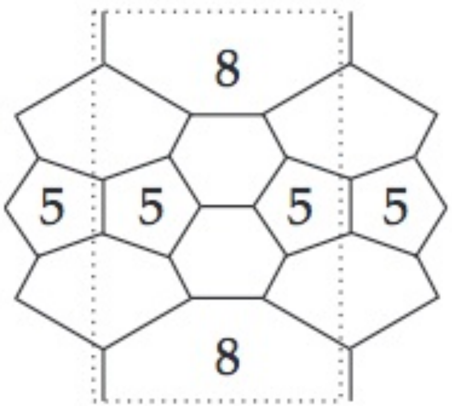
Curvature K is an *intrinsic* quantity !

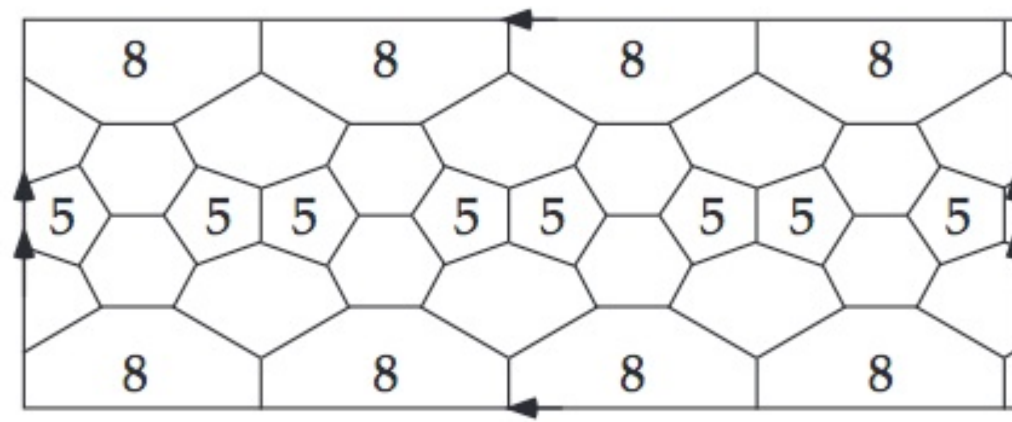


What are possible constant Gauss curvature geometries for smooth closed surfaces ?



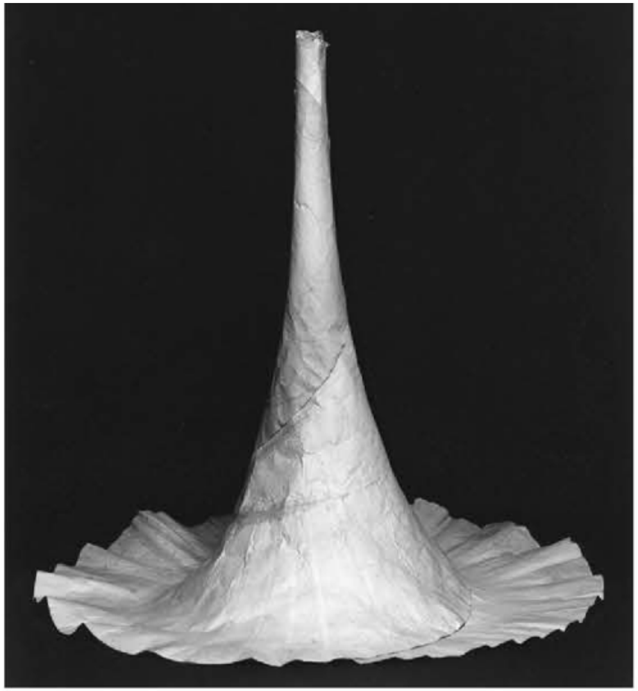
A possible construction for a torus

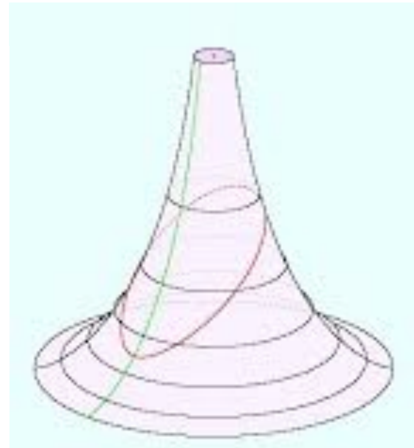




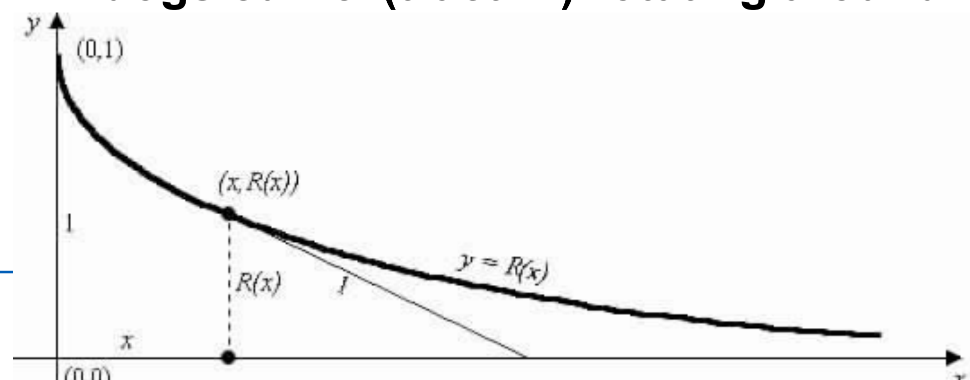
Note: V-E+F=0 for the torus

Eugenio Beltrami (1835-1900)





Pseudosphere (1866) : A local model via ‘lazy dogs curve’ (tractrix) rotating around x-axis



Curvature -1

Antonio Candido Capelo, Mario Ferrari, La "cuffia" di Beltrami: storia e descrizione, Bollettino di Storia delle Scienze Matematiche 2 (1982): 233–237.

David Hilbert (1862-1943)

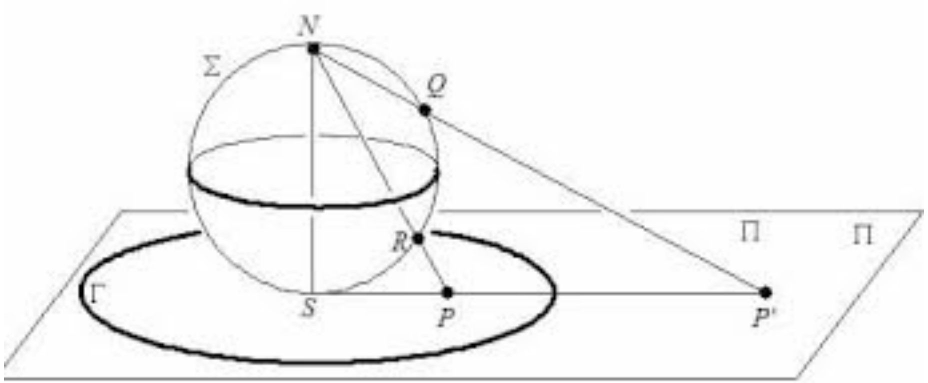
Answer (1901) to the question (*) posed by Riemann:

It is not possible to have an equation describe a surface in 3-space that has constant negative curvature and that is extended indefinitely in all directions.

Improvements by Erik Holmgren (1902), Marc Amsler (1955)

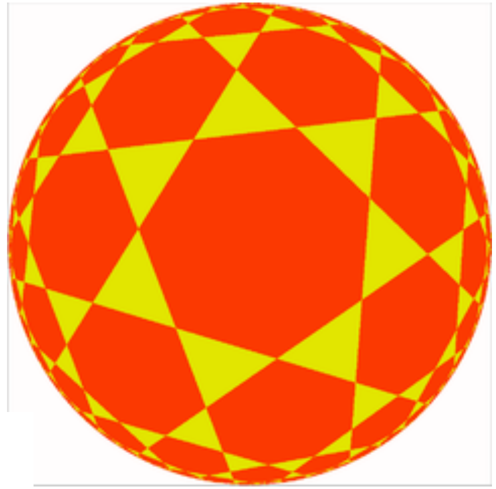
On various ways to map hyperbolic surfaces to Euclidean 3-space

Analogous problem as studying geography of our spherical planet by looking a flat map.



Ex: stereographic projection

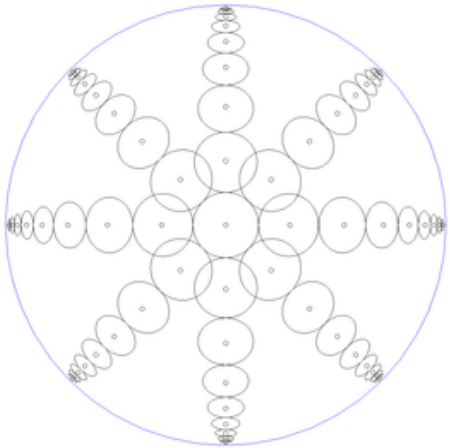
Beltrami-Klein model for the hyperbolic plane (1868, 1871)



Disk model, boundary not included

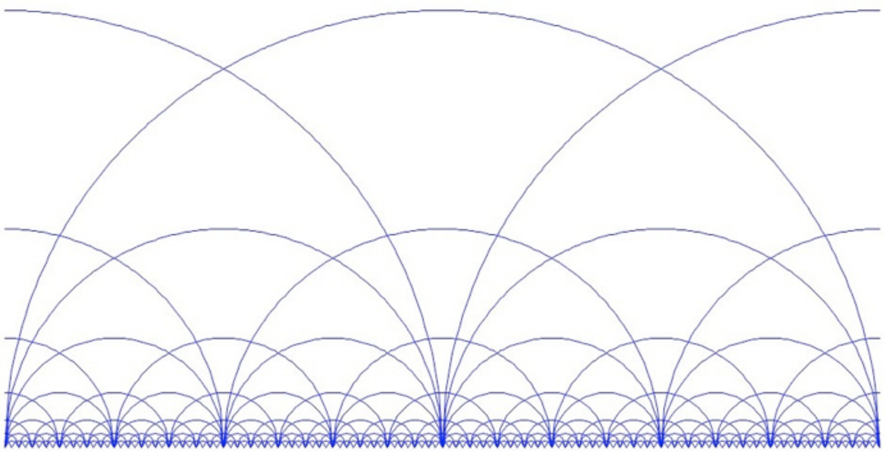
Advantage: shortest distances between points are straight lines

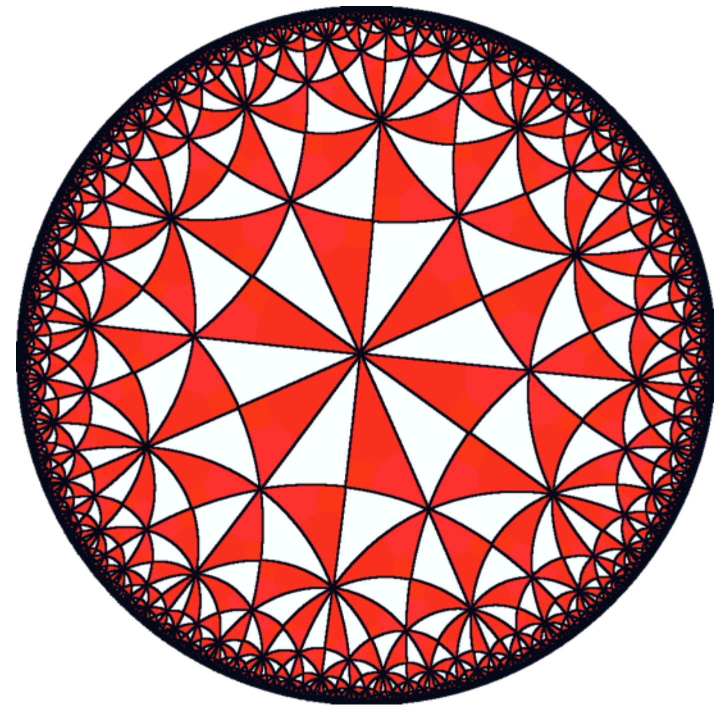
Weakness: Does not preserve angles,
Circles are not circular in general



Henri Poincaré (1854-1912) models

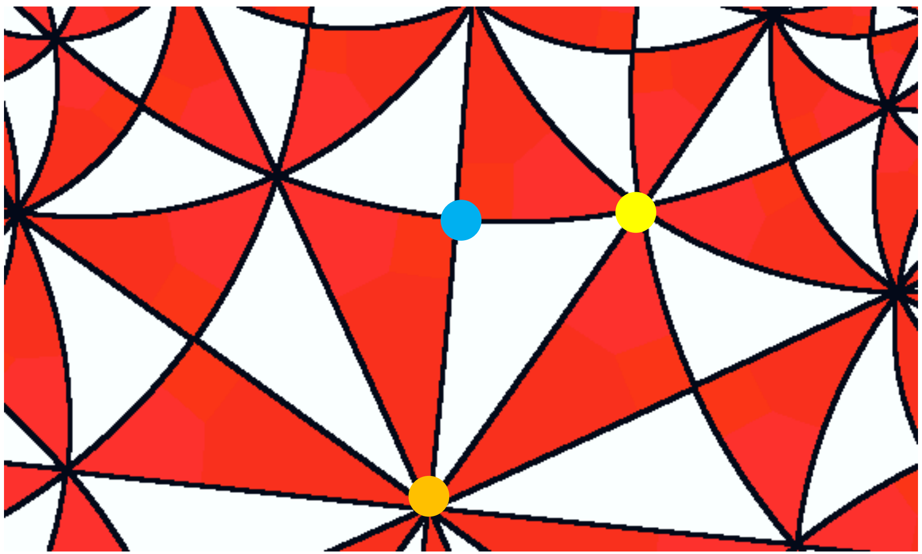
- Preserve angles (conformal model)
- Circular arcs perpendicular to the boundary realise shortest distances between points

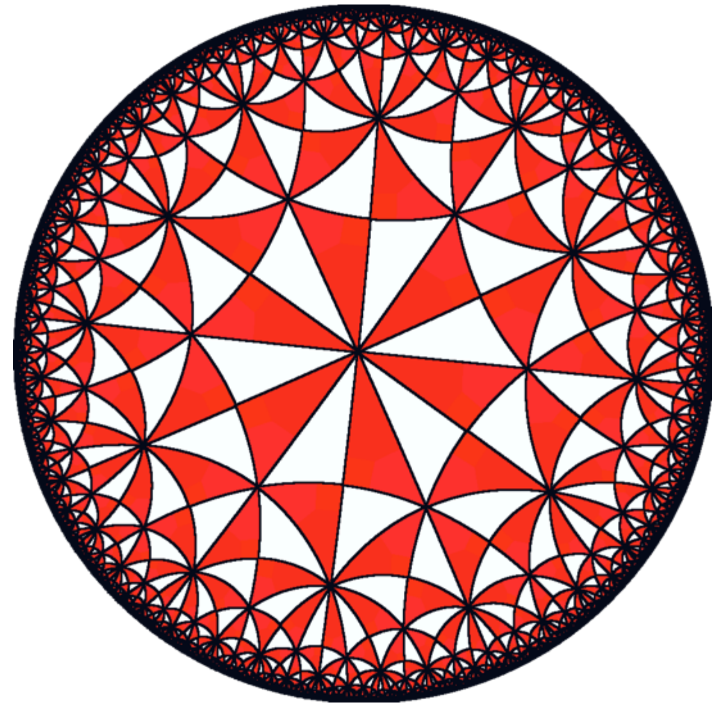




Isometries = Möbius maps preserving half plane/unit circle !

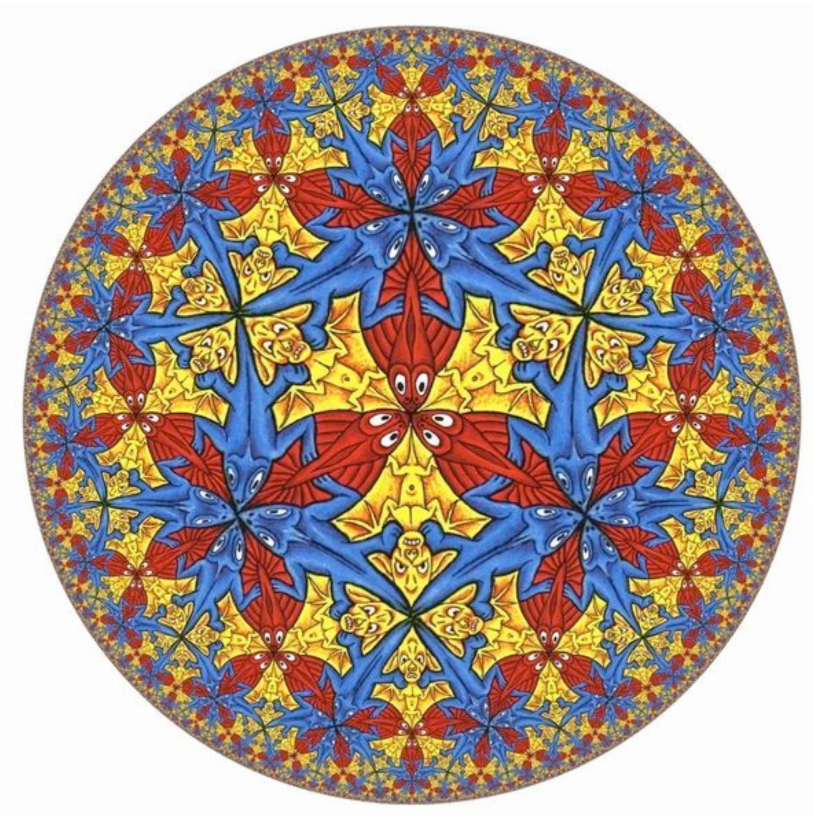
*642 if colours ignored
642 if not





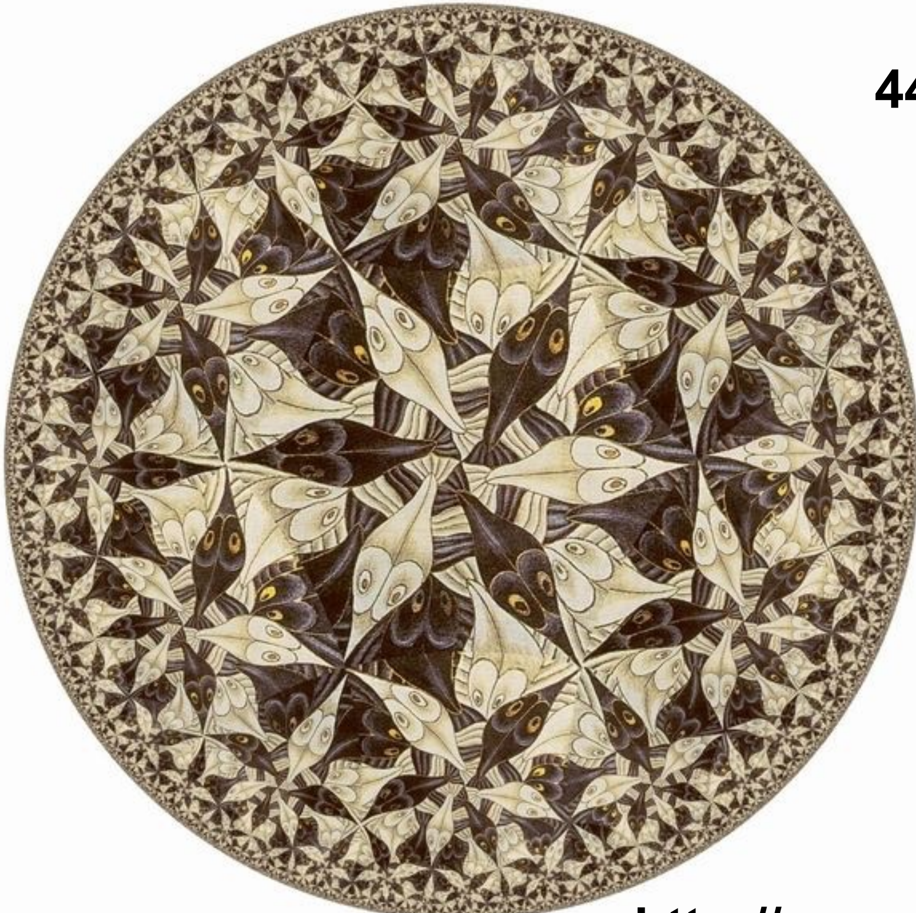
Price(*642)=1+5/12+3/8+1/4=2+2/48; Price(642)=5/6+3/4+1/2= 2+2/24

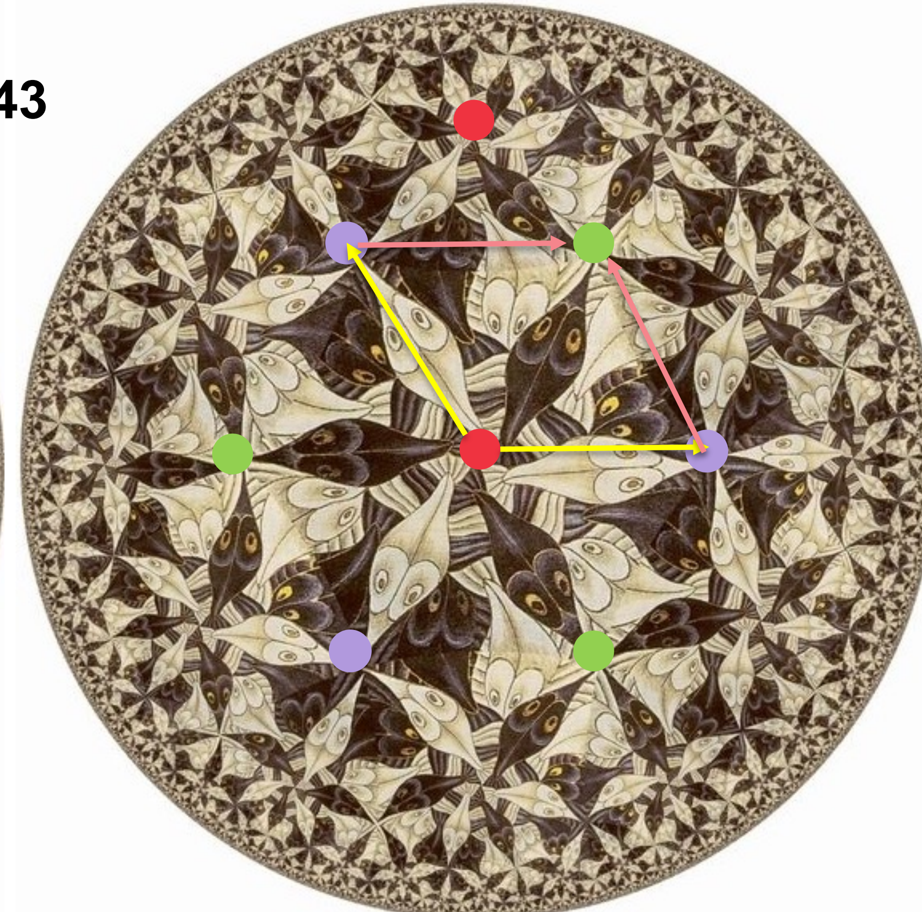
Source of inspiration for M.C. Escher (1898-1972)



***443 symmetry**
Cost: 1+3/8+3/8+2/6=2 1/12 >2 !

443

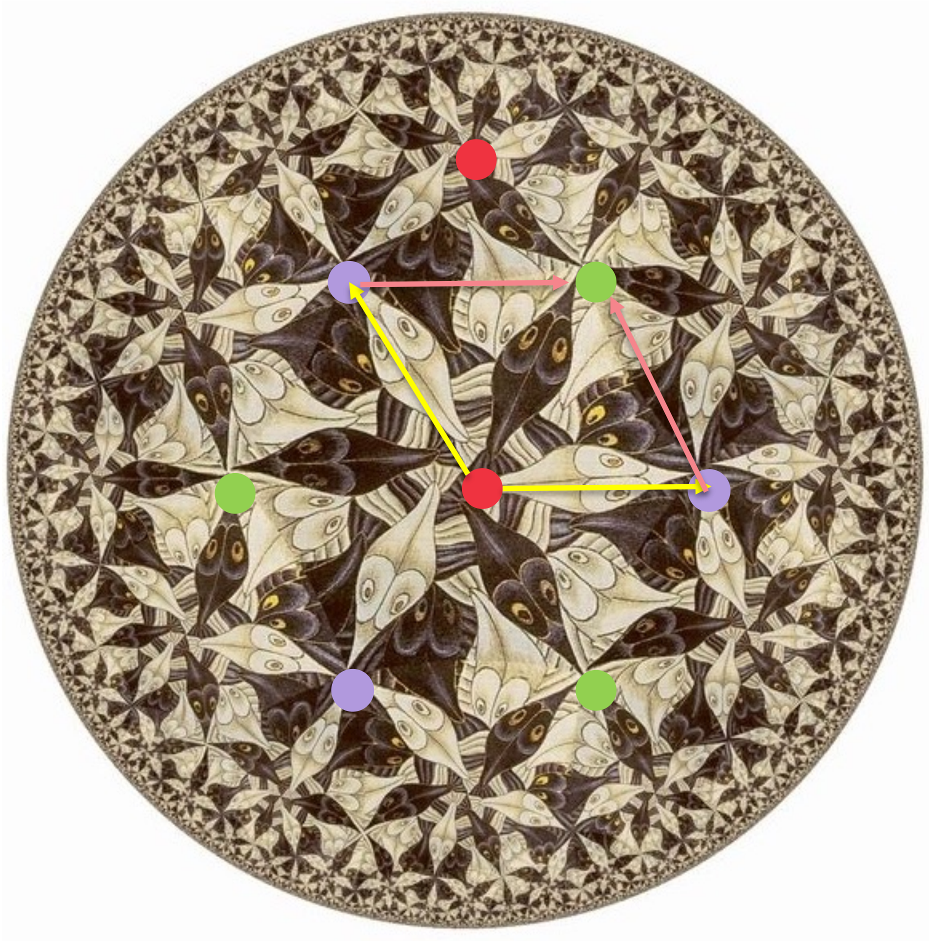




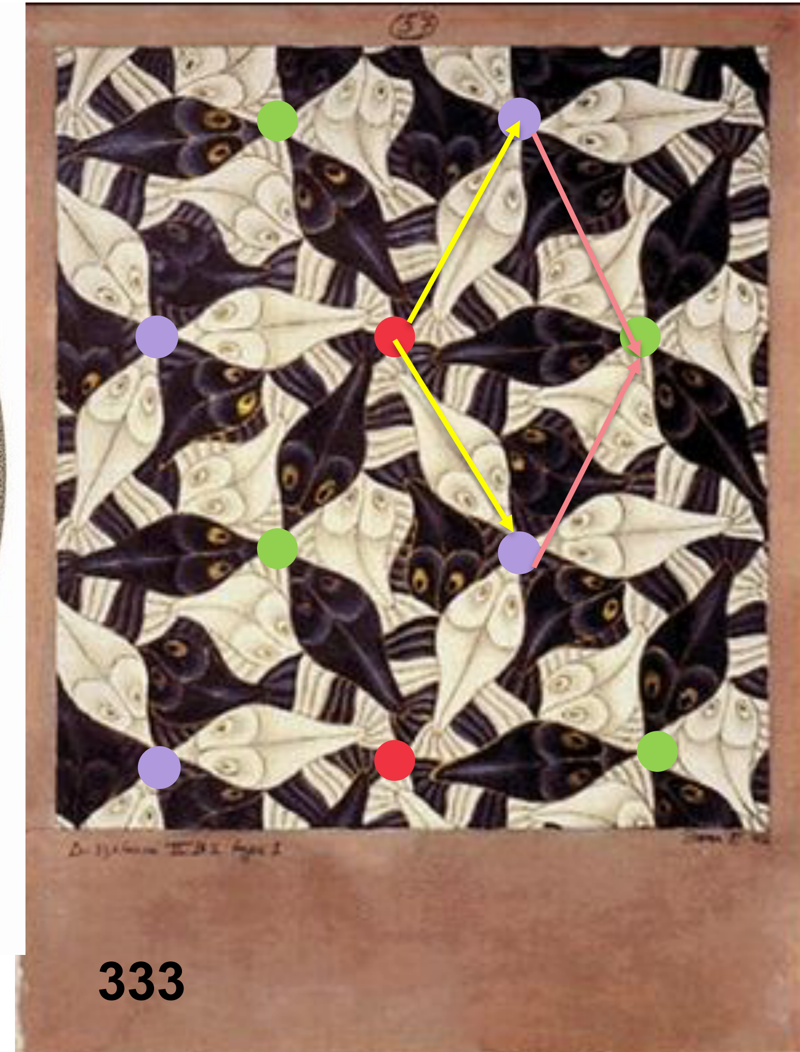
http://www.josleys.com/

Aalto University

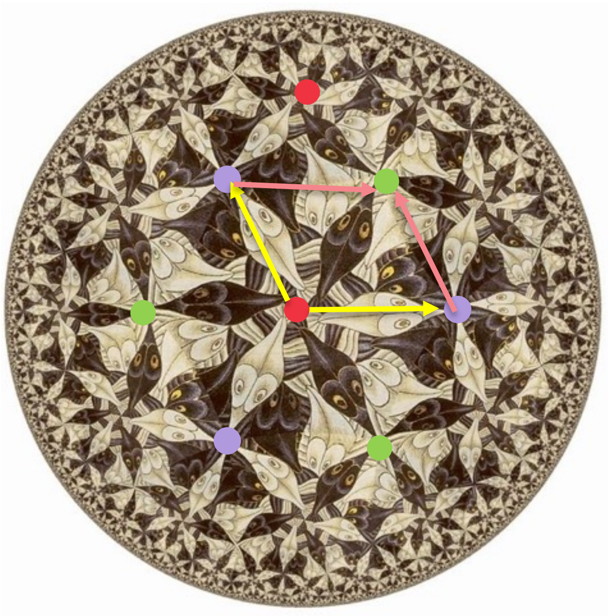
$\frac{3}{4} + \frac{3}{4} + 2/3 = 13/6 > 2$



443



333



Poicaré disk vs half plane model

4*3 total prize ¾+1+1/3 =2 1/12

4*3 or *3333 ?

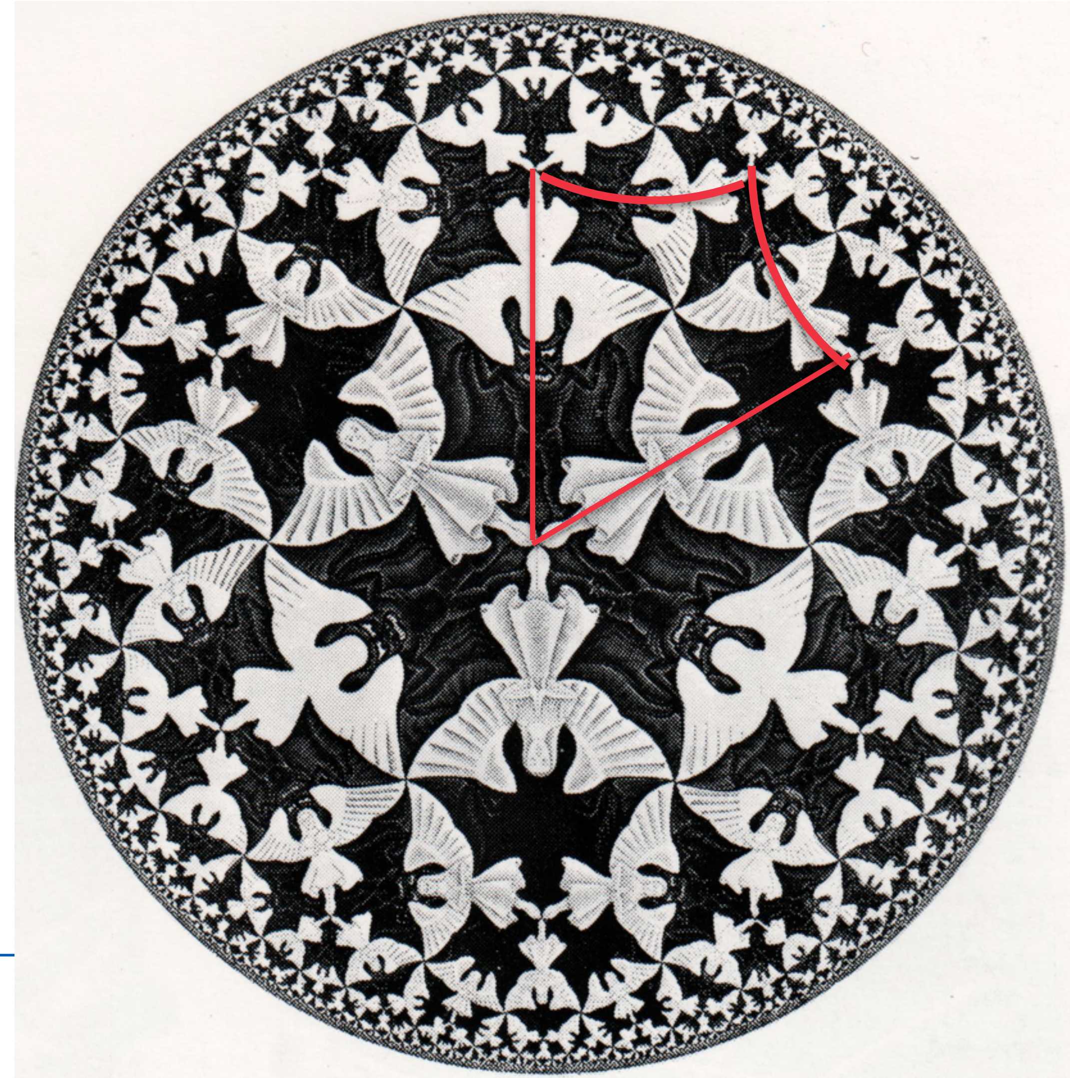
¾+1+1/3 =2 + 2/24

1+1/3 +1/3 + 1/3 + 1/3
= 2 1/3 = 2 + 2/6

Interpretation for 24 and 6 ?

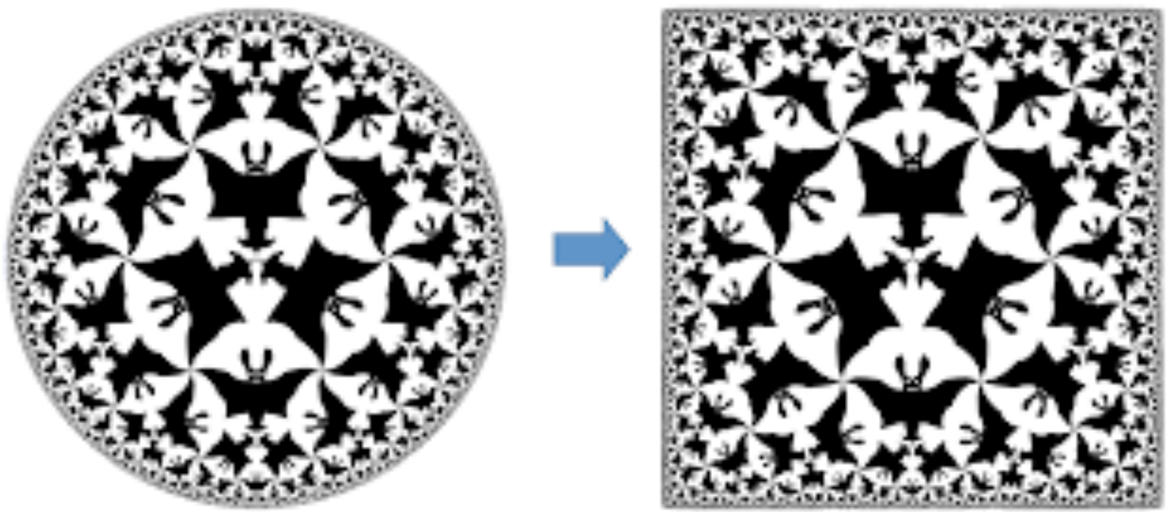
Magic theorem fails but

*3333 index 4 subgroup of 4*3

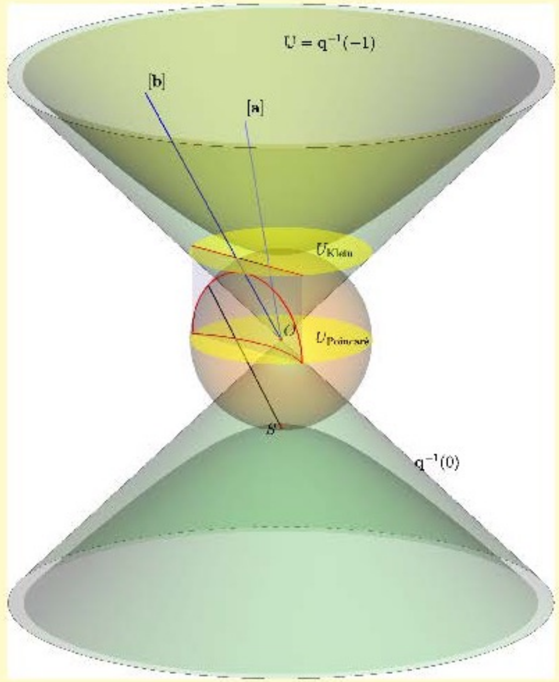


Riemann mapping theorem & Escher

Every domain (= open and connected set) without holes (=simply connected) whose exterior contains at least one point in the plane can be mapped conformally to a disk



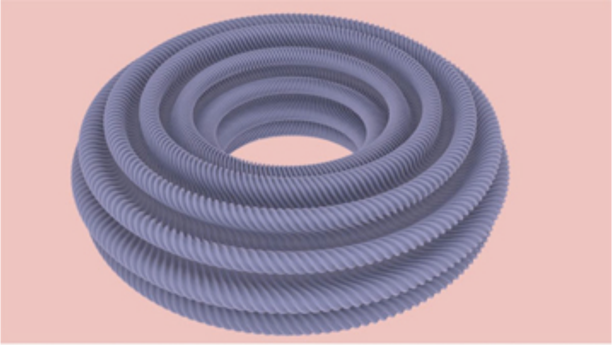
Space-time interpretation of the hyperbolic space

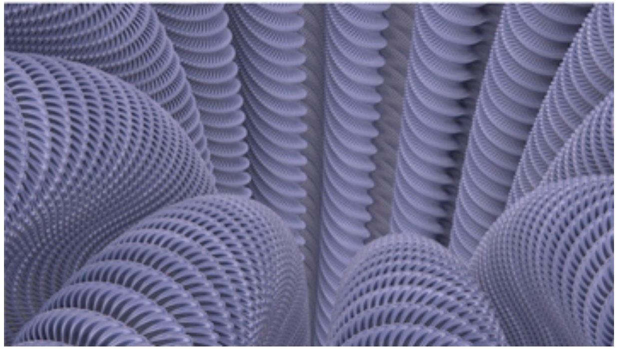


$$q(x)=x_1^2+ \dots +x_{n-1}^2 - x_n^2$$

What surfaces support hyperbolic structures ?

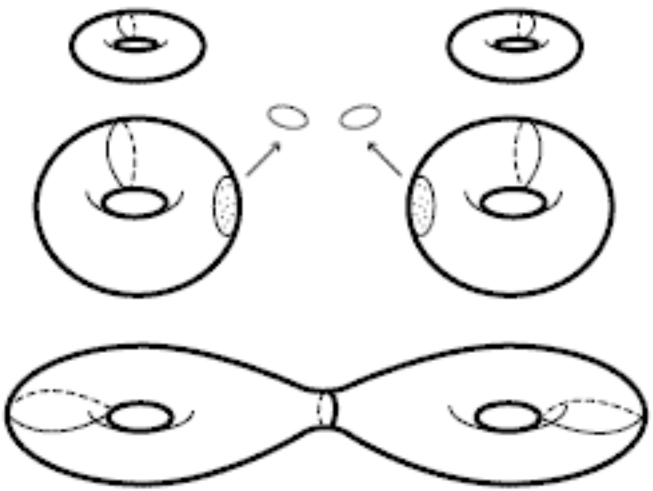
- **Sphere supports spherical structure**
- **Torus has a flat structure**
- **Surfaces of higher genus can be equipped with a hyperbolic structure**





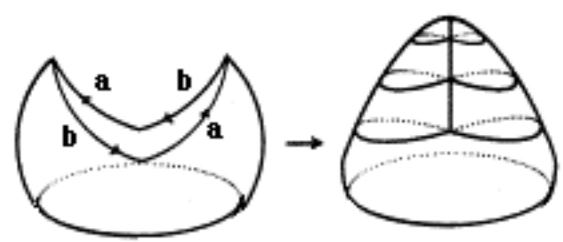
Classification of closed (compact without boundary) surfaces

Connected sum operation



All orientable surfaces are connected sums of a sphere and finitely many tori

All non-orientable surfaces are connected sums of a sphere and finitely many projective spaces



All surfaces can be described as polygons with boundary identifications (normal form)

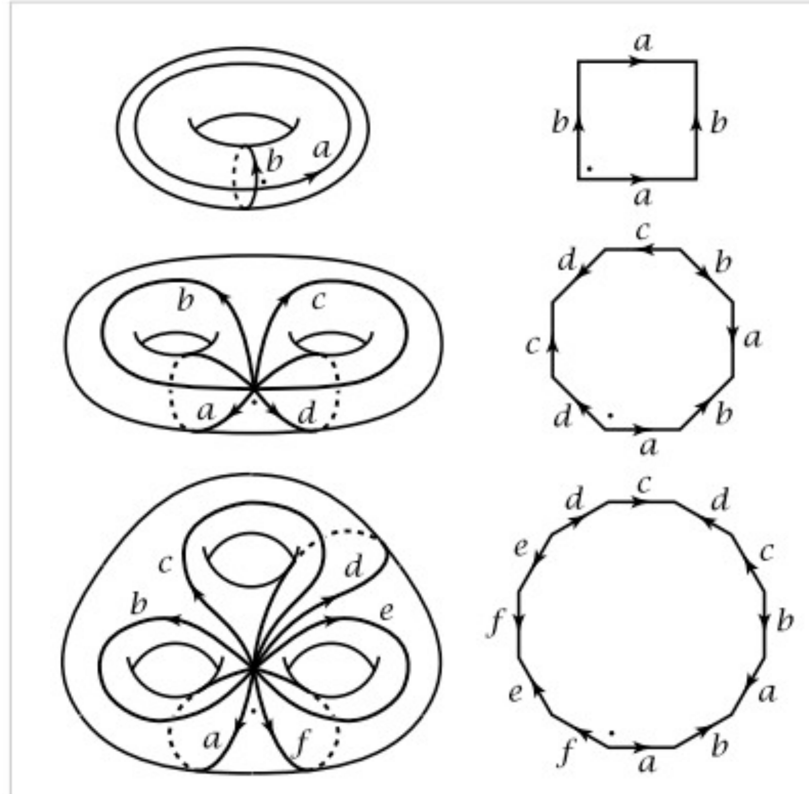
The *Euler characteristic* of a surface χ(S) =2-2g is a topological invariant !

g=0 sphere

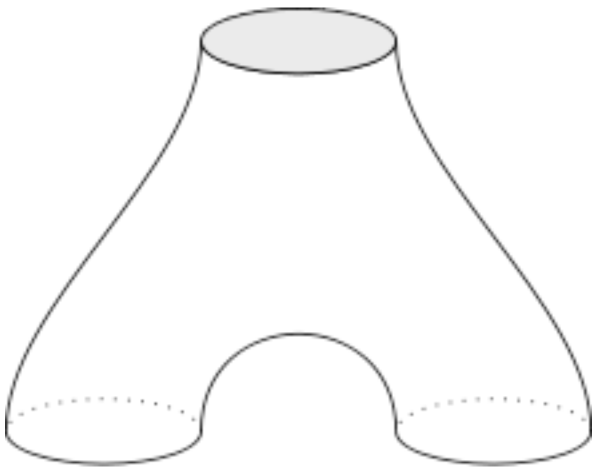
g=1 torus

χ(S) = V-E+F, **V**= number of vertices, **E**= number of edges, **F**=number of faces of any cell subdivision of the surface

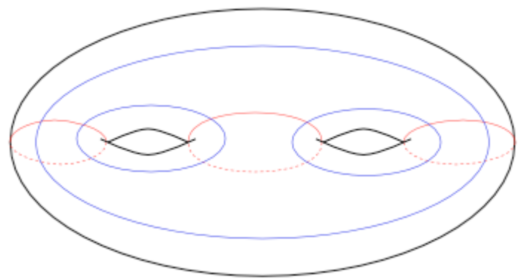
Especially **χ(S) =2** for *all polyhedrons !*



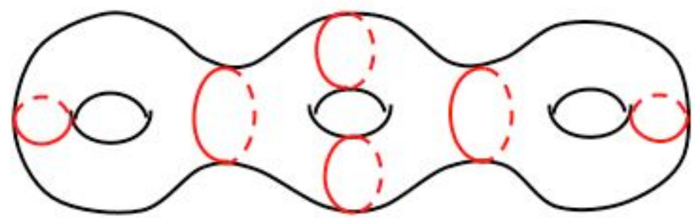
Pair of pants decompositions of surfaces



A pair of pants



Pair of pants decomposition of a two holed torus

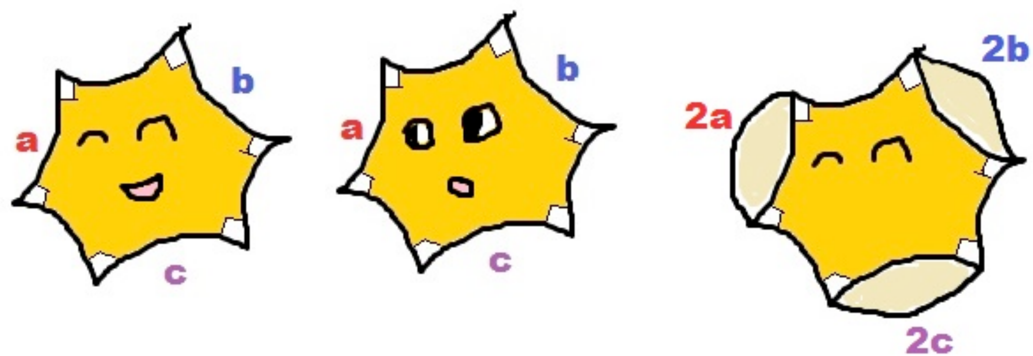


Pair of pants decomposition of a three holed torus

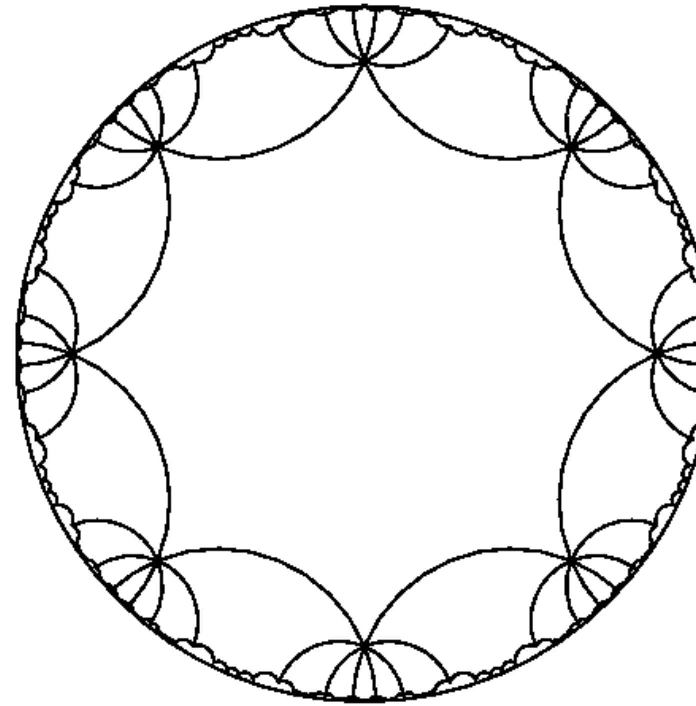
Gauss-Bonnet Theorem for compact orientable surfaces at least genus 2

Every two pair of pant decomposition of a surface S have the same number of pair of pants and this number is even. The number of pair of pants is $-\chi(S) = 2g-2$, $g \geq 2$ and the (normalized) hyperbolic surface area is $-2\pi\chi(S)$

Hyperbolic structure can be given for example through hexagons with 90 degree angles

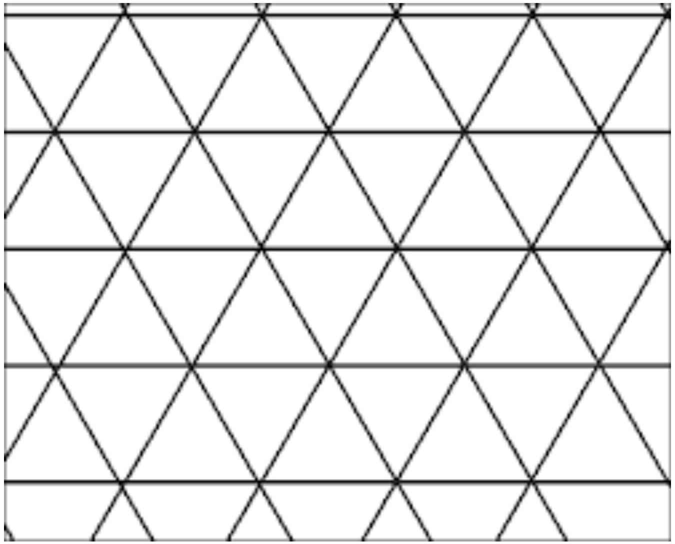


A construction by Daina Taimina



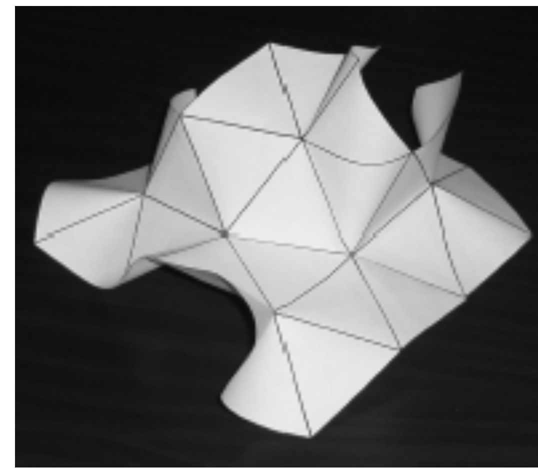
A hyperbolic octagon with 45 deg interior angles can be glued to a hyperbolic pair of pants

Thurston model to approximate hyperbolic plane



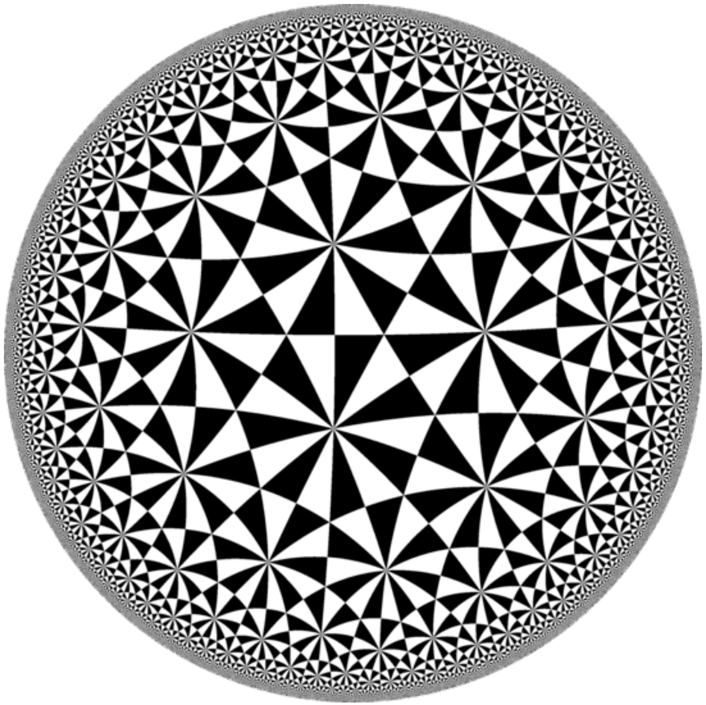
- Cut out a hexagon formed by 6 equilateral triangles
- Make a slit and tape one more triangle so that 7 triangles meet at a vertex

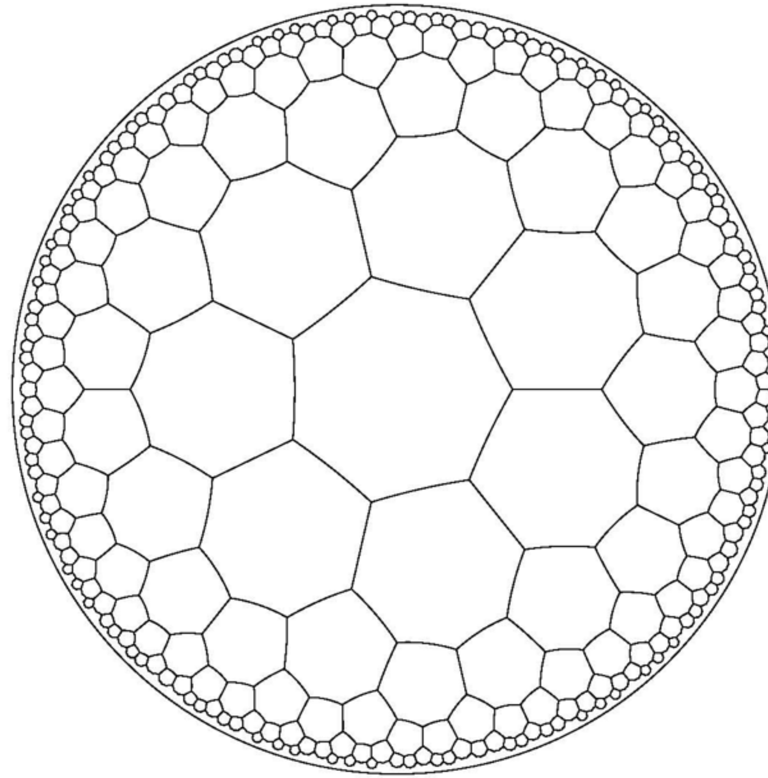
- Add at least two layers of triangles so that every vertex is adjacent to 7 triangles



Thurston vs Poincaré & Beltrami-Klein

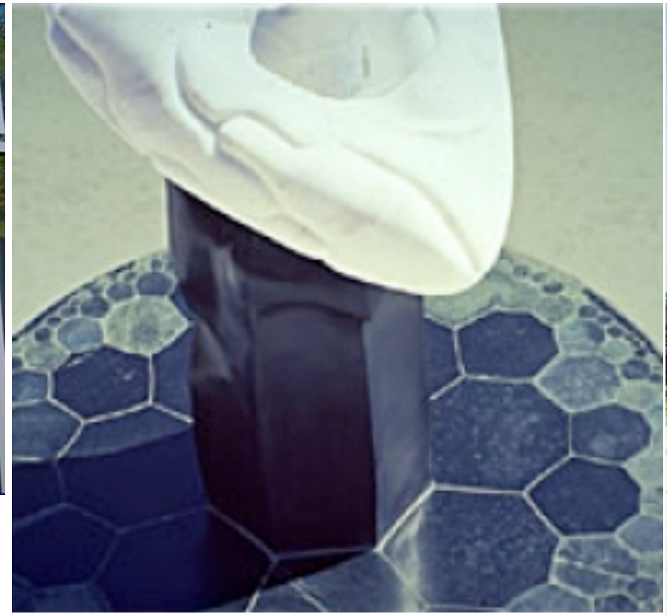
732 (or *732 by ignoring colours)

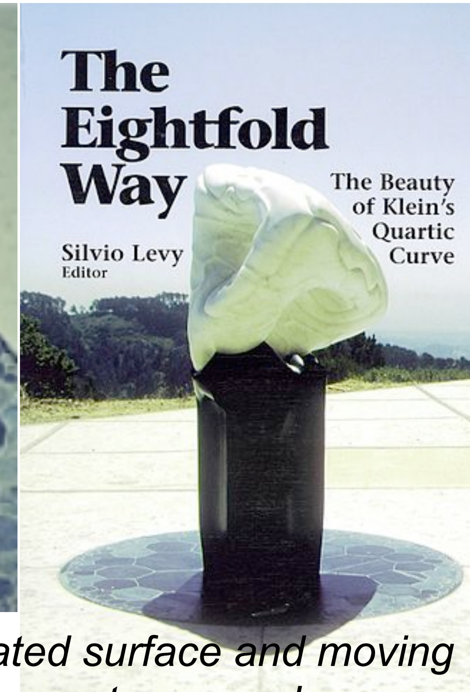




The Eightfold way by Helaman Ferguson

24 heptagons

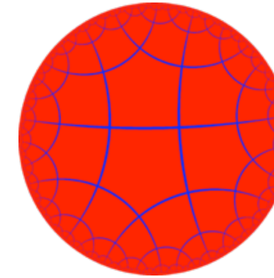




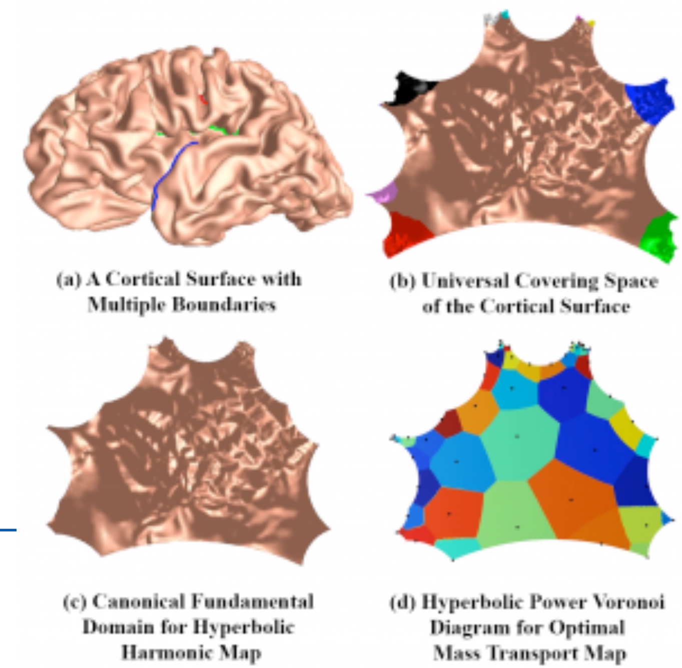
The title refers to the fact that starting at any vertex of the triangulated surface and moving along any edge, if you alternately turn left and right when reaching a vertex, you always return to the original point after eight edges.
http://library.msri.org/books/Book35/files/thurston.pdf

Klein Quartic $x^3y + y^3z + z^3x = 0.$ x,y,z complex (homogenous) coordinates

Why hyperbolic geometry ?



- Connections to cellular automata (Margenstern-Morita etc.)
- Visualizations of Web, Network security
- Modular functions in number theory (Fermat's last theorem)
- Algebraic geometry, differential geometry, complex variables, dynamical systems
- Biology



- Self folding materials in **chemistry**
- R. Nesper, S. Leoni structural chemistry
- **Medicine** ex. M. Steiner brain image analysis
- **Physics** J. Richard Gott III : multiply connected universe model
- Ergodic theory, string theory
- J. Vigoureux, R. Giust: Multilayers in optics
- **Music** Dmitri Tymoczko and his music chord orbifolds
- **Why not !**

Aalto University

Fractal Geometry

Phenomena that cannot be explained by classical geometry

Shapes in Action Tue 13th Oct

Contents

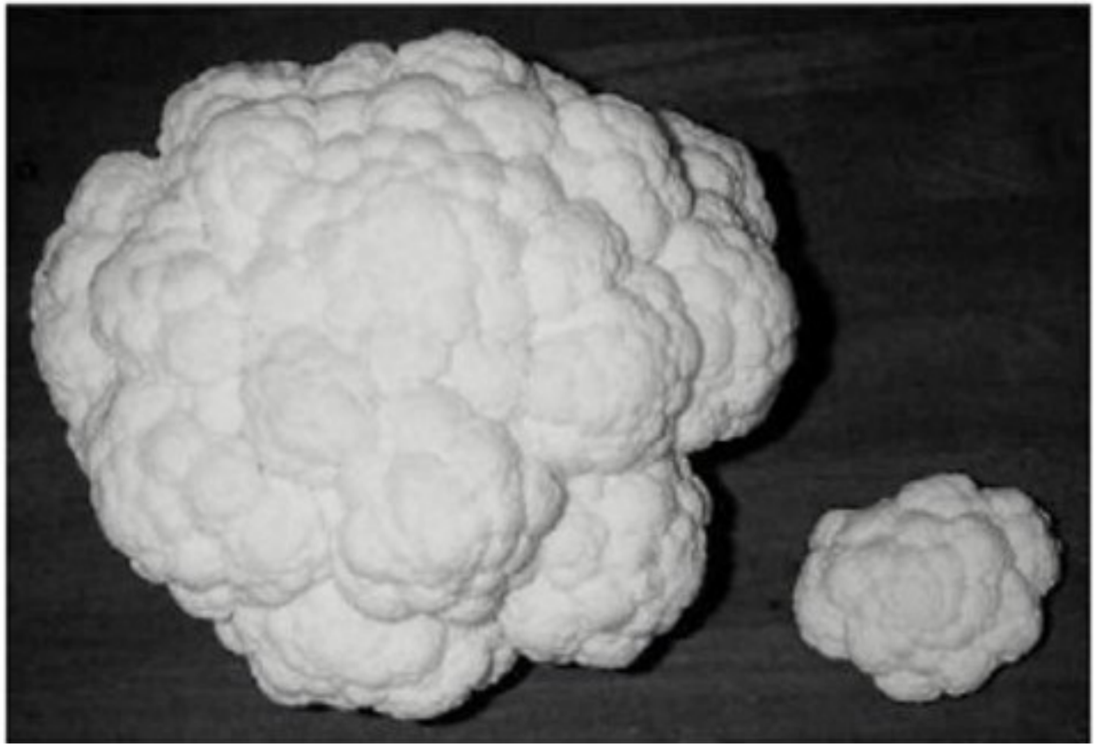
‘Natural’ vs. ‘man-made’ objects

What do these pictures present?

What happened ? Why does the trick work?

Many objects look the same in different scales.

How can one distinguish the correct size?



Who invented 'fractal geometry' in the sense of 'new geometry of nature'?

Many fundamental examples due to classical mathematics !

George Ferdinand Ludwig Philipp Cantor 1845-1918

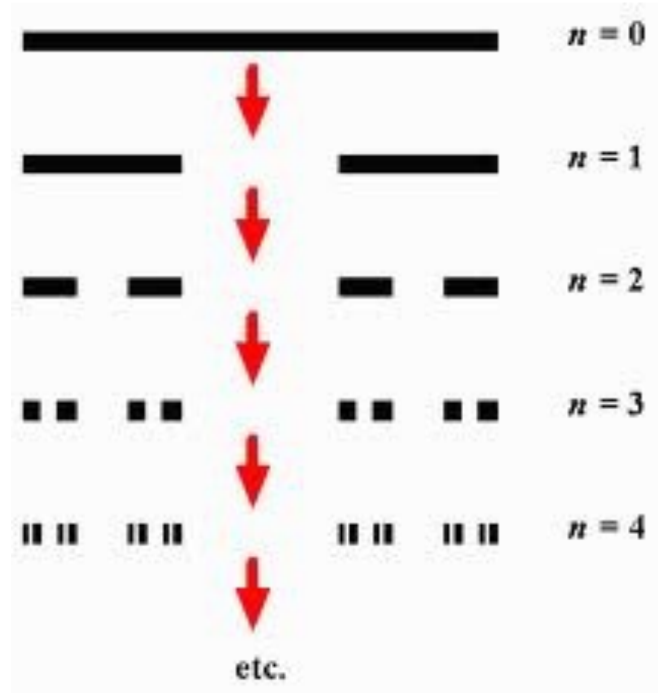
- *the crisis of the dimension*
- *exceptional objects*
- *'mathematical monsters'*
- *limits of fundamental notions*

('curve', 'continuous')

Abnormal Monsters or Typical Nature ?

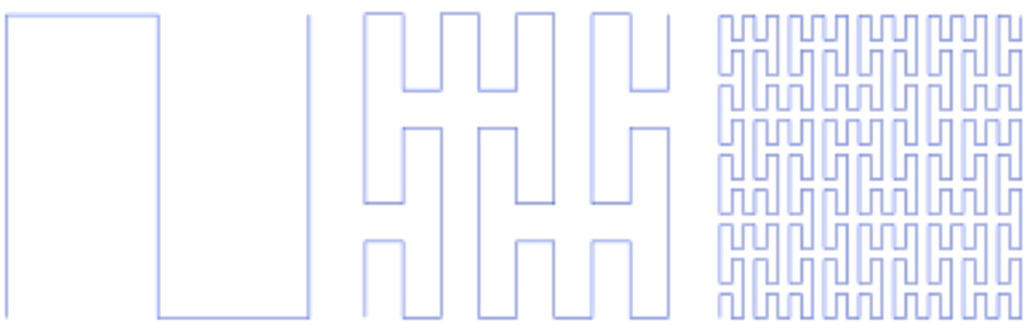
Cantor's middle third set (1883)

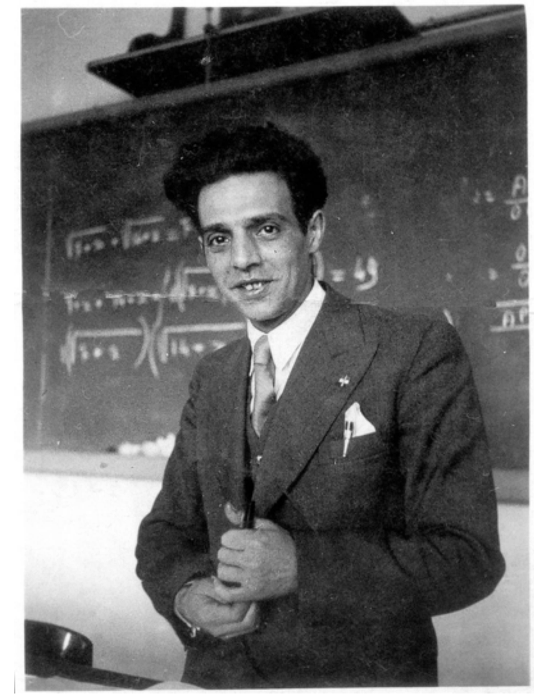
- **are there any points left in the limit?**
  - subintervals left $(2/3)^n \rightarrow 0$ as $n \rightarrow \infty$
  - endpoints never removed !
  - infinite decimal presentation of 0's and 2's in a base 3 (1/3=0.0222…!)
- **is it possible to numerate them ?**
- **size of the limit set vs [0,1]?**
- **dimension of the limit set?**
- **connectedness of the limit set?**
- ***a self-similar set***
- **a prototype of a fractal set**



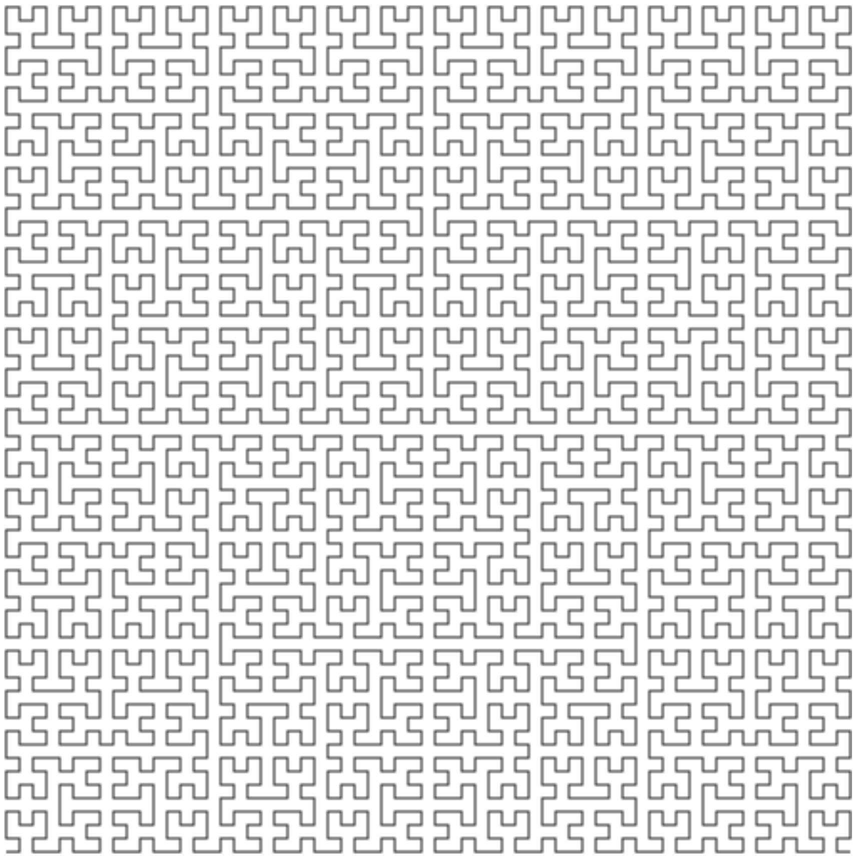
Giuseppe Peano, 1858-1932

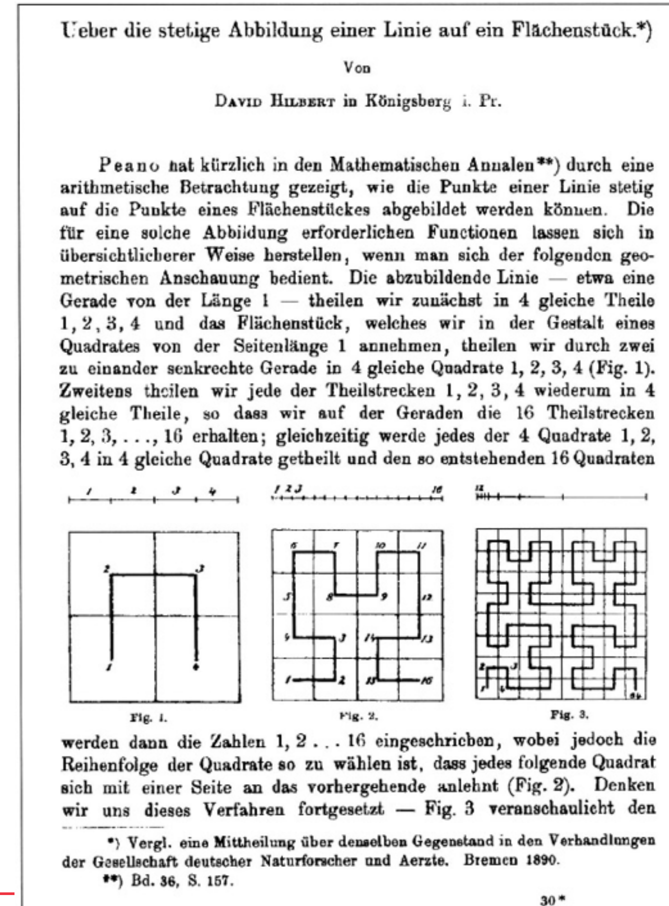
What is a curve?
What is the dimension of a curve?
Can a curve fill a square/cube/hypercube/…?





A Peano curve by David Hilbert (1862-1943)





Ueber die stetige Abbildung einer Linie auf ein Flächenstück.*)

Von

David Hilbert in Königsberg i. Pr.

Peano hat kürzlich in den Mathematischen Annalen**) durch eine arithmetische Betrachtung gezeigt, wie die Punkte einer Linie stetig auf die Punkte eines Flächenstückes abgebildet werden können. Die für eine solche Abbildung erforderlichen Functionen lassen sich in übersichtlicherer Weise herstellen, wenn man sich der folgenden geometrischen Anschauung bedient. Die abzubildende Linie — etwa eine Gerade von der Länge 1 — theilen wir zunächst in 4 gleiche Theile 1, 2, 3, 4 und das Flächenstück, welches wir in der Gestalt eines Quadrates von der Seitenlänge 1 annehmen, theilen wir durch zwei zu einander senkrechte Gerade in 4 gleiche Quadrate 1, 2, 3, 4 (Fig. 1). Zweitens theilen wir jede der Theilstrecken 1, 2, 3, 4 wiederum in 4 gleiche Theile, so dass wir auf der Geraden die 16 Theilstrecken 1, 2, 3, . . ., 16 erhalten; gleichzeitig werde jedes der 4 Quadrate 1, 2, 3, 4 in 4 gleiche Quadrate getheilt und den so entstehenden 16 Quadraten

Fig. 1. Fig. 2. Fig. 3.

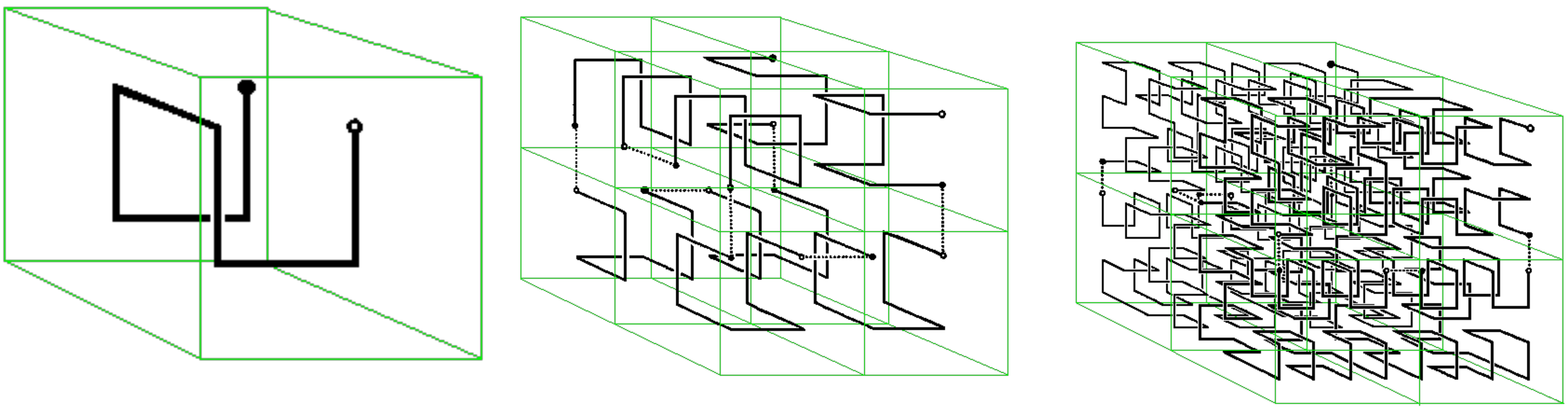
werden dann die Zahlen 1, 2 . . . 16 eingeschrieben, wobei jedoch die Reihenfolge der Quadrate so zu wählen ist, dass jedes folgende Quadrat sich mit einer Seite an das vorhergehende anlehnt (Fig. 2). Denken wir uns dieses Verfahren fortgesetzt — Fig. 3 veranschaulicht den

*) Vergl. eine Mittheilung über denselben Gegenstand in den Verhandlungen der Gesellschaft deutscher Naturforscher und Aerzte. Bremen 1890.

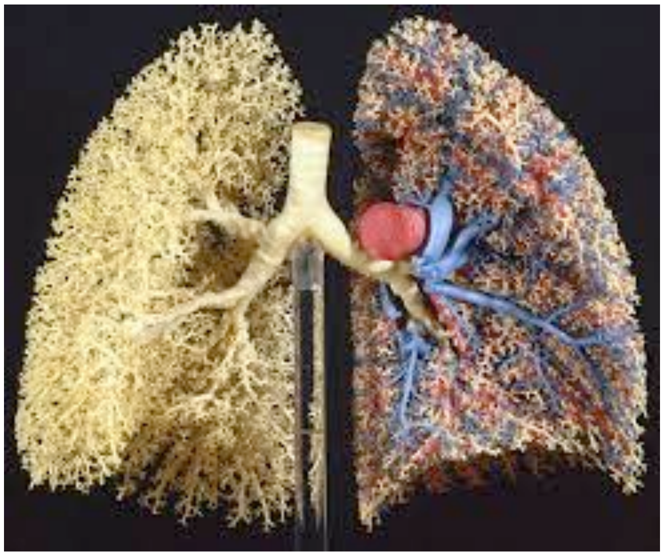
**) Bd. 36, S. 157.

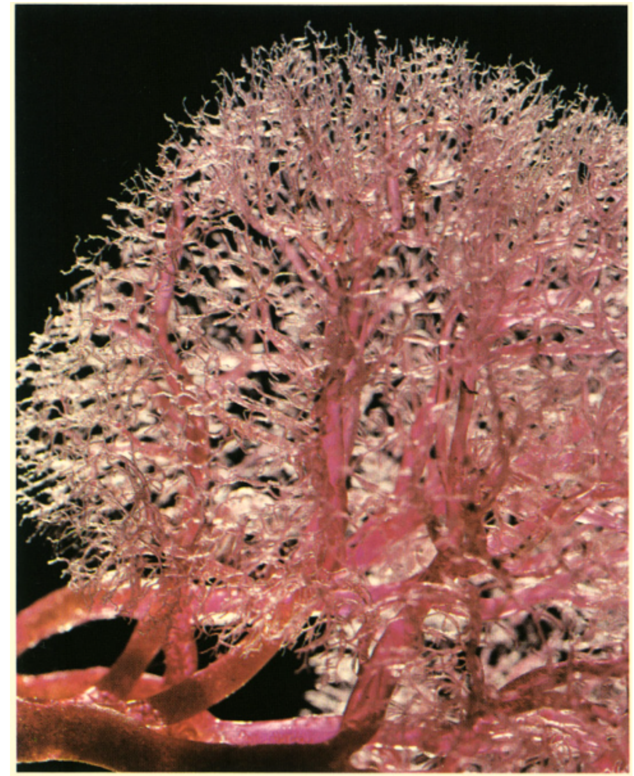
30*

A 3D generalisation of Hilbert's construction by W. Gilbert (Mathematical Intelligencer 6(3) (1984), page 78)

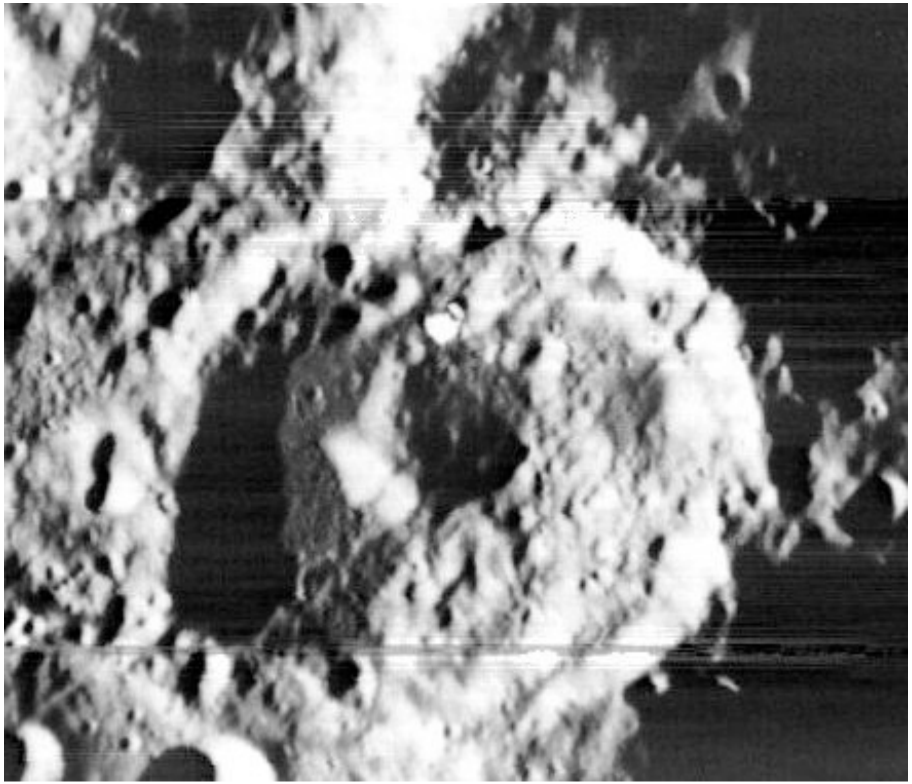


Space filling structures in Nature

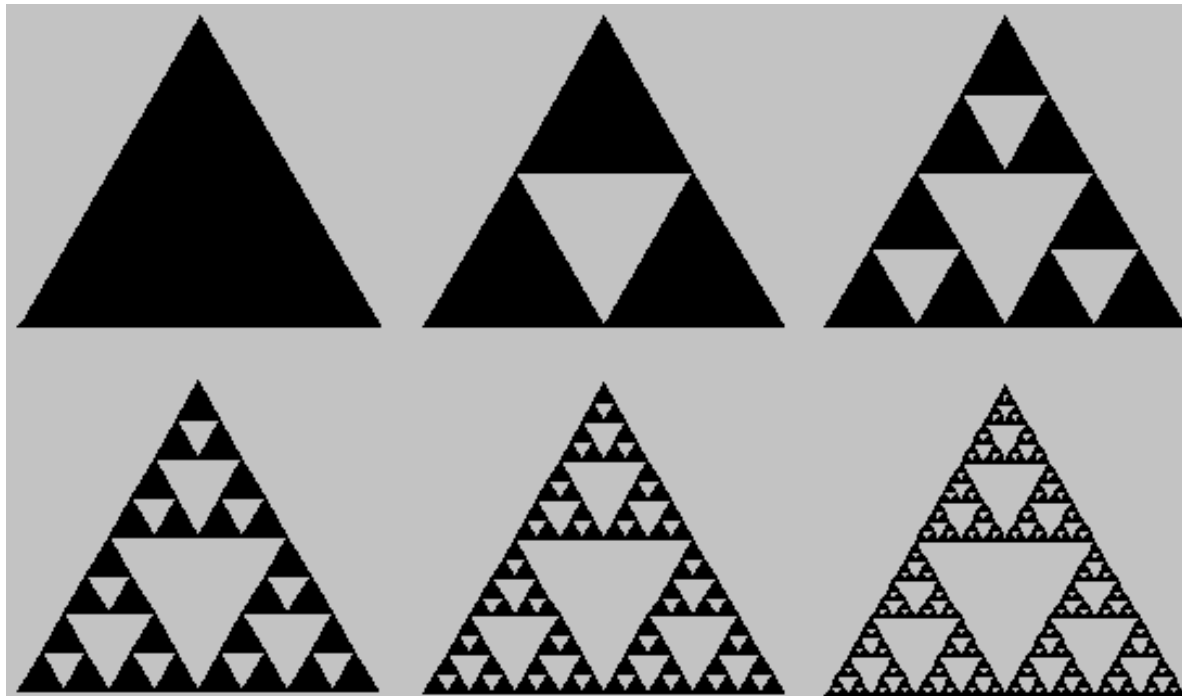




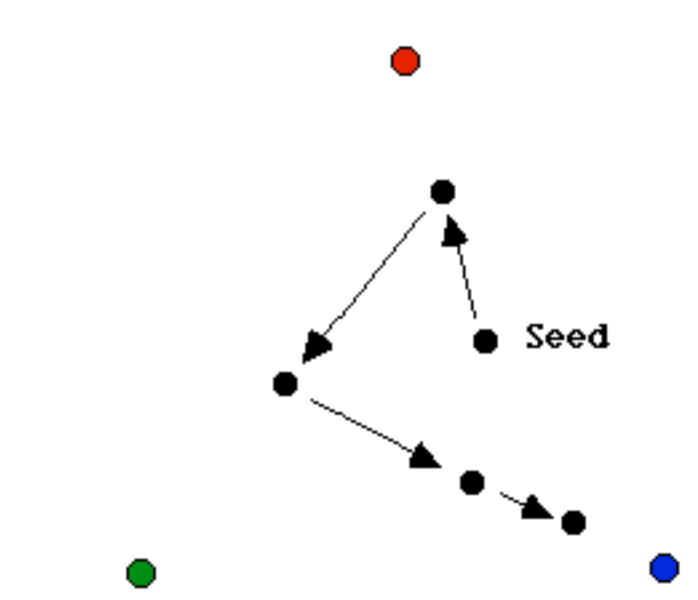
Wacław Franciszek Sierpiński 1882-1969

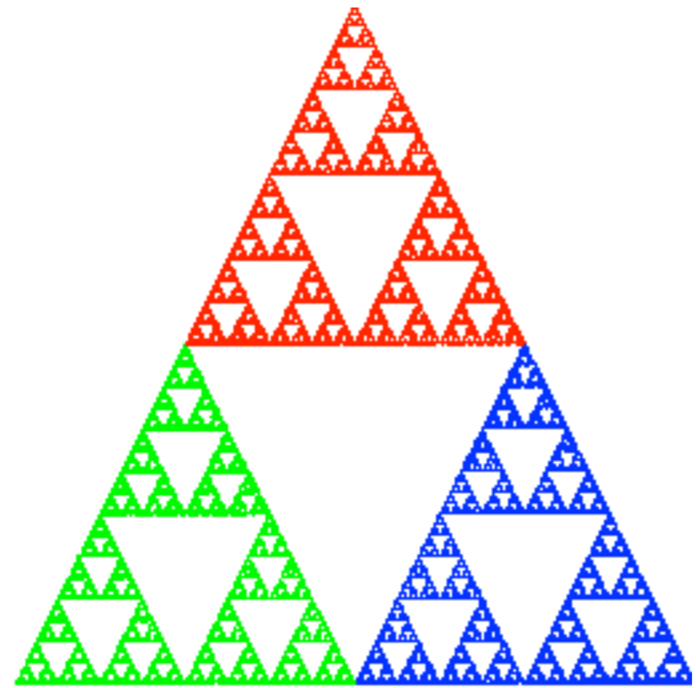


A self similar process in Sierpiński gasket (1916)



The Chaos Game (Barnsley)





Cathedral Anagni (Italy) 1104

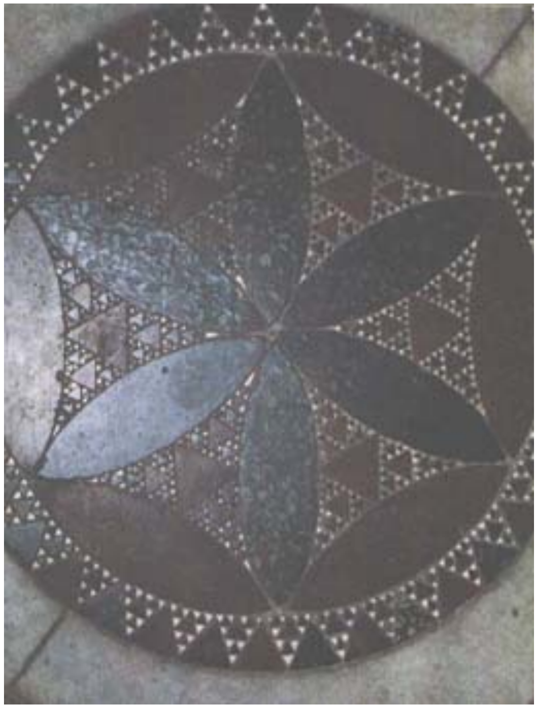
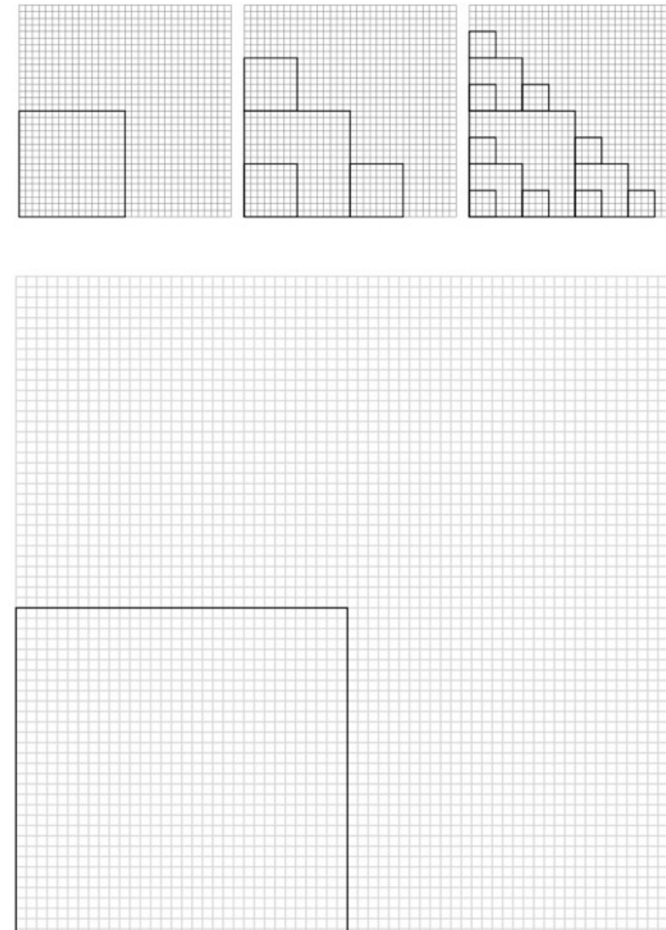
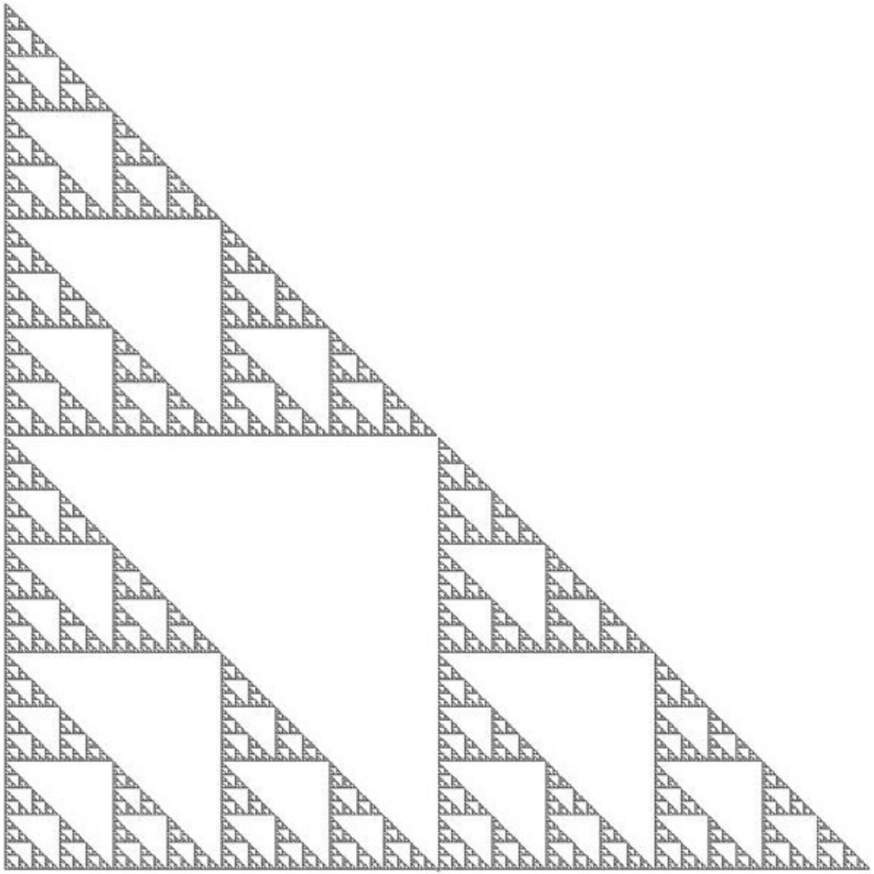


Fig. 6 SS. Giovanni e Paolo (13th century), Rome

Santa Maria in Cosmedin, Rome

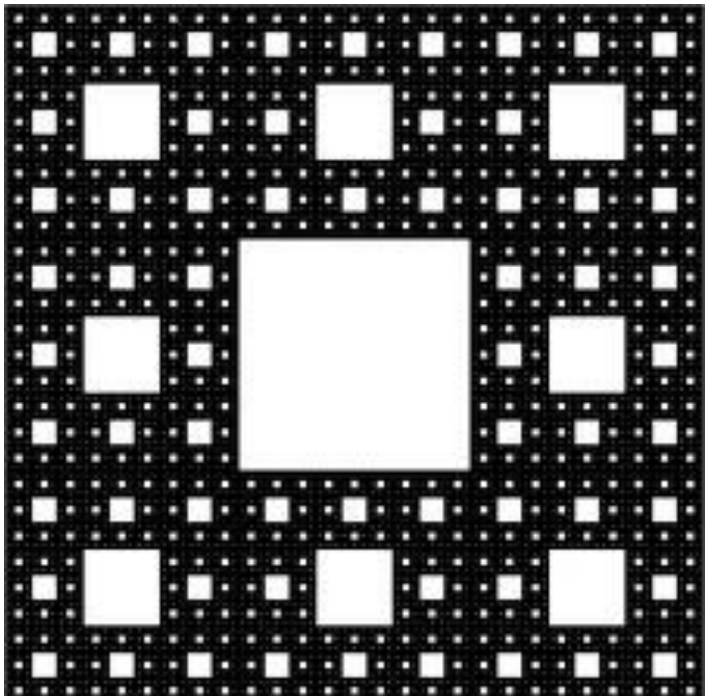


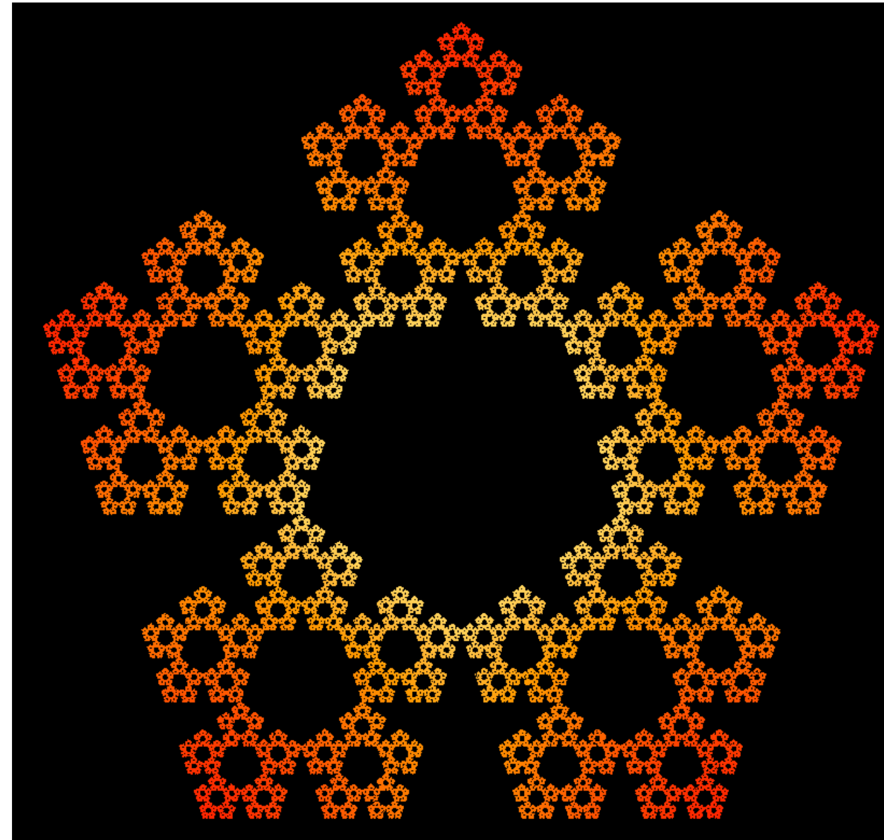


Escher’s studies of Sierpinski gasket-type patterns

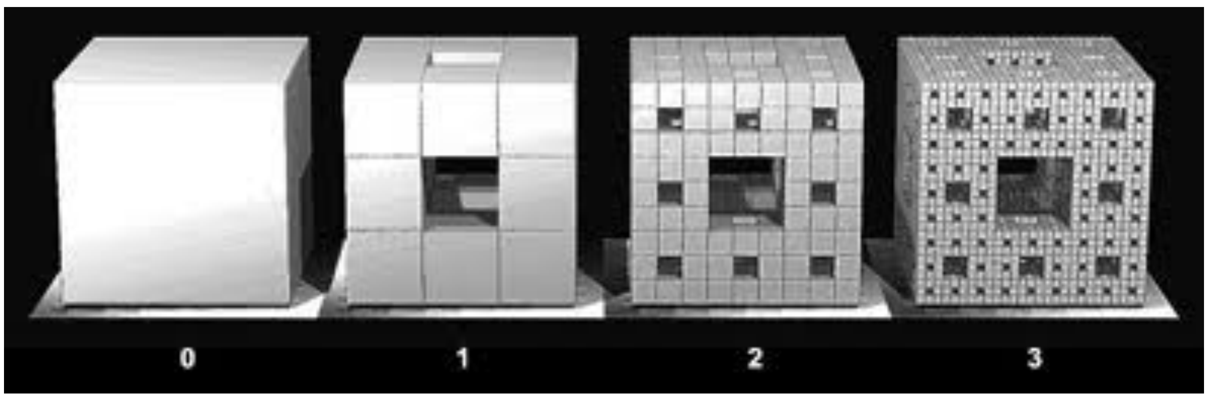
On twelfth-century pulpit of Ravello Cathedral, 1923

Sierpiński Carpet and generalizations

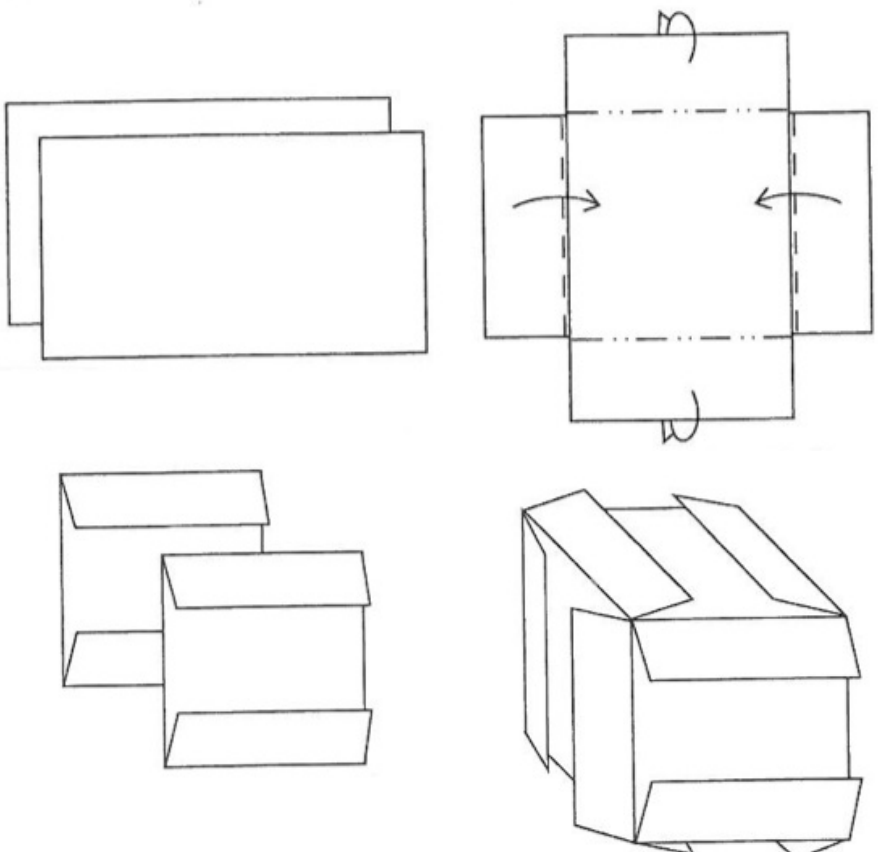


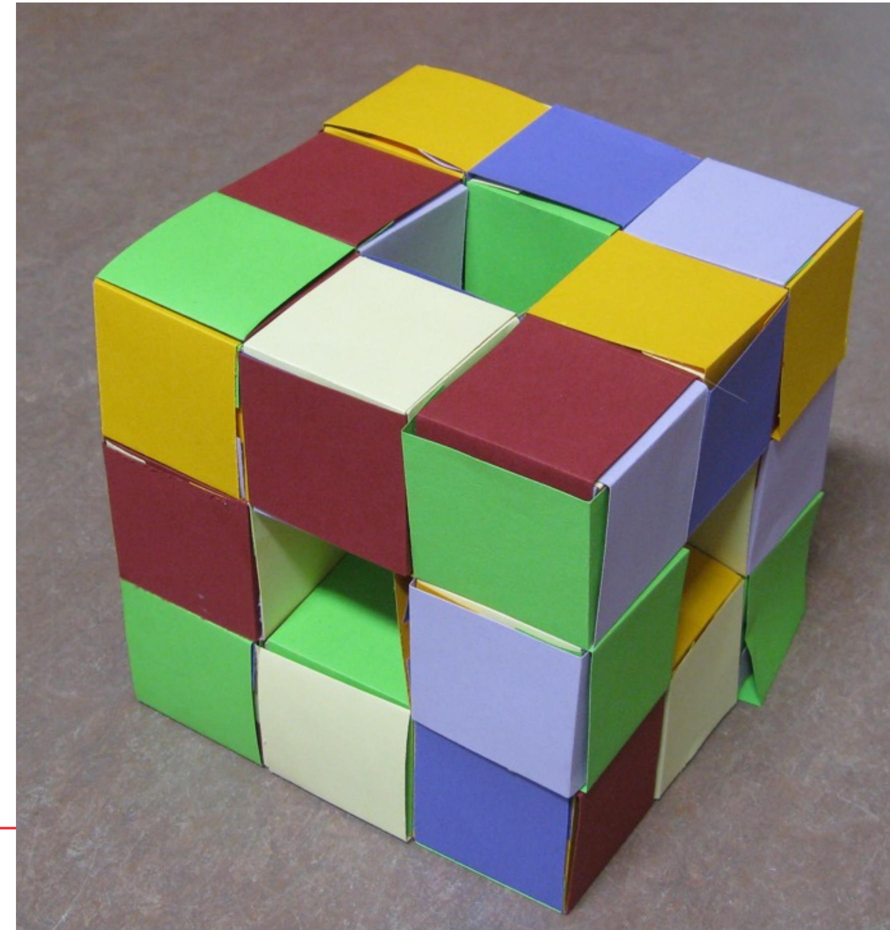


Karl Menger 1902-1985 and his sponge 1926

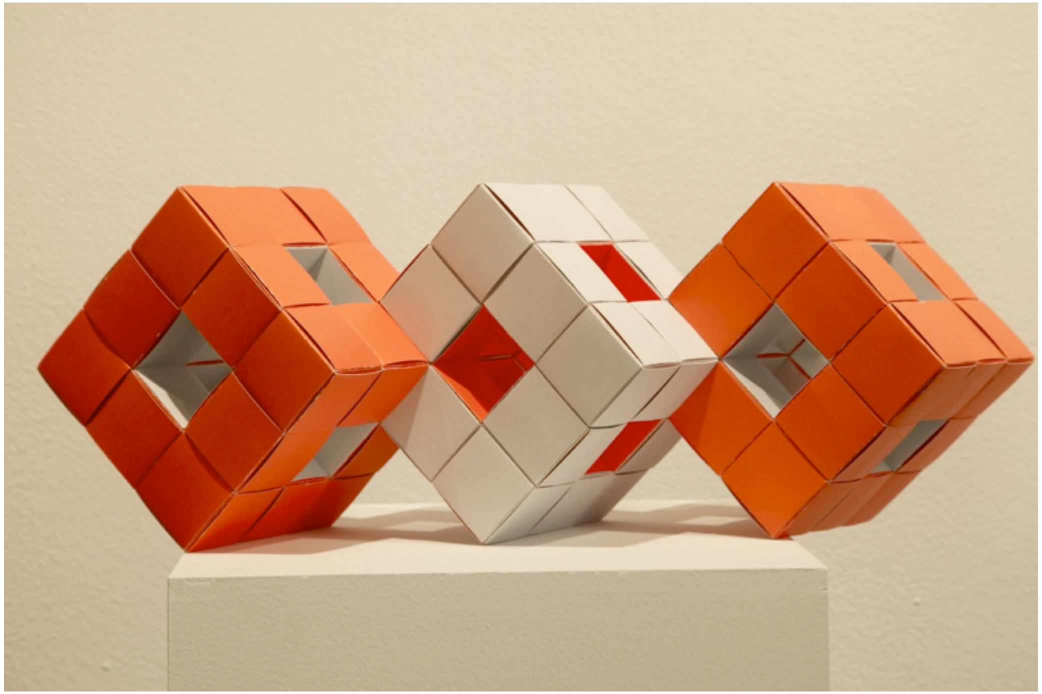


Menger sponge via business card origami





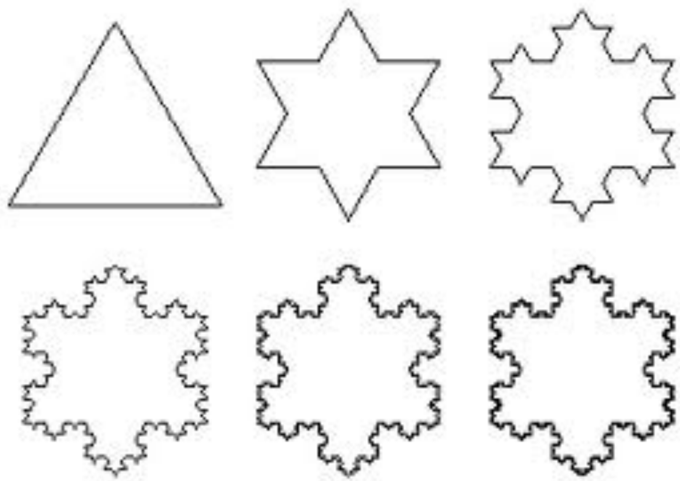
Aalto University



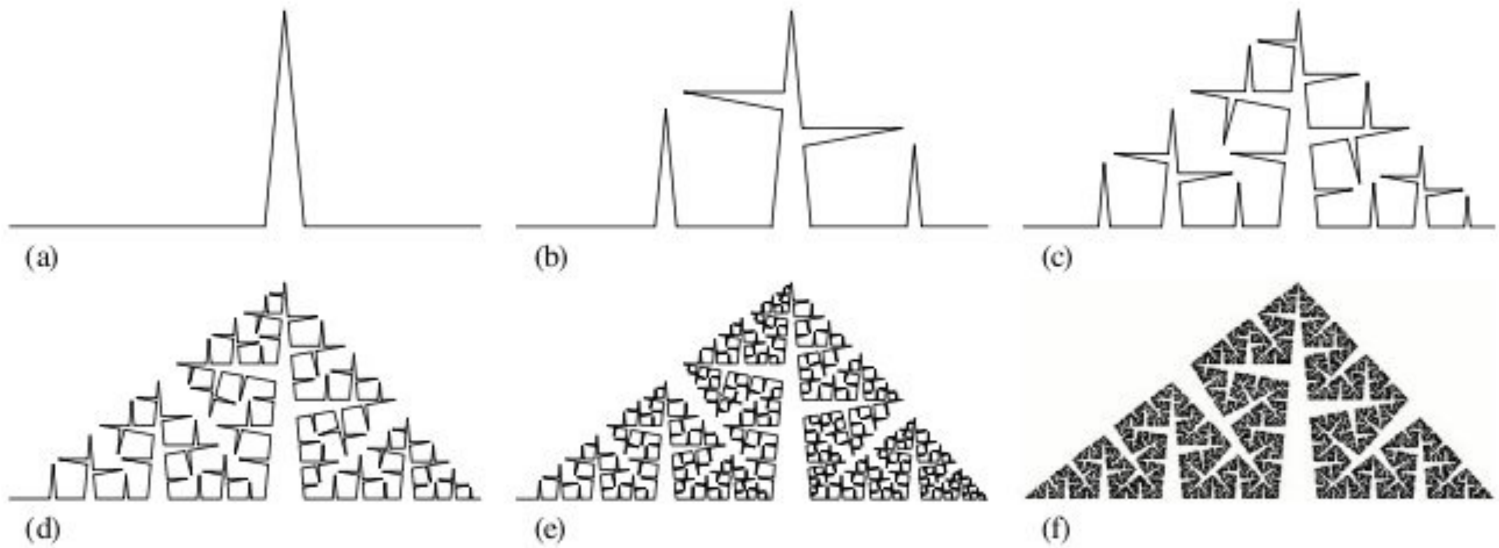
Three interlinked Level One Menger Sponges, by Margaret Wertheim.

Jeannine Mosely
66048 business cards

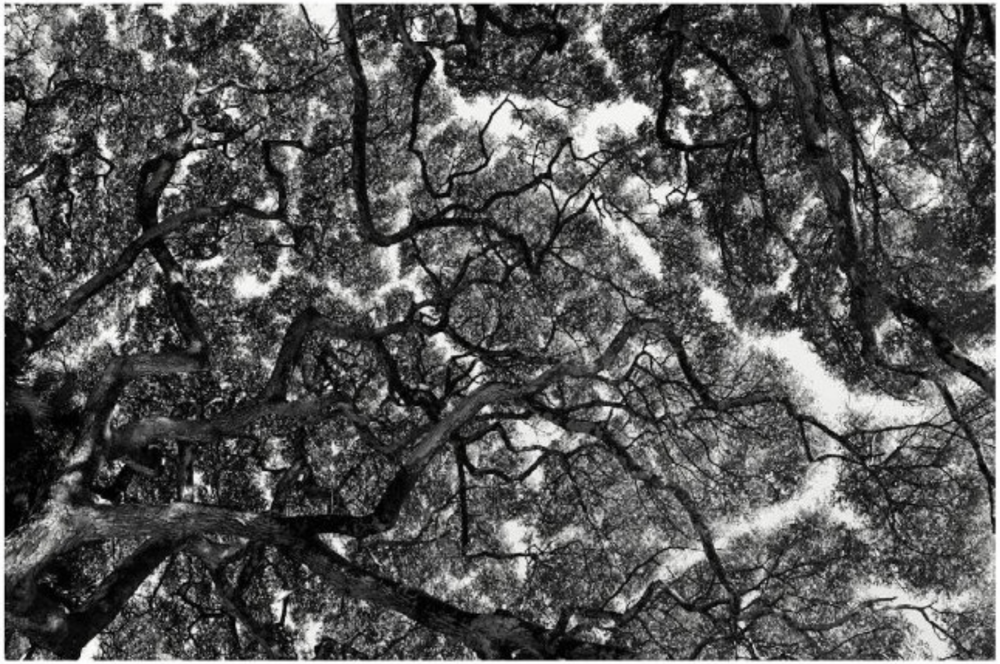
Niels Fabian Helge von Koch (1870-1924) and his snowflake (1904)



Evolution à la Mandelbrot



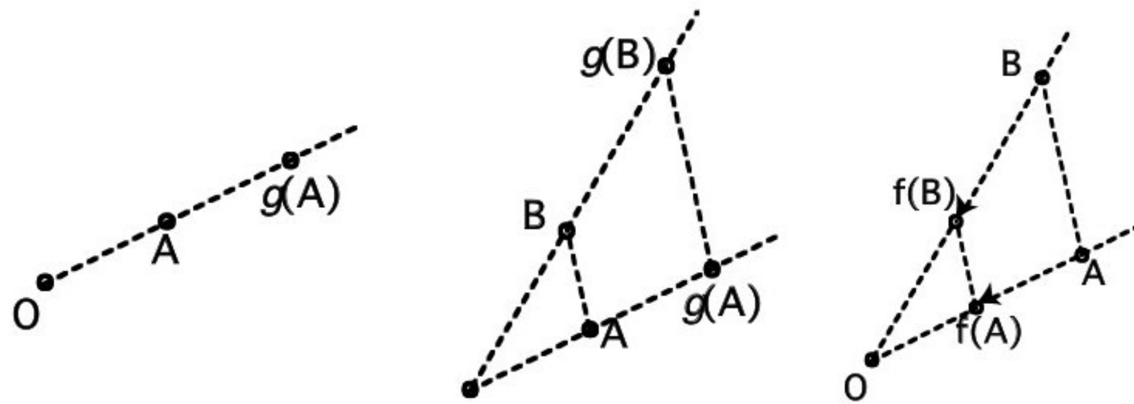
Canopy, by Craig Harris 2008



Similarity mapping

A plane transformation f is a *similarity* if there exists a positive number k such that for every point A and B, d(f(A),f(B))=kd(A,B). The number k Is a *stretching factor* of the similarity. Case k=1 gives a symmetry.

Similarities here are *central similarities* or *dilations*



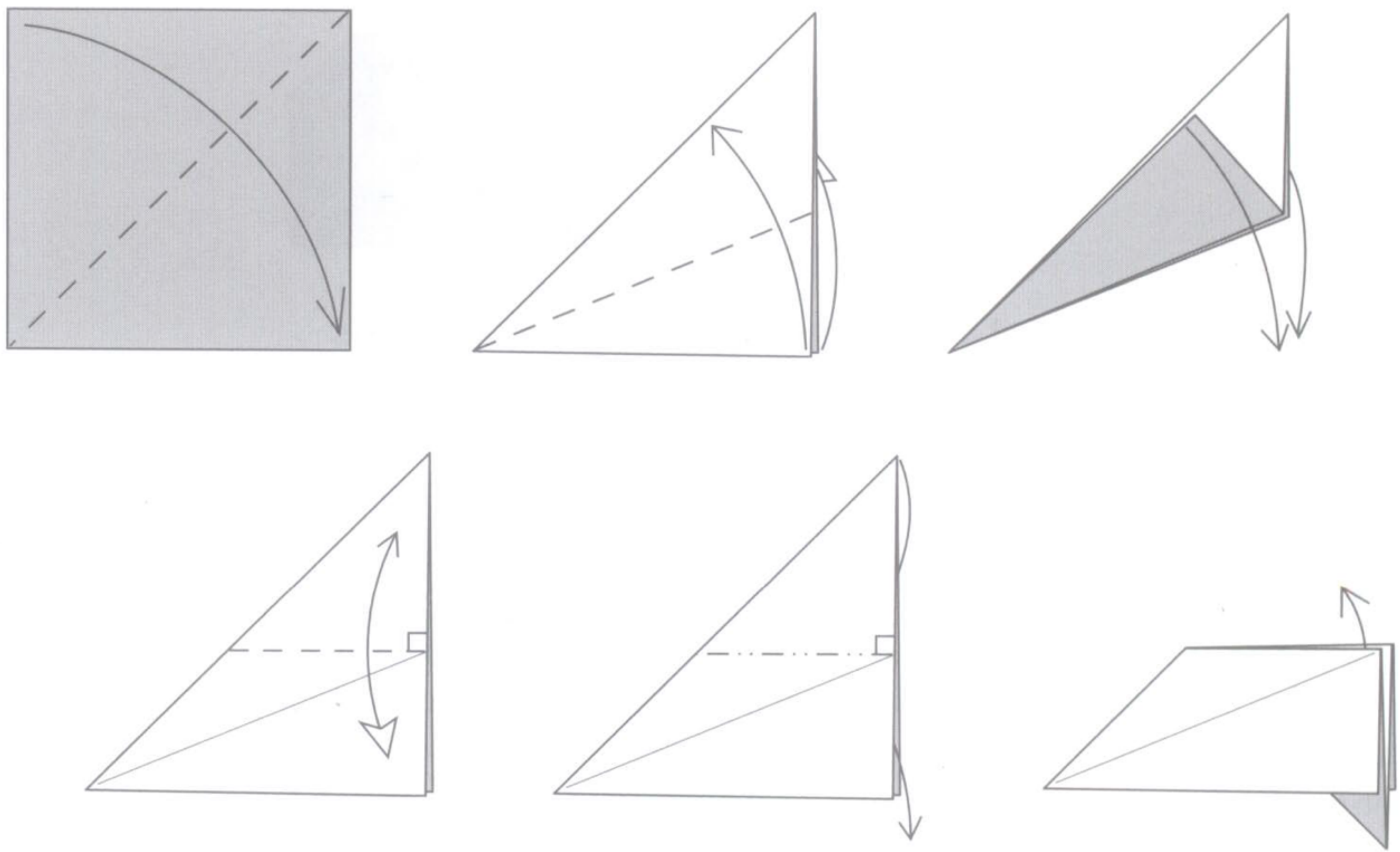
Classification of similarities in the plane

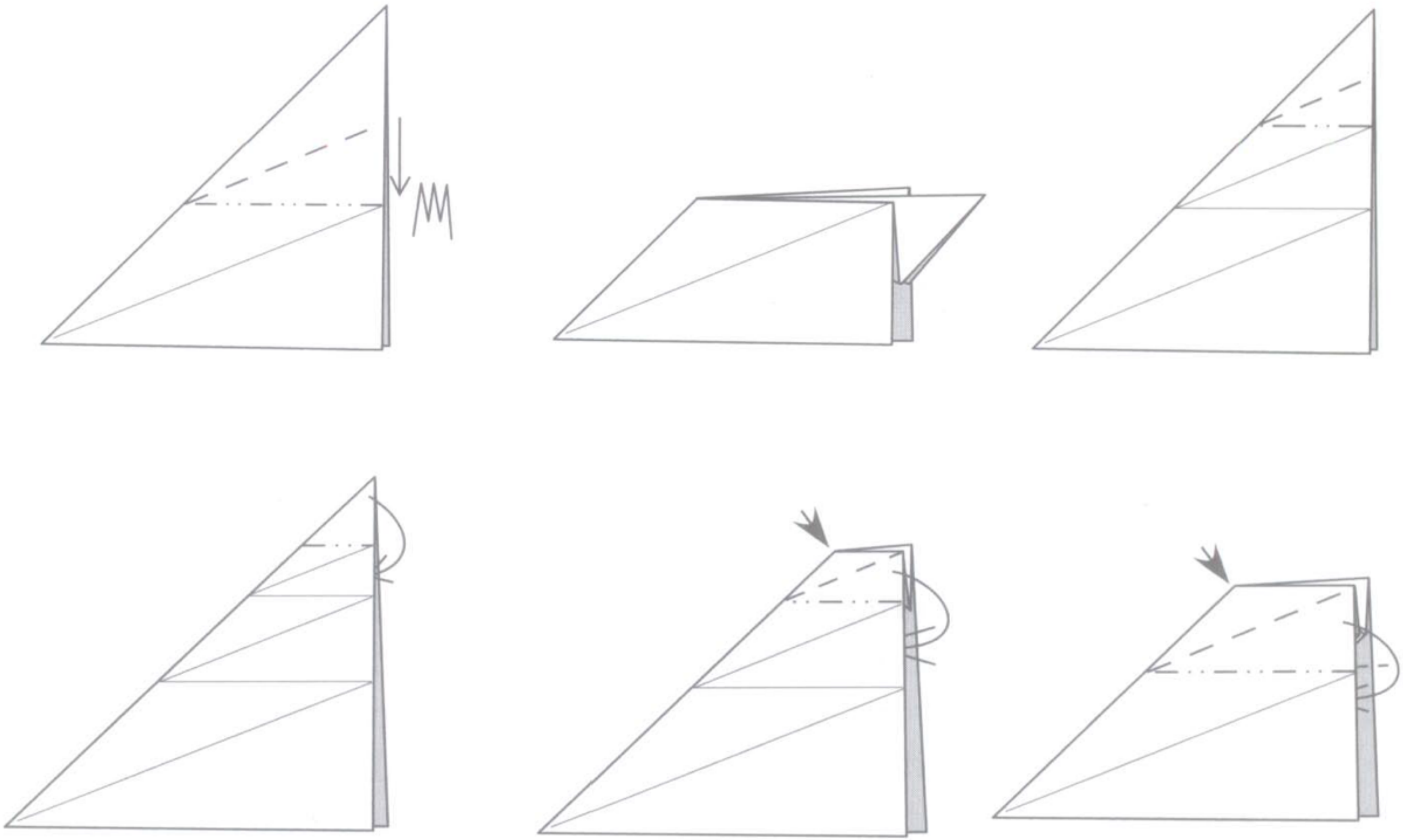
Spiral symmetry: rotation composed with a central similarity (w.r.t same point)

Dilative reflection: central similarity w.r.t. point O composed with a reflection w.r.t. a line going *through* O.

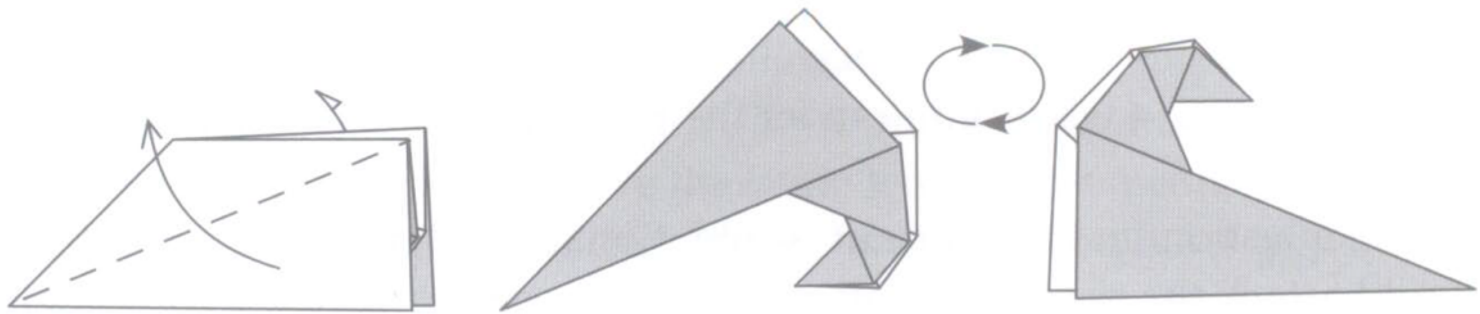
Can show: *Every similarity is a symmetry, a spiral similarity or a dilative reflection.*

The self similar wave by Tom Hull



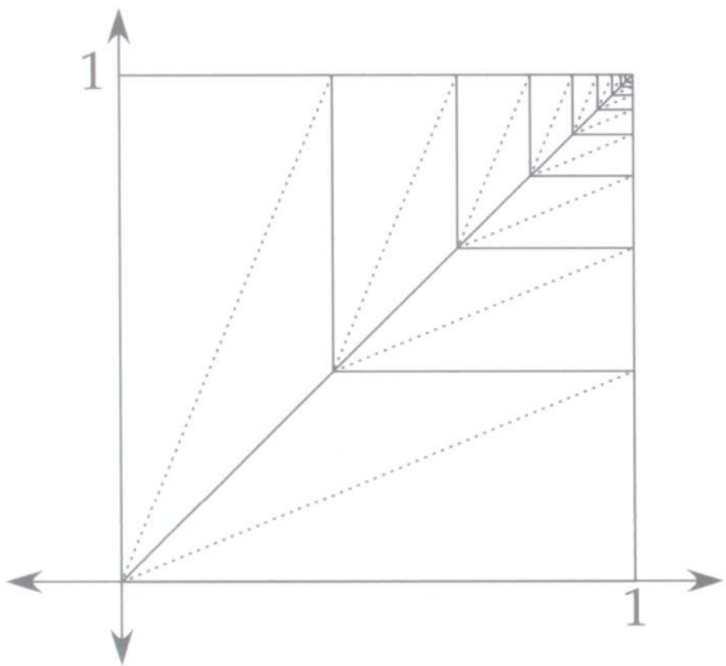


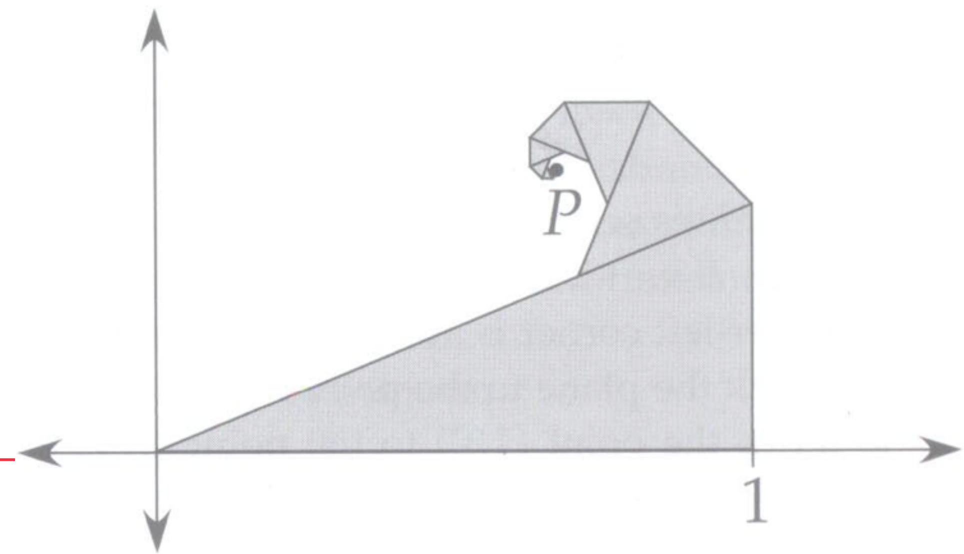
Is the outcome a fractal ?



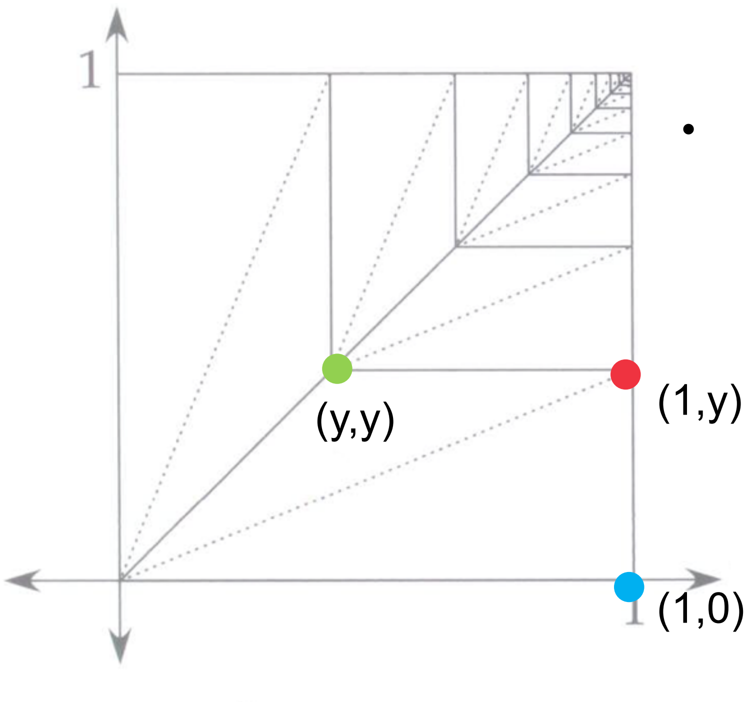
Question 1

Suppose we started with a square paper with side length 1 and folded the wave with an infinite number of levels, what would the coordinates of the limit point P of the spiral be ?

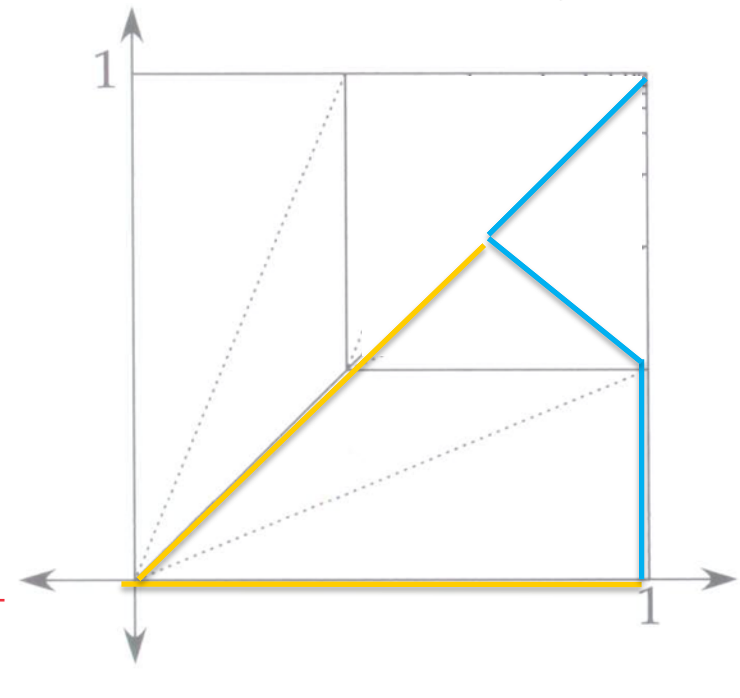




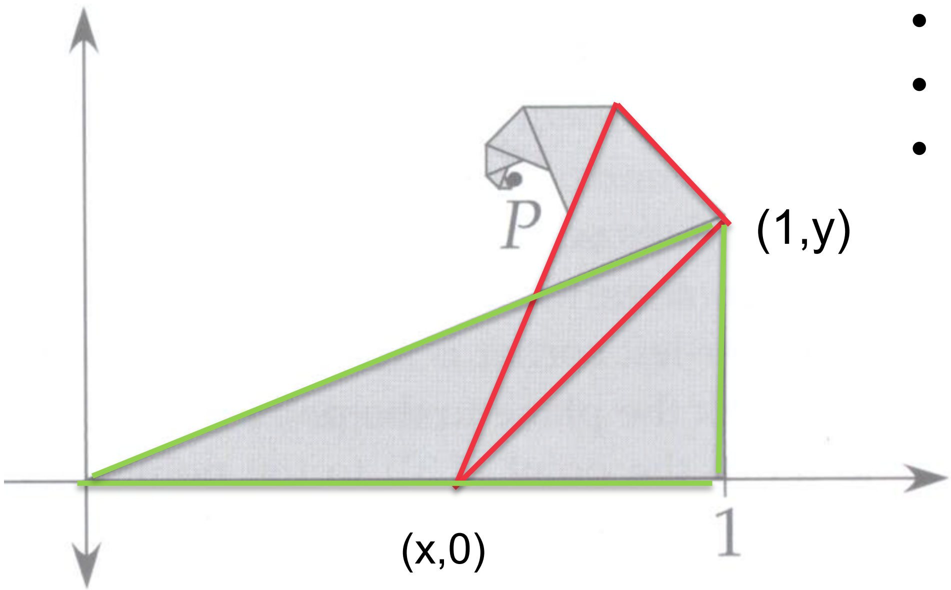
Geometric transformation solution



- Find affine mapping (scaling+ translation) taking (1,0) to (1,y) (and hence unit square to a square of side length 1-y
- y = √2 -1 from the picture=> scaling factor =1-y =2- √2



Self similarity in the folded wave pattern

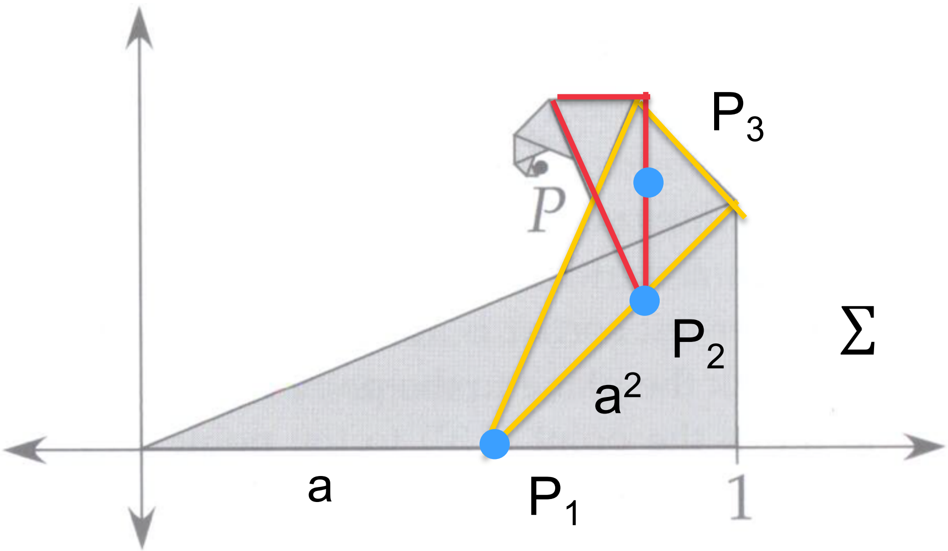


- (1,0) mapped (1,y) again
- (0,0) mapped to (x,0) x=1-y =2- √2
- P will become a fixed point of the mapping

⇒ Rotation of 45 ° counter clockwise wrt to the origin + scaling by factor 2- √2 + translation from origin to point (x,0)

⇒ P=(2/3, √2 /3)

Solution through complex (= nature's) numbers



$P_0 = 0,\ P_1 = a = 2-\sqrt{2}$

$P_2 = P_1 + a^2 e^{i\theta},\ \theta = \frac{\pi}{4}$

$P_3 = P_2 + a^3 e^{i2\theta}$

...

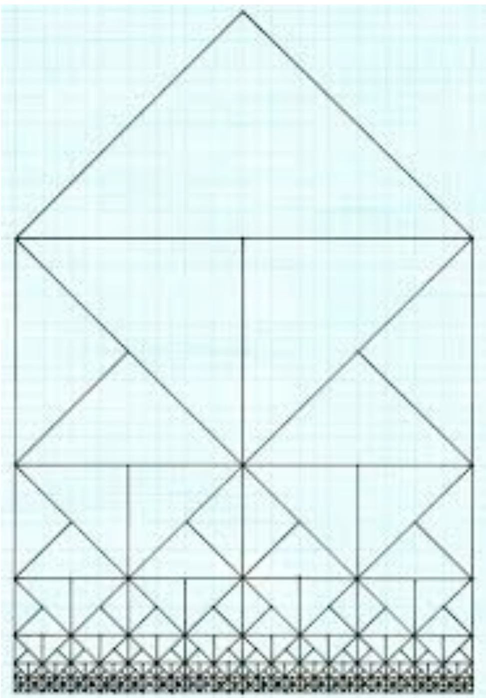
$P_n = P_{n-1} + a^n e^{i(n-1)\theta}$

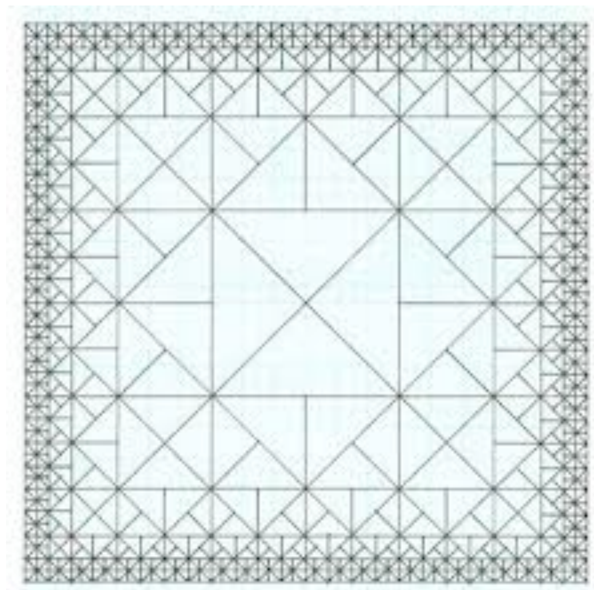
$\Rightarrow P = a\sum(ae^{i\theta})^n = 2/3 + \sqrt{2}/3\ i$

The spiral is logarithmic: $|P-P_n| = \sqrt{6}/3\ (2-\sqrt{2})^{n+1}$

Some other self similar origami patterns

Sometimes called ‘fractal origami’ but are they ***fractals*** really ?

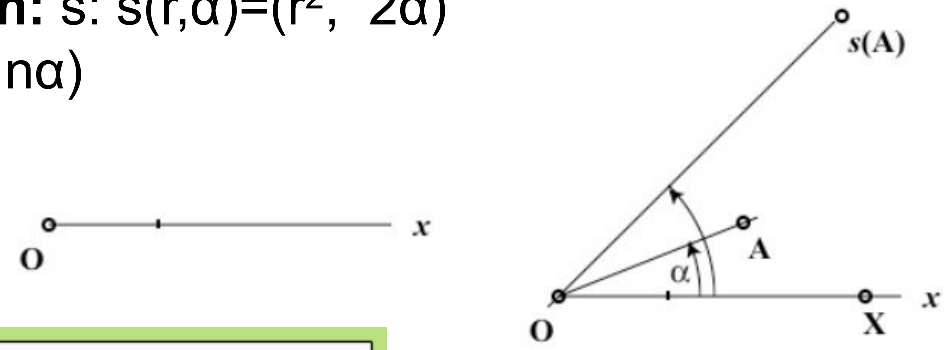


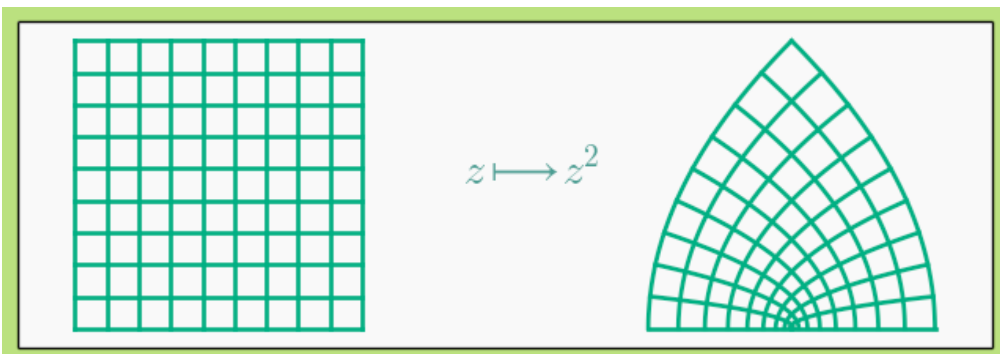


Gaston Maurice Julia 1893-1978

Iteration of planar rational functions

Squaring transformation: s: $s(r,\alpha)=(r^2, 2\alpha)$
Power n: pn: $pn(r,\alpha)=(r^n, n\alpha)$

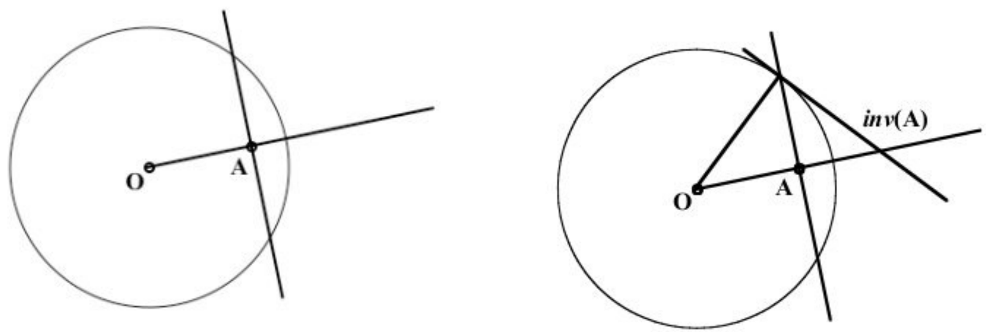




Preserves angles outside the origin !

… and (geometric) inversion in a circle

Planar rational maps are compositions of similarities, powers and inversions.



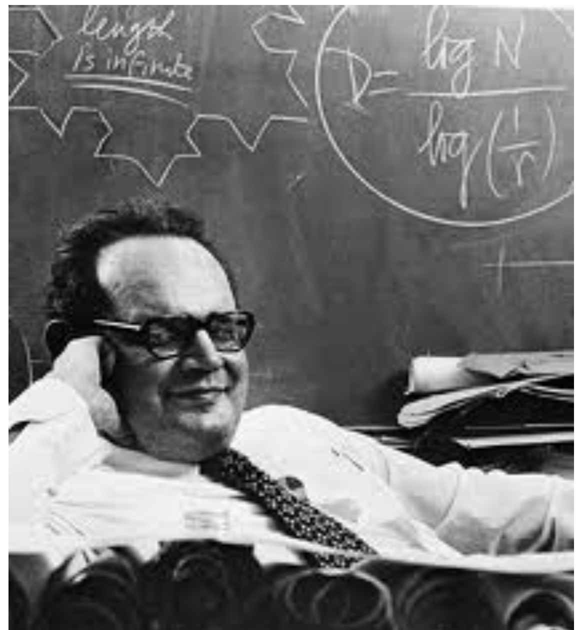
Pierre Joseph Louis Fatou 1878-1929

- ‘Fatou set’
- Holomorphic dynamics

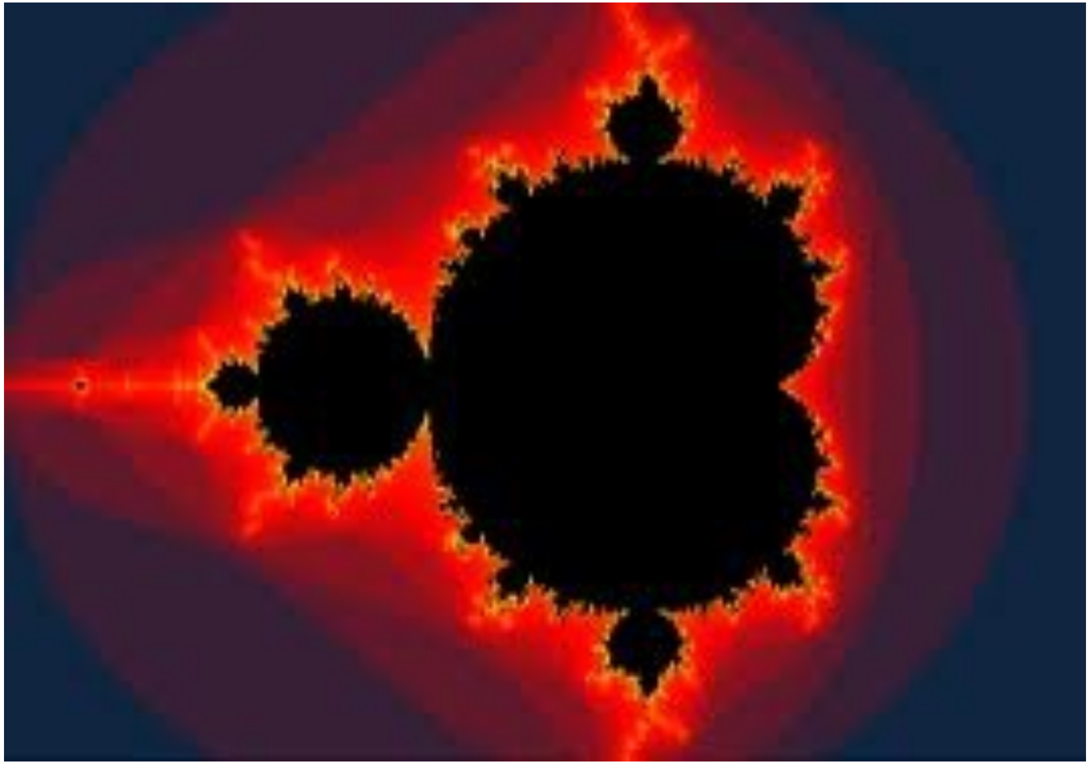
Benoit Mandelbrot 1924-2010

Mandelbrot coined (70's) the word 'fractal' to explain self similar objects

Fractus= fractured, broken



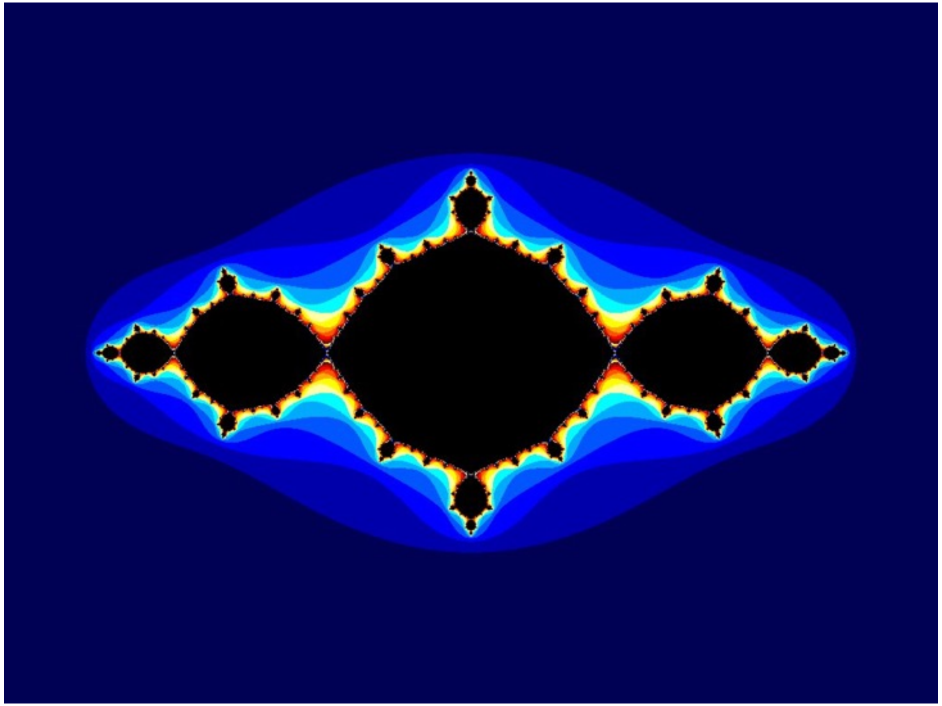
Mandelbrot set



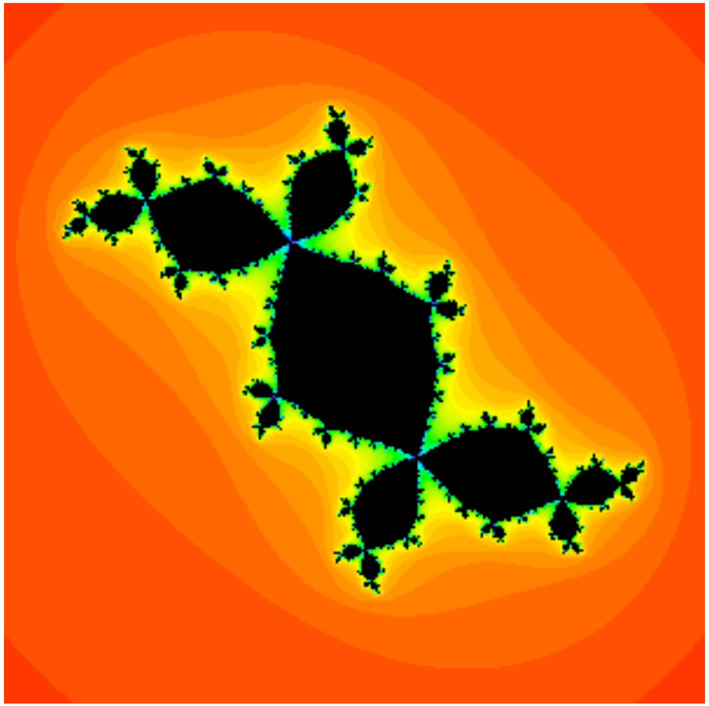
Parameter space for $C=(C_x,C_y)$ under f: $f(r,\alpha)=(r^2,2\alpha)+C$

Look at C=0 once more!

C=-1, Julia/Fatou set

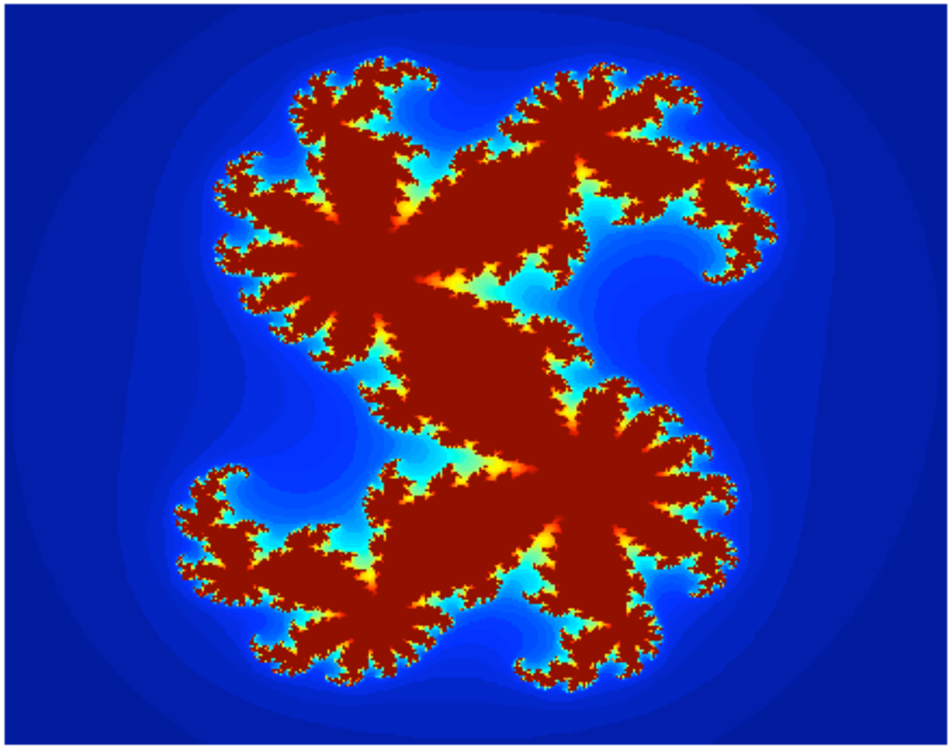


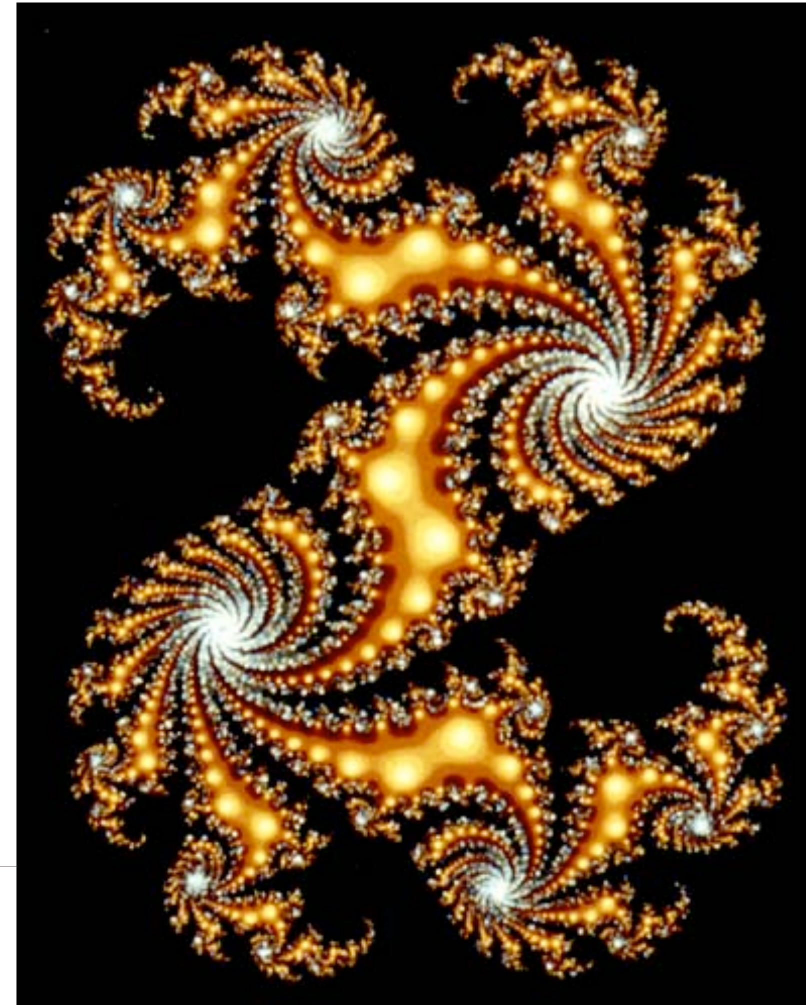
Douady’s rabbit (Adrien Douady 1935-2006)



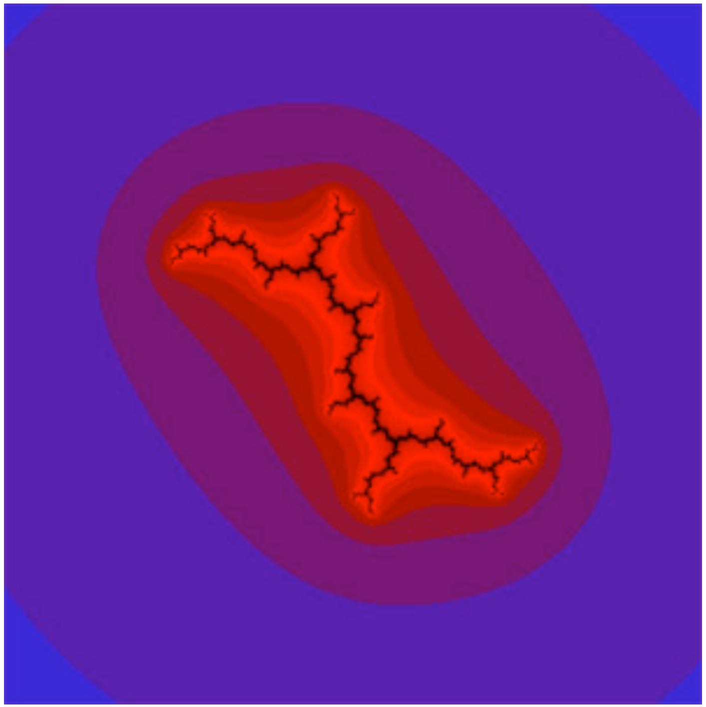
C = -0.12256+0.74486*i*.

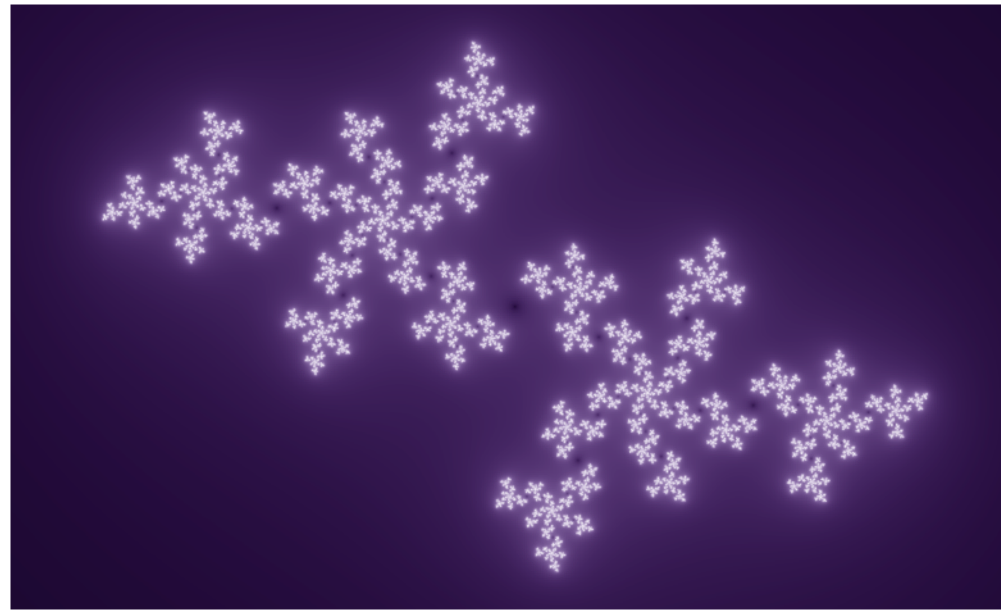
Dragon c=0.360284+0.100376i



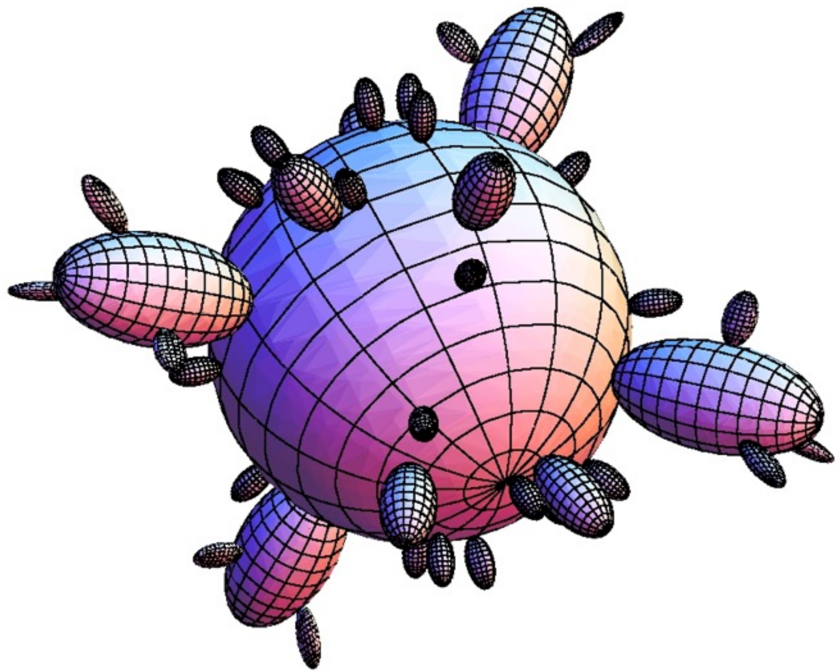


Dendrite and Cantor dust

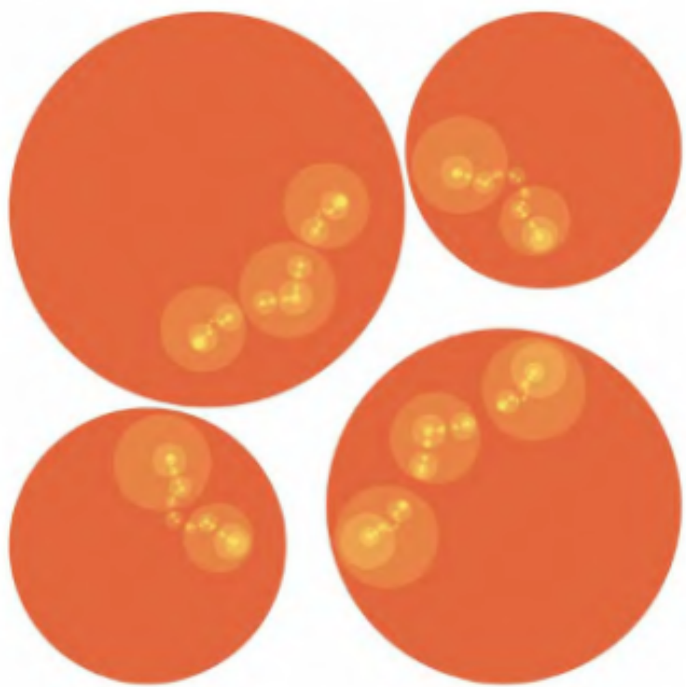




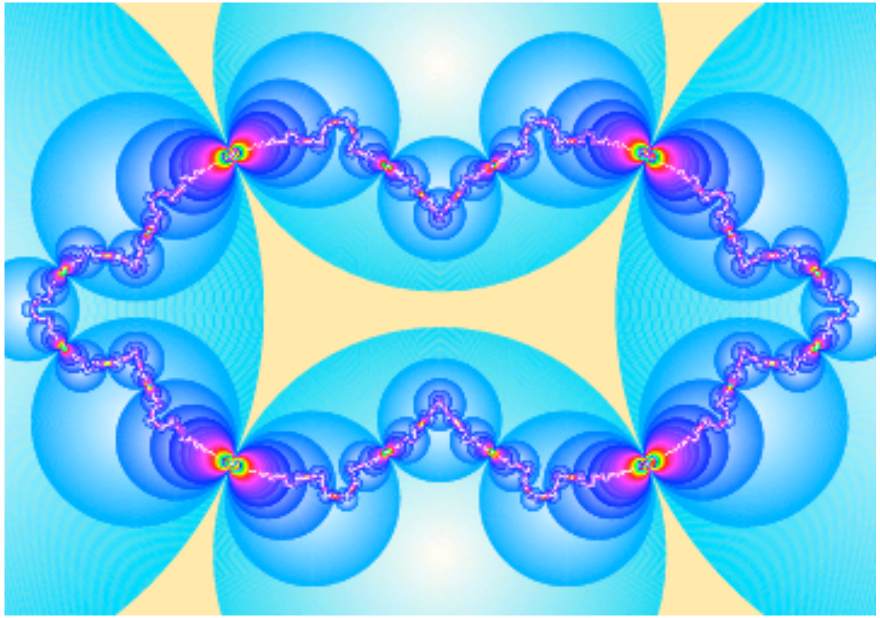
Higher dimensional analogues of complex polynomials (joint work in progress with G. Martin)



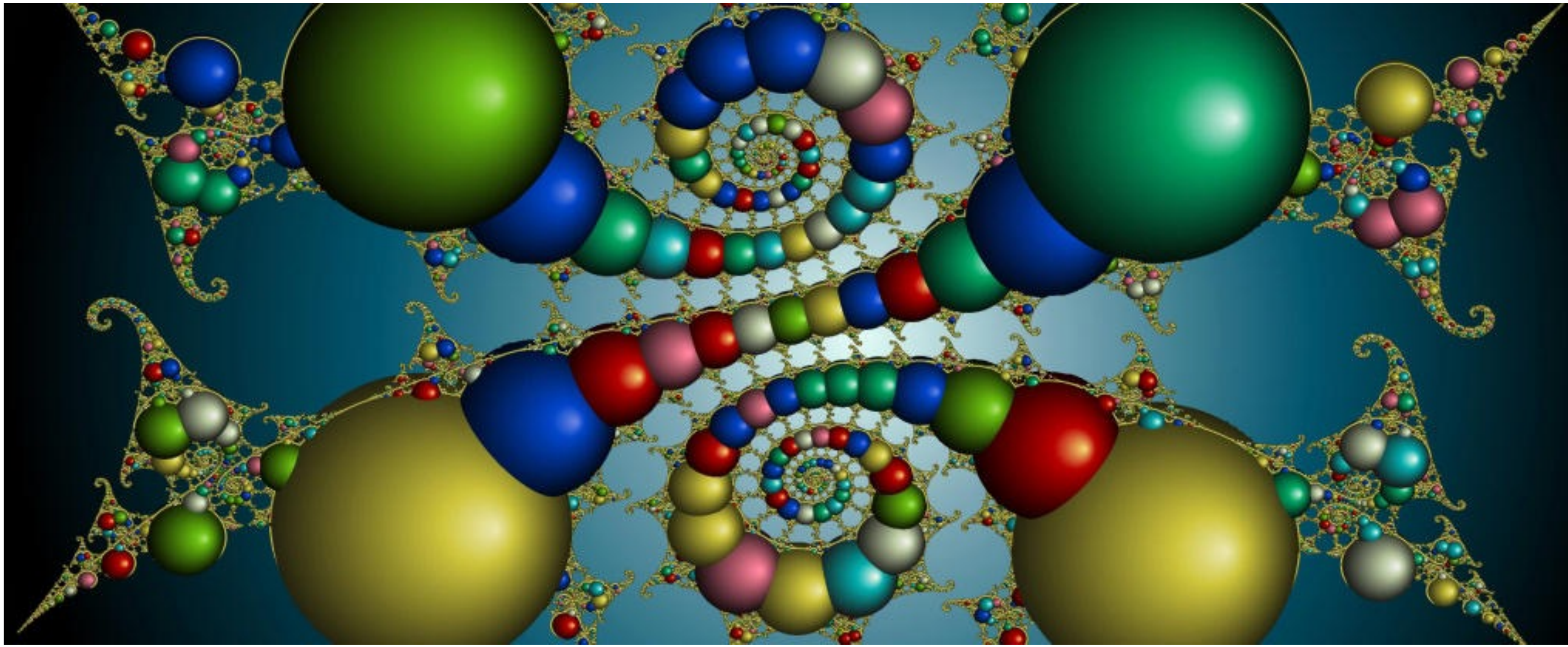
Kleinian groups



Ex: pairing of circles under Möbius transformations

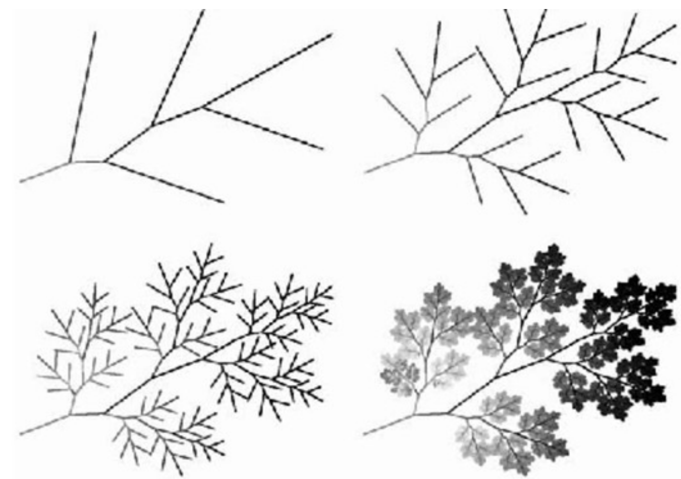


An artistic interpretation by Jos Leys

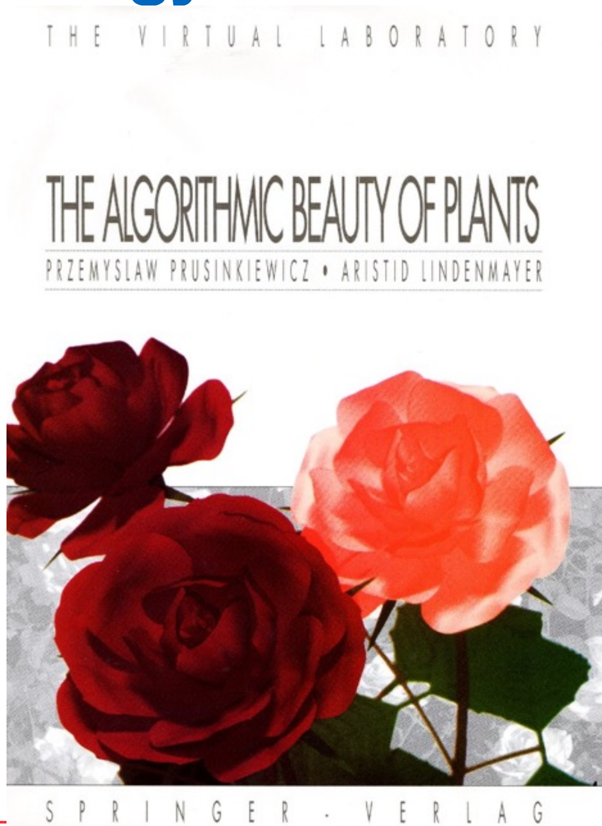


Fractals in approximating natural forms

Change from mechanical/geometrical to organic by using mathematical algorithm



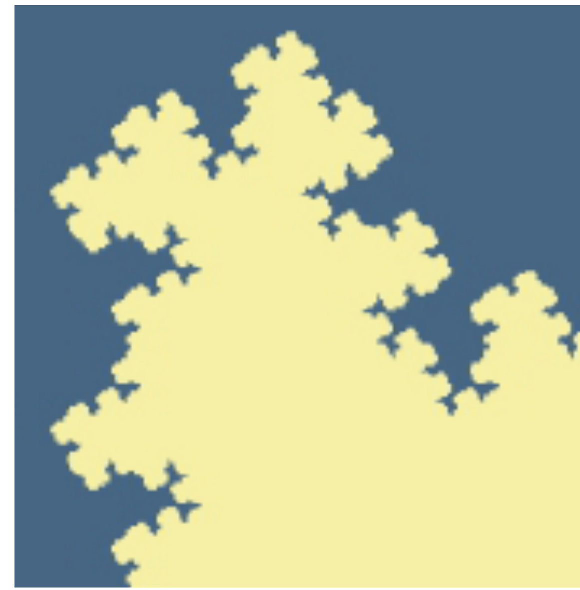
Aristid Lindenmayer 1925-1989 (L-systems) in plant biology



Artistic inventions of fractals a bit earlier and its reproduction by a process called Iterated Function System IFS.

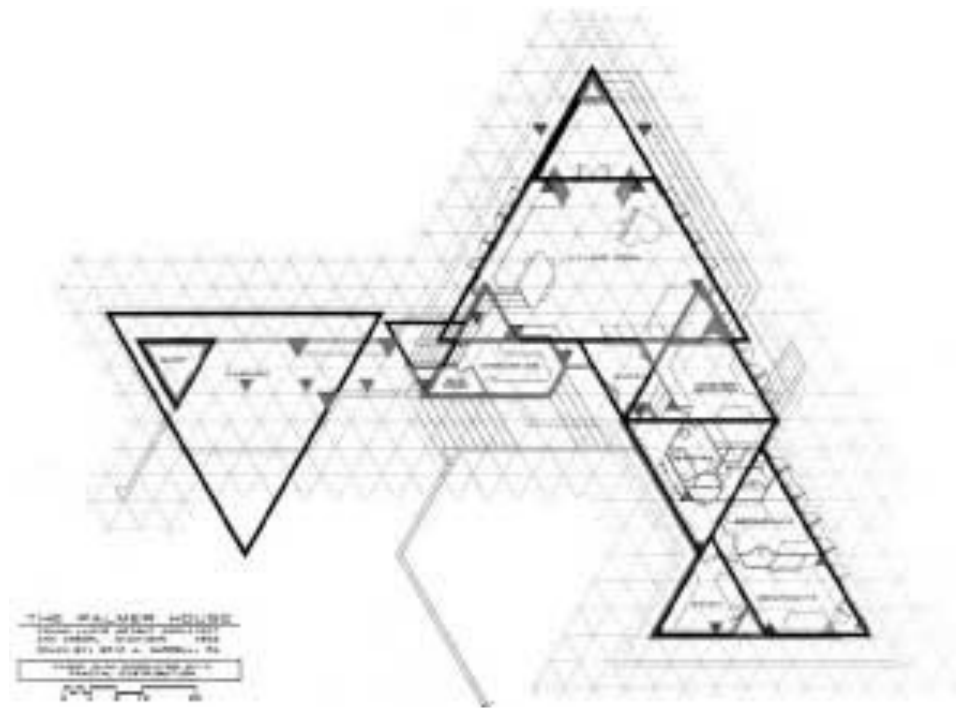
‘Driving Rain’ by Ando Hiroshige (1797-1858)

‘A Thousand Pictures of the Sea’ by Katsushika Hokusai (1817-1859) and IFS again

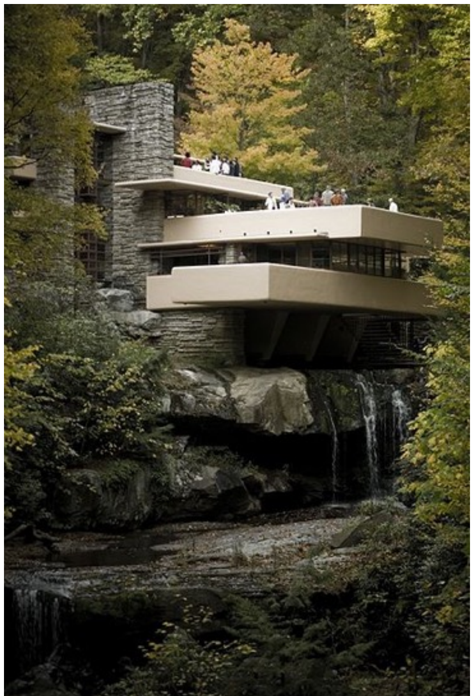


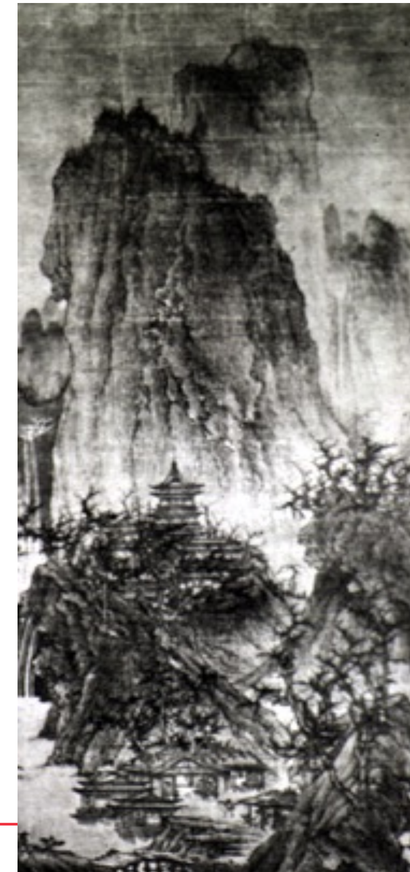
Frank Lloyd Wright (1867-1959)

Palmer house in Michigan (1950-51)



Fallingwater, Pennsylvania (1937) and Li Cheng (960-1127): Solitary Temple





African fractals: Ron Eglash

http//www.ted.com/talks/ron_eglash_on_african_fractals.html