First Intermediate Exam

- First intermediate exam on Thursday 22.10.2020, **14:00**-16:30, Distant Open Book exam. 3 problems, max 5+5+5=15 points. Instructions will come in course pages. Shortly: the exam will be given as an Assignment open between 14:00-16:30. Submission like in a homework Assignment as a pdf file. All material is available, but you are not allowed to be in contact with any person by any means during the exam.

- You do not have to register to the exam.

- Note that on the next week (evaluation period) there are no lectures and no exercises of the course. Only the exam.

Fundamental Limitations in Control Design

Is there a limit to how good compensator it is **possible** to design for a given process?

Signal scaling:

Example. Room temperature dynamics

$$\dot{z}^f = K_1(x_1^f - z^f) - K_2(z^f - w^f)$$

$$\dot{x}_1^f = K_3(u^f - x_1^f) - K_4(x_1^f - z^f)$$

z is the room temperature
x_1 is the temperature of the heating radiator
w is the outdoor temperature (disturbance)
u is the temperature of the heating water (control)

The superscript f indicates that the variable is in physical (unscaled) units.

Constants $K_1 = K_2 = K_4 = 0.7, \quad K_3 = 35$

Possible stationary point

$$x_1^f = 50^\circ C, \quad z^f = 20^\circ C, \quad w^f = -10^\circ C, \quad u^f = 50.6^\circ C$$

Purpose of control: keep room temp within $\pm 1^\circ C$ when the outdoor temp varies as $\pm 10^\circ C$;control range $\pm 20^\circ C$

In what follows the variables denote variations from the steady state.

$$z^f = \frac{0.5}{(0.03s+1)(0.7s+1)} u^f + \frac{0.01s+0.5}{(0.03s+1)(0.7s+1)} w^f$$

The time constants of the radiator and room are 0.03 and 0.7 (hours).

Because the outdoor temperature cannot change arbitrarily fast, let us model it as

$$w^f = \frac{1}{s+1} d^f$$

where d^f is within the range $\pm 10^\circ C$

Use the scalings $u = u^f / 20, \quad z = z^f, \quad d = d^f / 10$

to obtain $z = \frac{10}{(0.03s+1)(0.7s+1)} u + \frac{0.1s+5}{(0.03s+1)(0.7s+1)(s+1)} d$

Formalize the procedure:

Physical system

$$z^f(t) = G^f(p)u^f(t) + G_d^f(p)d^f(t)$$
$$y^f(t) = z^f(t) + n^f(t)$$
$$e^f(t) = r^f(t) - z^f(t)$$

scaling matrices $D_u u = u^f$, $D_d d = d^f$

$$Dz = z^f, \quad Dy = y^f, \quad Dn = n^f, \quad Dr = r^f, \quad De = e^f$$

”D”:s are diagonal matrices, with which different components of the variables are changed into the same scale.

Scaled system variables

$$z(t) = G(p)u(t) + G_d(p)d(t)$$
$$y(t) = z(t) + n(t)$$
$$e(t) = r(t) - z(t)$$

where $G = D^{-1}G^f D_u, \quad G_d = D^{-1}G_d^f D_d$

After proper scaling the transfer functions
G and G_d are fully comparable as functions of frequency.

Earlier that would have been impossible, because the functions are related to different physical variables.

Performance limitations:

- unstable systems
- systems with delay
- non-minimum phase systems
- limitations in control signal range
- system inverse

Meaning of the system inverse

Let the system be

$$z(t) = G(p)u(t) + w(t)$$
$$y(t) = z(t) + n(t)$$

(for simplicity, assume $n = 0$)

controller $\quad u = F_r r - F_y y$

It follows that

$$u = \frac{F_r}{1+F_y G} r - \frac{F_y}{1+F_y G} w = G^{-1}\left[G_c r - (1-S)w\right]$$

Perfect control, if $G_c = 1$ and $S = 0$

in which case

$$\boxed{u = G^{-1}(r - w)}$$

Note. If *w* were measurable, this result could have been obtained directly from the system model.

$$u = G^{-1}(r - w)$$

Generally:

- perfect control means using the process inverse
- in practice, control methods are based on the search for the (approximative) inverse model
- this explains,why systems with delay and nonminimum phase systems are difficult to control

Ex. Consider the system

$$y = Gu + G_d d$$

in which the variables have been scaled such that

$$|d(t)| \le 1, \quad |u(t)| \le 1$$

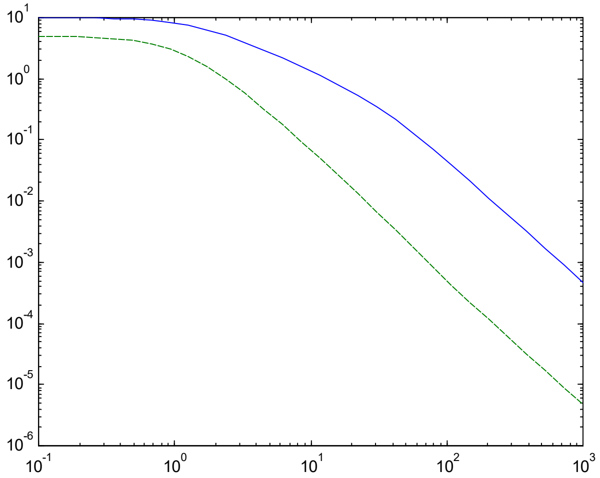
Perfect control $\quad u = -G^{-1} G_d d$

A necessary (but not sufficient) condition for the existence of a control that compensates all allowed disturbances is

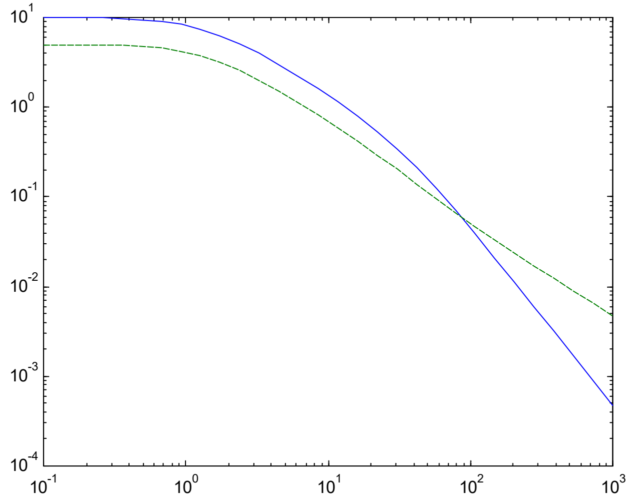
$$|G(i\omega)| \ge |G_d(i\omega)|, \quad \forall \omega$$

Let us return to the room temperature control example

$$z = \frac{10}{(0.03s+1)(0.7s+1)}u + \frac{0.1s+5}{(0.03s+1)(0.7s+1)(s+1)}d$$



s+1 with compensation Ok for all frequencies



s+1 not with compensation not perfect in high frequencies

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Loop gain:

$$S + T = 1$$ (consider the SISO-case)

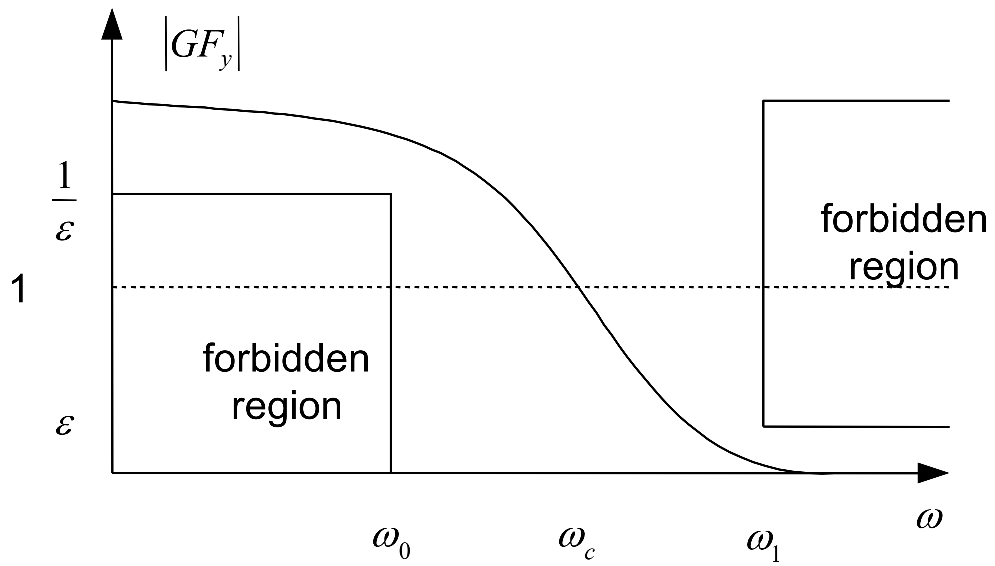
-keep S small in low frequencies
-keep T small in high frequencies

But the loop gain GF_y determines uniquely these functions

$$|S| < \varepsilon \Leftrightarrow |GF_y| > \frac{1}{\varepsilon}$$

$$|T| < \varepsilon \Leftrightarrow |GF_y| < \varepsilon$$

(approximative)



The change should be fast (as S must grow, let it happen fast in a small frequency range, in order to force T to be small).

But the loop gain and phase are interconnected!

Bode equations

For a minimum phase system it holds

$$\arg G(i\omega) \approx \frac{\pi}{2} \frac{d}{d\log\omega} \cdot \log\left|G(i\omega)\right|$$

Stability requirement: if at the gain crossover frequency the gain decreases

$20 \cdot \alpha \ dB$ (per decade), the phase is (about) $-\alpha \cdot \frac{\pi}{2}$

In order to have a positive phase margin, the gain must not drop faster than 40 dB /decade

But this is against the above requirements!

Assume that the loop gain $|L| = |GF_y|$ decreases as fast as $|s|^{-2}$ as $|s|$ tends to infinity. Then the Bode integral holds :

$$\int_0^{\infty} \log |S(i\omega)| \, d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}\,(p_i)$$

where p_i :s are the RHP poles of the loop gain $G(s)F_y(s)$

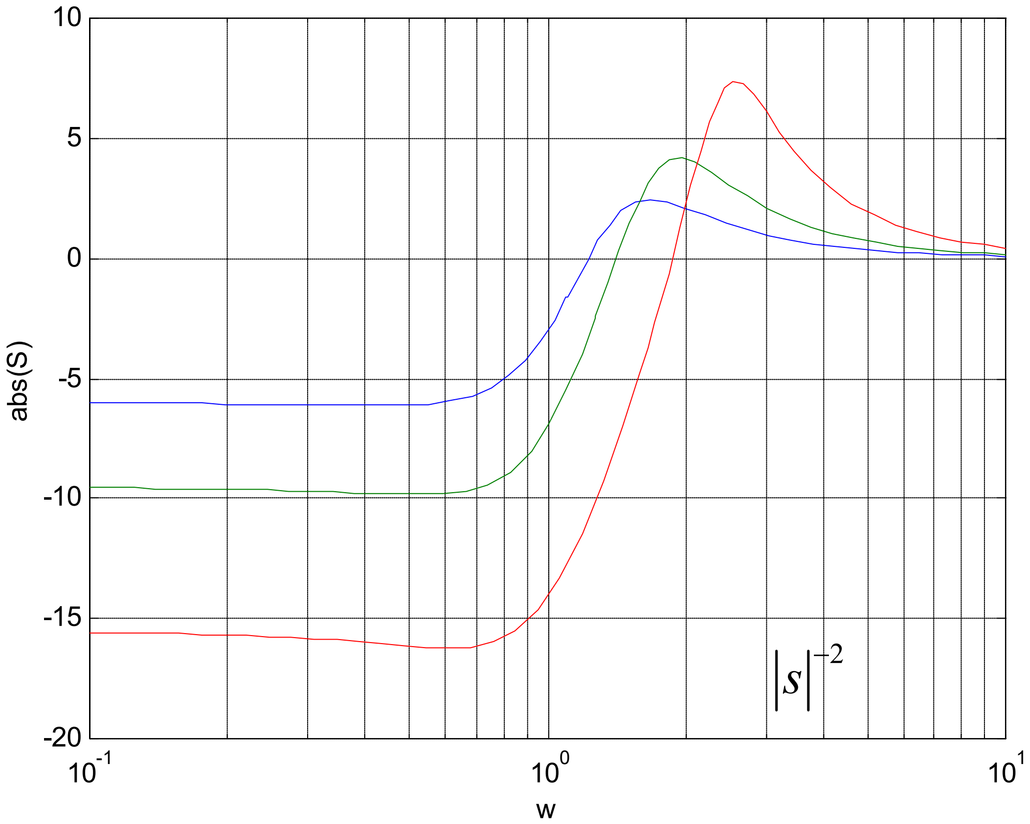
(here log means the natural logarithm (ln)).

If there are no RHP-poles, it follows

$$\int_0^{\infty} \log |S(i\omega)| \, d\omega = 0$$

These are again fundamental limitations.

Sometimes the phrase ”waterbed formula” is used in the literature



$$G(s) = \frac{1}{s^2 + s + 1}, \quad F_y = K$$

Note that the condition: gain $|L| = |GF_y|$ must decrease as fast as $|s|^{-2}$ as $|s|$ tends to infinity holds for practically all physical systems. (Both elements in L are at least 1st order systems).

Concequences:

1. Let the process have an unstable *pole* $p_1 > 0$

For the bandwidth the (approximative) bound

$$\omega_c > p_1$$

can be set, in order to be able to control the unstable mode.

2. Let the process have *delay* T_d

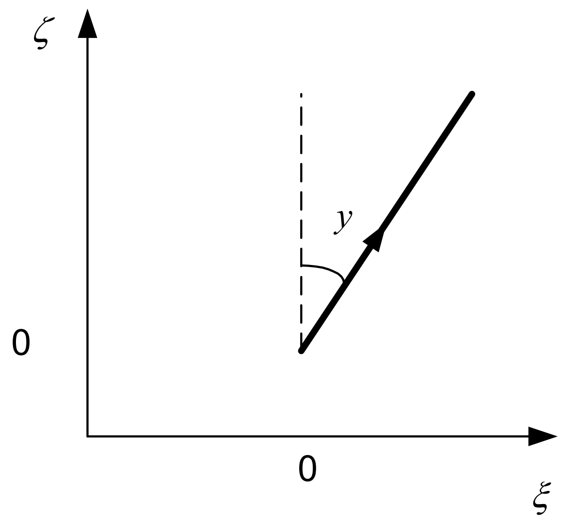
For the gain crossover frequency ω_c

$$\omega_c < (\text{appr})\frac{1}{T_d}$$

3. Let the process have a nonminimum phase *zero, z*

$$\omega_c < \frac{z}{2}$$

Ex. 1. Control of the inverted pendulum



control

$$u = \ddot{\xi}$$

$$\xi + \frac{l}{2}\sin y$$

$$\frac{l}{2}\cos y$$

Dynamic equations

$$F\cos y - mg = m\frac{d^2}{dt^2}\left(\frac{l}{2}\cos y\right) = m\frac{l}{2}\left(-\ddot{y}\sin y - \dot{y}^2\cos y\right)$$

$$F\sin y = m\ddot{\xi} + m\frac{d^2}{dt^2}\left(\frac{l}{2}\sin y\right) = mu + m\frac{l}{2}\left(\ddot{y}\cos y - \dot{y}^2\sin y\right)$$

By eliminating F $\quad \frac{l}{2}\ddot{y} - g\sin y = -u\cos y$

and linearizing with respect to small deviations

$$G(s) = \frac{-2/l}{s^2 - \frac{2g}{l}}$$

The poles are $\pm\sqrt{\frac{2g}{l}}$ (unstable)

The bandwidth should exceed $\sqrt{2g/l}$

say, $2\pi\sqrt{2g/l}$

It is seen that a short pendulum is more difficult to control than a long one.

Ex. 2. Process with delay

$$G(s) = G_1(s)e^{-sT_d}$$

Again, for the control it can be written

$$u = G^{-1}\left[G_c r - (1 - S)w\right]$$

Perfect control $G_c \to 1, \quad S \to 0$ is impossible, because

it would mean

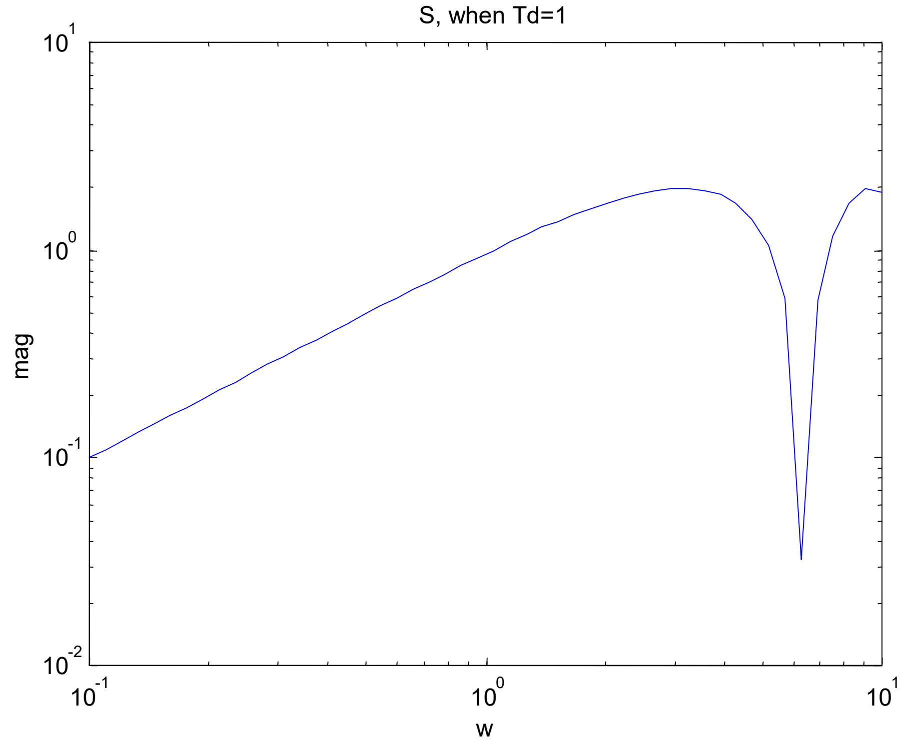
$$u = G_1^{-1}(s)e^{sT_d}(r - w)$$

which contains anticipation.

But choose ideally $G_c = e^{-sT_d}$; $T = e^{-sT_d}$ (= 1-S)
so that the delay term is cancelled from control equation.

An ”ideal” sensitivity function is then

$$S = 1 - e^{-sT_d}$$



For small frequencies

$$S(i\omega) \approx i\omega T_d$$

Then the amplitude of the (ideal) sensitivity function is smaller than one in frequencies

$$\omega < 1/T_d$$

This approximates the bandwidth, so that

$$\omega_B < \frac{1}{T_d}$$

Ex. Consider again the delay but now by means of the Padé-approximation

$$e^{-sT} \approx \frac{1 - sT/2}{1 + sT/2}$$ 1. degree Padé-approximation

But this transfer function has a non-minimum phase zero

$$z = 2/T$$

But by earlier results

$$\omega_c < \frac{z}{2} = \frac{1}{T}$$ same result!

Interpolation constraints

Let z be a RHP zero of the loop transfer function $L(z) = 0$.
Then (SISO case)

$$S(z) = \frac{1}{1+L(z)} = 1 \qquad \text{(Interpolation condition 1)}$$

In control: $\|W_S S\|_\infty \leq 1 \Leftrightarrow |S| \leq \frac{1}{|W_S|}, \quad \forall \omega$

$$\Rightarrow |W_S(z)| \leq 1$$

Let p_1 be a RHP pole of the loop transfer function *L*, $L(p_1) = \infty$

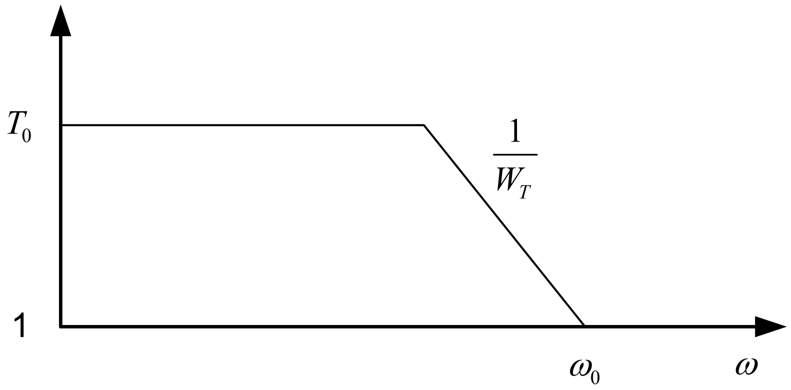
Then

$$T(p_1) = \frac{L(p_1)}{1 + L(p_1)} = \frac{1}{1 + \frac{1}{L(p_1)}} = 1 \qquad \text{(Interpolation condition 2)}$$

In control:

$$\|W_T T\|_\infty \le 1 \Leftrightarrow |T| \le \frac{1}{|W_T|}, \quad \forall \omega$$

$$\Rightarrow |W_T(p_1)| \le 1$$



We want T to lie below this curve.

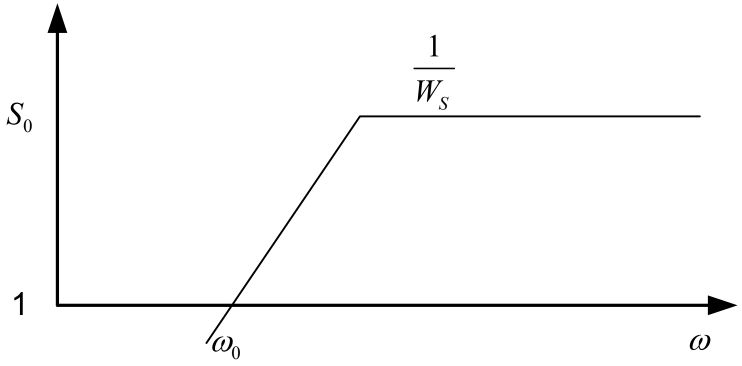
$$\|W_T T\|_\infty \le 1 \Leftrightarrow |T(i\omega)| \le \frac{1}{|W_T(i\omega)|}$$

Let $$W_T = \frac{s}{\omega_0} + \frac{1}{T_0} \Rightarrow \frac{1}{W_T} = \frac{1}{(\frac{1}{T_0} + \frac{s}{\omega_0})}$$

Use interpolation condition 2:

$$|W_T(p_1)| \le 1 \Rightarrow \frac{p_1}{\omega_0} + \frac{1}{T_0} \le 1 \Rightarrow \omega_0 \ge \frac{p_1}{1 - 1/T_0}$$

If we choose $T_0 = 2 \quad \Rightarrow \omega_0 \ge 2p_1$



We want S to lie below this curve.

$$\|W_S S\|_\infty \leq 1 \Leftrightarrow |S(i\omega)| \leq \frac{1}{|W_S(i\omega)|}$$

Let $W_S = \dfrac{s+\omega_0 S_0}{S_0 s} \Rightarrow \dfrac{1}{W_S} = \dfrac{S_0 s}{s+\omega_0 S_0}$

Use interpolation condition 1:

$$|W_S(z)| \leq 1 \Rightarrow \frac{z+\omega_0 S_0}{S_0 z} \leq 1 \Rightarrow \omega_0 \leq (1-\frac{1}{S_0})z$$

If we accept $S_0 = 2 \Rightarrow \omega_0 \leq \dfrac{z}{2}$

But $S_0 \rightarrow \infty \Rightarrow \omega_0 \leq z$