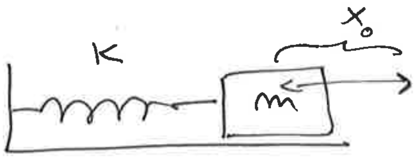
Harmoninen värähtelijä



$$\sum F = -kx$$

NII $\Rightarrow$ $ma = m \cdot \frac{d^2x(t)}{dt^2} = -kx(t)$

$$\Rightarrow \boxed{\frac{d^2x(t)}{dt^2} = -\frac{k}{m}x(t)}$$

2.kl homogeeninen dy

$\Rightarrow$ $x(t) = A\cos(\omega t) + B\sin(\omega t)$; $\omega = \sqrt{k/m}$.

Alkuehdot : $\begin{cases} x(0) = x_0 & \text{(alkupoikkeutus)} \\ \frac{d}{dt}x(t)\Big|_{t=0} = 0 & \text{(alkunopeus)} \end{cases}$

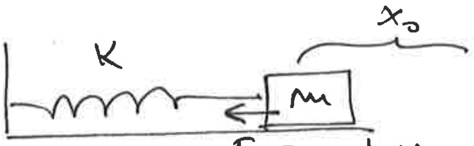
$\Rightarrow$ $x(0) = x_0 = A \cdot \cos(0) + B\sin(0) = A$ $\Rightarrow$ $\underline{A = x_0}$

$$\frac{d}{dt}x(t)\Big|_{t=0} = \Big[-A\omega\sin(\omega t) + B\omega\cos(\omega t)\Big]\Big|_{t=0}$$

$$= B\omega = 0 \quad \Rightarrow \quad \underline{B = 0}$$

$$\Rightarrow \quad \underline{x(t) = x_0\cos\left(\sqrt{\frac{k}{m}}\,t\right)}$$

Lisätään lineaarinen (nopeudessa) vastusvoima:



$F_\lambda = -bv.$

$$\sum F = -kx - b\cdot v = -kx(t) - b\frac{dx(t)}{dt}.$$

<u>NII</u>:

$$m\frac{d^2x(t)}{dt^2} = -kx(t) - b\frac{dx(t)}{dt}$$

$$\Rightarrow \quad \frac{d^2x(t)}{dt^2} + \frac{b}{m}\frac{dx(t)}{dt} + \frac{k}{m}x(t) = 0.$$ homogeeninen 2. kl dy

Vastaava karakteristinen yhtälö:

$$\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m}\lambda = 0.$$

$$\Rightarrow \quad \lambda_{1,2} = \frac{-\frac{b}{m} \pm \sqrt{\left(\frac{b}{m}\right)^2 - 4\frac{k}{m}}}{2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}.$$

$$\Rightarrow \quad x(t) = \begin{cases} Ae^{\lambda_1 t} + Be^{\lambda_2 t}. & ; \text{ jos } \lambda_1 \neq \lambda_2 \\ Ae^{\lambda t} + Bte^{\lambda t} & ; \text{ jos } \lambda_1 = \lambda_2 =: \lambda. \end{cases}$$ "Kaksoisjuuri"

<u>Ylivaimennettu värähtely</u>: $\left(\frac{b}{2m}\right)^2 > \frac{k}{m} = \omega_0^2$ | Vastaavan vaimenemattoman värähtelijän kulmataajuus $\omega_0 = \sqrt{\frac{k}{m}}$.

$\Rightarrow$ juuret reaaliset ja erisuuret. Olkoon $\sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} =: \omega.$

$$x(t) = Ae^{\left(-\frac{b}{2m}+\omega\right)t} + Be^{\left(-\frac{b}{2m}-\omega\right)t}.$$

Huom: $\frac{b}{2m} > 0, \ \omega > 0 \quad \Rightarrow \quad -\frac{b}{2m} + \omega < 0$

$-\frac{b}{2m} - \omega < 0$

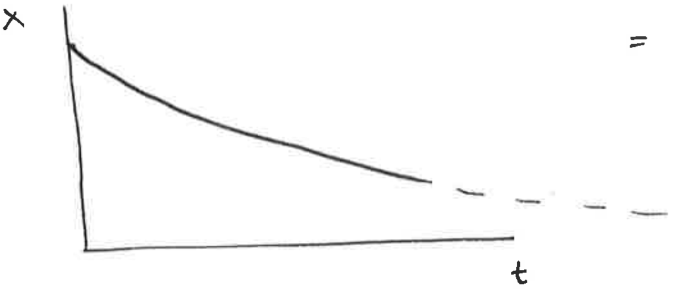
ja erityisesti $-\frac{b}{2m} - \omega < -\frac{b}{2m} + \omega.$

Ratkaisun osat

$$\underbrace{Ae^{\left(-\frac{b}{2m}+\omega\right)t}}_{\text{vaimenee nollaan hitaasti}} + \underbrace{Be^{\left(-\frac{b}{2m}-\omega\right)t}}_{\text{menee nollaan hyvin nopeasti}}$$

Kuinka hitaasti?

$$-\frac{b}{2m}+\omega = -\frac{b}{2m}+\sqrt{\left(\frac{b}{2m}\right)^2-\frac{k}{m}}$$



$$= -\frac{b}{2m}+\frac{b}{2m}\underbrace{\sqrt{1-\frac{4km}{b^2}}}_{\approx 1-\frac{1}{2}\frac{4km}{b^2}}$$

; jos b suuri ($\frac{km}{b^2} \ll 1$)

$$\approx -\frac{b}{2m}+\frac{b}{2m}-\frac{\cancel{b}}{\cancel{2m}}\frac{2k\cancel{m}}{\cancel{b^2}} = -k/b$$

⇒ jos b suuri:

$$x(t) \approx \underbrace{Ae^{-\frac{kt}{b}}}_{\text{Vaimenemisen aikaskaala}} + Be^{-\frac{b}{m}t}$$

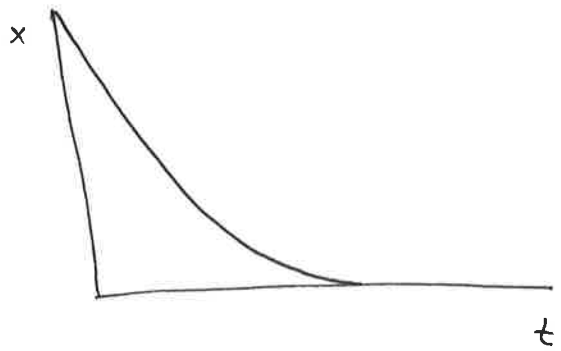
$\tau \sim \frac{b}{k}$.

Kriittinen vaimeneminen:

$$\frac{b}{2m} = \frac{k}{m}$$

⇒

$$x(t) = \underbrace{Ae^{-\frac{b}{2m}t}}_{\text{vaimenee nopeammin}} + \underbrace{Bte^{-\frac{b}{2m}t}}_{\text{hitaammin}}$$



Alivaimennettu värähtely:

$$\left(\frac{b}{2m}\right)^2 < \frac{k}{m}:$$

Pelkkä: $x(t) = Ae^{-\frac{b}{2m}t}\cos(\omega t) + Be^{-b/2m\,t}\sin(\omega t)$; $\omega = \sqrt{\frac{k}{m}-\left(b/2m\right)^2} = \sqrt{\omega_0-\left(\frac{b}{2m}\right)^2}$