Exercise session 2

keskiviikko 28. lokakuuta 2020 14.05

Robust estimation

$$y(n) = \underbrace{\alpha_3 x(n)^3 + \alpha_2 x(n)^2 + \alpha_1 x(n) + \alpha_0}_{} + v(n)$$

$$v(n) \sim N(0, \sigma^2)$$

$$y(n) = z(n) + v(n)$$

$$z(n) = \begin{bmatrix} 1 & x(n) & x(n)^2 & x(n)^3 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \vec{x}(n)\vec{\alpha}$$

$$y(n) \sim N(z(n), \sigma^2)$$

$$\mathcal{L}(\vec{\alpha}) = \frac{1}{\sqrt{2\pi\sigma^2}^{N}} e^{-\frac{1}{2\pi\sigma^2} \underbrace{\sum_{n=1}^{N} (y(n) - \vec{x}(n)\vec{\alpha})^2}_{}}$$

$$\min_{\alpha} \sum_{n=1}^{N} (y(n) - \vec{x}(n)\vec{\alpha})^2$$

$$= \min_{\alpha} \|\vec{y} - \vec{X}\vec{\alpha}\|_2^2 \qquad \vec{y} = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix} \qquad \vec{X} = \begin{bmatrix} \vec{x}(1) \\ \vdots \\ \vec{x}(N) \end{bmatrix}$$

$$\frac{\partial}{\partial \vec{\alpha}} \|\vec{y} - \vec{X}\vec{\alpha}\|_2^2 = \frac{\partial}{\partial \vec{\alpha}} (\vec{y} - \vec{X}\vec{\alpha})^T (\vec{y} - \vec{X}\vec{\alpha})$$

$$= \frac{\partial}{\partial \vec{\alpha}} \left(-2y^T X \alpha + \alpha^T X^T X \alpha\right)$$

$$= -2X^T (y - X\alpha) = 0$$

$$X^T X \alpha = X^T y$$

$$\vec{\alpha} = (X^T X)^{-1} X^T y$$

$$\min_{\alpha} \sum_{n=1}^{N} w(n) \left(y(n) - \vec{x}(n)\alpha\right)^2$$

$$\min_{\alpha} \underbrace{(y - X\alpha)^T W (y - X\alpha)}_{} \qquad W = \begin{bmatrix} w(1) & 0 & 0 & \\ 0 & w(2) & 0 & \\ 0 & 0 & w(3) & \\ & & & \ddots \end{bmatrix}$$

$$\frac{\partial}{\partial \alpha} \cdot = -2X^T W (y - X\alpha) = 0$$

$$X^T W X \alpha = X^T W y$$

$$\alpha = (X^T W X)^{-1} X^T W y$$

$$X^T W X \alpha = X^T W y$$

$$\alpha = \underbrace{(X^T W X)^{-1} X^T W y}$$

$$w(n) = \psi(\tilde{y}(n))$$

$$\tilde{y}(n) = \vec{x}(n)\vec{\alpha}_{WLS} - y(n)$$

$$\psi(\tilde{y}) = \begin{cases} \sin(\tilde{y}/a), & |\tilde{y}| \leq a\pi \\ 0, & |\tilde{y}| > a\pi \end{cases}$$

$$3\sigma$$

$$|\tilde{y}| = 3\sigma = a\pi$$

$$a = \frac{3\sigma}{\pi}$$

Iterative Reweighted least squares IRLS

1. Init $\vec{\alpha}_{WLS}$
2. Find weights
3. Find $\vec{\alpha}_{WLS}$

EM algorithm for Bernoulli mixtures

$$X \sim B(q_k)$$

$$p(x \mid q_k) = q_k^{x}(1-q_k)^{1-x} \qquad x \in \{0,1\}$$

$$y(n) = \{x(n), \vec{z}(n)\} \qquad \vec{z}(n) = \begin{bmatrix} z_1(n) \\ \vdots \\ z_K(n) \end{bmatrix}$$

$z_k(n) \in \{0,1\}$

x(n) was not sampled from distribution k

otherwise

$$p(x(n) \mid \theta) = \sum_{k=1}^{K} \pi_k \, B(x(n); q_k)$$

$$\sum_{k=1}^{K} z_k(n) = 1$$

$$p(x(n), z(n) \mid \theta) = \underbrace{p(z(n) \mid \theta)}\,\underbrace{p(x(n) \mid z(n), \theta)}$$

$$= \prod_{k=1}^{K} \pi_k^{z_k(n)} \prod_{k=1}^{K} B(x(n); q_k)^{z_k(n)}$$

$$= \prod^{K} \left[\pi_k \, B(x(n); q_k)\right]^{z_k(n)}$$

$$= \underbrace{\prod_{k=1}^{K} \left[\pi_k \, B\left(x(n) \, ; \, q_k\right) \right]^{z_k(n)}}_{A}$$

$$\mathcal{L}(\theta \mid x, z) = \prod_{n=1}^{N} A$$

$$\log\left(\mathcal{L}(\theta \mid x, z)\right) = \sum_{n=1}^{N} \sum_{k=1}^{K} z_k(n) \left[\log \pi_k + \log B(x(n) \, ; \, q_k) \right]$$

$$\mathbb{E}_z\left[\log\left(\mathcal{L}(\theta \mid x, z)\right)\right] = \sum_{n=1}^{N} \sum_{k=1}^{K} \mathbb{E}\left[z_k(n)\right] \left[\log \pi_k + \log\left(B(x(n) \, ; \, q_k)\right) \right]$$

$$\gamma_k(n) = \mathbb{E}\left[z_k(n)\right]$$

E-step:

- q_k^{old}
- π_k^{old}

$$\gamma_k(n) = p\left(z_k(n) = 1 \mid x(n), \theta\right)$$

$$= \boxed{\frac{\pi_k \, p(x(n) \mid q_k)}{\sum_{j=1}^{K} \pi_j \, p(x(n) \mid q_k)}}$$

$$\frac{\partial \mathbb{E}_z\left(\mathcal{L}(\theta \mid x, z)\right)}{\partial q_k} = \frac{\partial}{\partial q_k} \sum_{n=1}^{N} \sum_{k=1}^{K} \gamma_k(n) \left[\log \pi_k + x(n) \log q_k + (1 - x(n)) \log(1 - q_k) \right]$$

$k \neq k$

$$= \sum_{n=1}^{N} \gamma_k(n) \left[\frac{x(n)}{q_k} - \frac{1 - x(n)}{1 - q_k} \right] = 0$$

M-step:

- $\gamma_k^{old}(n)$

$$\boxed{q_k = \frac{\sum_{n=1}^{N} \gamma_k(n) \, x(n)}{\sum_{n=1}^{N} \gamma_k(n)}}$$

$$\max_{\pi} \; \mathbb{E}_z\left(\mathcal{L}(\theta \mid x, z)\right)$$

$$\text{s.t.} \quad \sum_{k=1}^{K} \pi_k = 1$$

Solve using Lagrangian multipliers.

$$\boxed{\pi_k = \frac{\sum_{n=1}^{N} \gamma_k(n)}{N}}$$