MATRIX ALGEBRA 2020 (Strang, MIT)

VECTORS

$$\mathbb{R}^n = \underbrace{\mathbb{R}\times\mathbb{R}\times\dots\times\mathbb{R}}_{n}$$

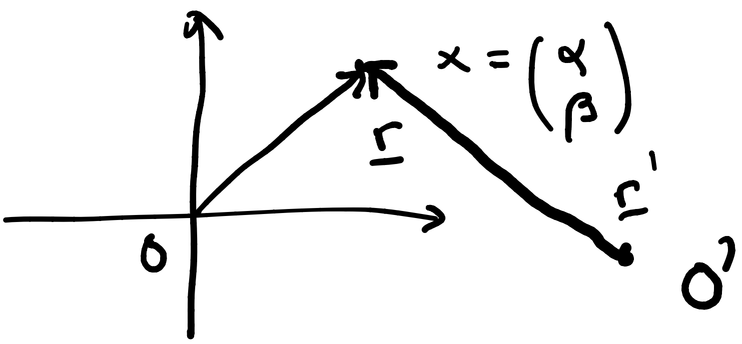
$$= \{ (\xi_1, \xi_2, \dots, \xi_n) \mid \xi_i \in \mathbb{R} \}$$

The elements of $\mathbb{R}^n$ are vectors; column vectors:

$$x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^2$$

"Physical vectors" : $\underline{a}$ $\quad (\bar{a}, \vec{a})$

$\mathbb{R}^2$: Origin : $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$



$\mathbb{R}^n$: (i) addition : $x, y \in \mathbb{R}^n$; $x + y \in \mathbb{R}^n$

(ii) scalar multiplication :

$\alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$; $\alpha x \in \mathbb{R}^n$

$$x = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}, \quad y = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}, \quad x + y = \begin{pmatrix} \xi_1 + \eta_1 \\ \xi_2 + \eta_2 \\ \vdots \\ \xi_n + \eta_n \end{pmatrix}$$

$$\alpha x = \begin{pmatrix} \alpha \xi_1 \\ \alpha \xi_2 \\ \vdots \\ \alpha \xi_n \end{pmatrix}$$

<u>Agreement</u> :

$$\underline{r} = x_1 \underline{i} + x_2 \underline{j} + x_3 \underline{k} \quad \hat{=} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\{ 0, \underline{i}, \underline{j}, \underline{k} \}$ coincides with

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \in \mathbb{R}^3$$

Scalar product (dot product, inner product)

$$\underline{a} \cdot \underline{b} = \begin{cases} \|\underline{a}\| \, \|\underline{b}\| \cos \measuredangle(\underline{a}, \underline{b}), & \text{if } \underline{a} \neq \underline{0}, \underline{b} \neq \underline{0} \\ 0 & \text{, if } \underline{a} = \underline{0} \text{ or } \underline{b} = \underline{0} \end{cases}$$

Orthogonality: $\underline{a} \cdot \underline{b} = 0$, i.e., $\underline{a} \perp \underline{b}$

The scalar component of $\underline{b}$ in the direction of $\underline{a}$:

$$\|\underline{b}\| \cos \measuredangle(\underline{a}, \underline{b}) = \frac{\underline{a}}{\|\underline{a}\|} \cdot \underline{b} = \underline{a}^{\circ} \cdot \underline{b}$$

The vector component: $(\underline{a}^{\circ} \cdot \underline{b})\, \underline{a}^{\circ}$

If $\underline{a} = \alpha_1 \underline{i} + \alpha_2 \underline{j} + \alpha_3 \underline{k}$

$\underline{b} = \beta_1 \underline{i} + \beta_2 \underline{j} + \beta_3 \underline{k}$

$$\underline{a} \cdot \underline{b} = \alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3$$

Let us denote: $\underline{a} \mathrel{\hat{=}} a$, $\underline{b} \mathrel{\hat{=}} b$

Define

$$a^T b = \sum_{i=1}^{3} \alpha_i \beta_i ,$$

where $a^T = (\alpha_1 \ \alpha_2 \ \alpha_3)$ (row vector)

Notice: $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

So, $a^T b$ is the scalar product of two vectors in $\mathbb{R}^n$.

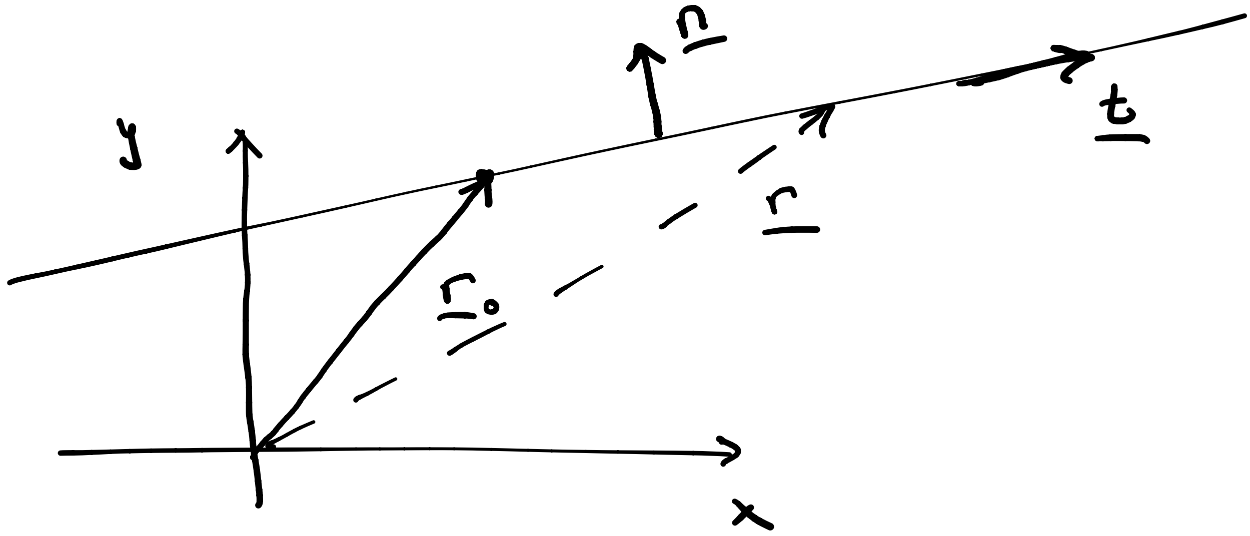
General: $a^T b = \sum_{i=1}^{n} \alpha_i \beta_i$.

<u>Lines and plane</u>

Line: $\underline{r} = \underline{r}_0 + \tau \underline{t}$, $\underline{t} \neq \underline{0}$, $\tau \in \mathbb{R}$

Plane: $\underline{r} = \underline{r}_0 + \sigma \underline{s} + \tau \underline{t}$, $\underline{s}, \underline{t} \neq 0$, $\sigma, \tau \in \mathbb{R}$

A line on a plane:



$\underline{n} \cdot (\underline{r} - \underline{r}_0) = 0$; $\underline{r} = x\underline{i} + y\underline{j}$

Let $\underline{n} = n_1 \underline{i} + n_2 \underline{j} + n_3 \underline{k}$;

Define a plane :

$$n_1 x + n_2 y + n_3 z = d \,, \quad \text{where } d = \underline{n} \cdot \underline{r}.$$

A straight line in space is an intersection of two planes.

<u>Linear combination of vectors</u>

$$a = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}, \quad b = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \qquad (a, b \text{ not parallel})$$

Every vector : $\xi a + \eta b$ lies on the plane spanned by a and b

$\xi, \eta \in \mathbb{R}$

Vector $\xi a + \eta b$ is a linear combination of a and b.

Formally : $\xi a + \eta b \in \operatorname{span}(\{a, b\})$

Linear Equations

$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases}$$

or $$x\begin{pmatrix}2\\1\end{pmatrix} + y\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$$

If the solution exists: $\begin{pmatrix}1\\5\end{pmatrix} \in \operatorname{span}\left(\left\{\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}\right)$

Three options:

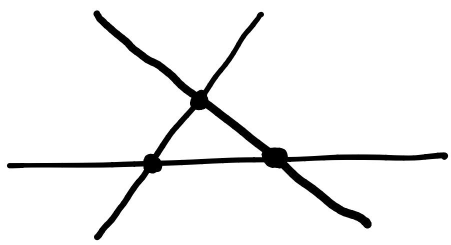
(A) One unique solution

(B) No solutions

(C) Infinite number of solutions

Three planes in $\mathbb{R}^3$:

(B) two or more are parallel.
every pair of planes intersects in a line, and those lines are parallel



(C)