GAUSSIAN ELIMINATION

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$

Add equations: $4x + 0 \cdot y = 12 \Rightarrow x = 3$

Substitute back:

$$3 - 2y = 1 \Rightarrow y = 1$$

Alternative: $x = 1 + 2y$

Substitute:

$$3(1 + 2y) + 2y = 11$$

$$\Leftrightarrow \quad 8y = 8 \Rightarrow y = 1$$

$$\Rightarrow x = 3$$

$$\begin{cases} x + 2y + 3z = 6 \\ 2x + 5y + 2z = 4 \\ 6x - 3y + z = 2 \end{cases}$$

The simplest possible problem:

$$x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The second most simple problem:

$$x\begin{pmatrix}1\\1\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix} \longrightarrow \text{FORWARD SUBSTITUTION}$$

or

$$x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix} \longrightarrow \text{BACKWARD SUBSTITUTION}$$

OUR GOAL:

An algorithm that transforms the original linear system to one of the simple ones.

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & 11 \end{array}\right) \xrightarrow{?} \left(\begin{array}{cc|c} 1 & -2 & 1 \\ \underline{\underline{0}} & * & ** \end{array}\right)$$

Row operation:

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & 11 \end{array}\right) \begin{array}{l} \big\downarrow -3 \end{array}$$

Two observations:

(a) the order of equations is arbitrary

(b) one can multiply and add equations

Notation: $\downarrow -3$

$$-3(x-2y) + 3x + 2y = -3 \cdot 1 + 11$$

$$\Leftrightarrow \quad 8y = 8$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & 11 \end{array}\right] \downarrow -3$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 8 & 8 \end{array}\right] \longrightarrow \text{BACKWARD SUBSTITUTION} \Longrightarrow \begin{array}{l} y = 1 \\ x = 3 \end{array}$$

How to choose the right row operation:

$$\left[\begin{array}{cc|c} \underline{1} & -2 & 1 \\ 3 & 2 & 11 \end{array}\right]$$

$\uparrow$
0

$\underline{1}$ is the so-called pivot.

the scalar should be:

$$-\left(\frac{3}{1}\right) = -3$$

Linear system: n unknowns $\longrightarrow$ n^2 coefficients

double precision: 8 bytes per coeff.

$$n = 10^4 \Longrightarrow n^2 = 10^8$$
$$n = 10^6 \Longrightarrow n^2 = 10^{12}$$

Moral: In engineering the linear systems have a structure that can be exploited.

$\longrightarrow$ there are more zeros than non-zeros
"the systems are sparse"

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right) \quad -2 \quad 1 = -\left(\frac{-2}{2}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right) \quad -1$$

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right)$$

BACKWARD SUBSTITUTION $\Rightarrow \begin{cases} x = -1 \\ y = 2 \\ z = 2 \end{cases}$

Remainder: Number of solutions : $0, 1, \infty$

1) Parallel straight lines

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & -6 & 11 \end{array}\right) \quad -3$$

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 8 \end{array}\right)$$

Contradiction!

$0 \cdot y = 8$ FALSE!

2) Overlapping straight lines

$$\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & -6 & 3 \end{array} \quad \downarrow -3$$

$$\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array} \quad \Rightarrow \quad 0 = 0 \text{ True!}$$

↳ the second unknown can be chosen freely

$\Rightarrow \quad x = 1 + 2y \ , \ y \in \mathbb{R}$

$\Rightarrow$ infinite number of solutions

3) Permuting equations

$$\begin{array}{cc|c} 0 & 2 & 4 \\ 3 & -2 & 5 \end{array}$$

$$\begin{array}{cc|c} 3 & -2 & 5 \\ 0 & 2 & 4 \end{array} \quad \Rightarrow \begin{cases} y = 2 \\ x = \frac{1}{3}(5 + 2 \cdot 2) = 3 \end{cases}$$

MATRIX

$$\begin{cases} 2x + 3y = 11 \\ 3x + 4y = 2 \end{cases}$$

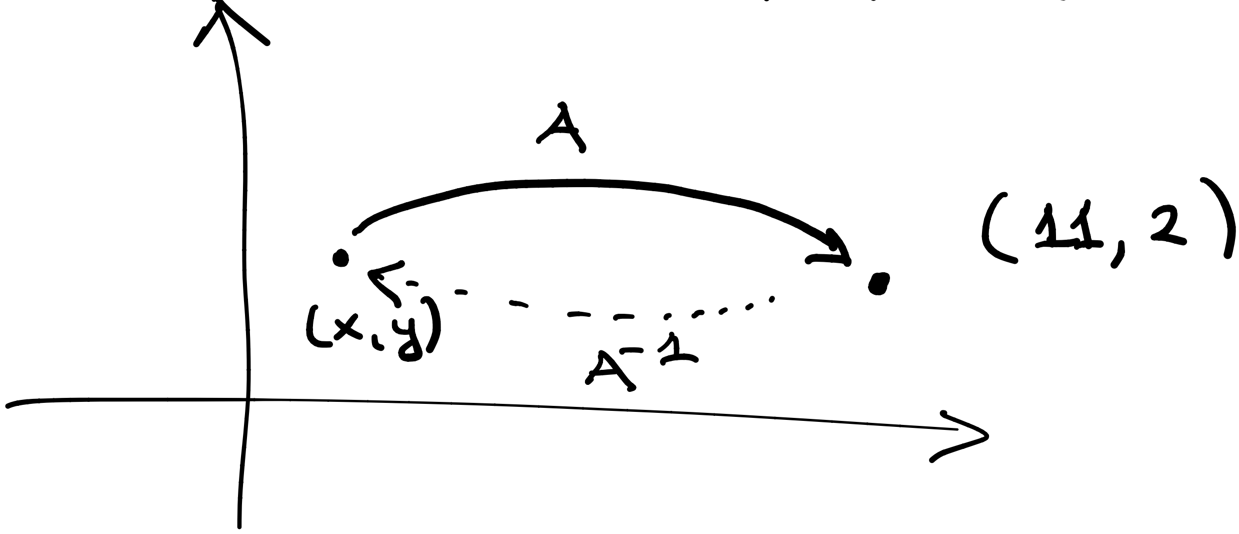
$$\underbrace{x\begin{pmatrix}2\\3\end{pmatrix} + y\begin{pmatrix}3\\4\end{pmatrix}} = \begin{pmatrix}11\\2\end{pmatrix} \longleftarrow (*)$$

$$\underbrace{\begin{pmatrix}2 & 3\\3 & 4\end{pmatrix}}_{\substack{\text{matrix}\\ = A}}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}11\\2\end{pmatrix}$$

column vectors

(*) matrix - vector multiplication

$A : \mathbb{R}^2 \to \mathbb{R}^2$



$Ax = b$ The most important problem in engineering!