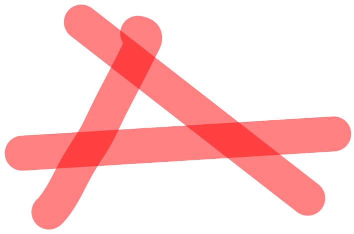
Winnie the Pooh



"To-morrow," said Eeyore. "Or the next day." Piglet came a little closer to see what it was. Eeyore had three sticks on the ground, and was looking at them. Two of the sticks were touching at one end, but not at the other, and the third stick was laid across them. Piglet thought that perhaps it was a Trap of some kind.

"Oh, Eeyore," he began again, "I just--"

"Is that little Piglet?" said Eeyore, still looking hard at his sticks.

"Yes, Eeyore, and I--"

"Do you know what this is?"

"No," said Piglet.

"It's an A."

"Oh," said Piglet.

"Not O--A," said Eeyore severely. "Can't you hear, or do you think you have more education than Christopher Robin?"

"Yes," said Piglet. "No," said Piglet very quickly. And he came closer still.

"Christopher Robin said it was an A, and an A it is--until somebody treads on it," Eeyore added sternly.

Piglet jumped backwards hurriedly, and smelt at his violets.

"Do you know what A means, little Piglet?"

"No, Eeyore, I don't."

"It means Learning, it means Education, it means all the things that you and Pooh haven't got. That's what A means."

"Oh," said Piglet again. "I mean, does it?" he explained quickly.

"I'm telling you. People come and go in this Forest, and they say, 'It's only Eeyore, so it doesn't count.' They walk to and fro saying 'Ha ha!' But do they know anything about A? They don't. It's just three sticks to them. But to the Educated--mark this, little Piglet--to the Educated, not meaning Poohs and Piglets, it's a great and glorious A. Not," he added, "just something that anybody can come and breathe on."

Piglet stepped back nervously, and looked round for help.

MATRIX

Let $u = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$$\xi_1 u + \xi_2 v + \xi_3 w = \begin{pmatrix} \xi_1 \\ \xi_2 - \xi_1 \\ \xi_3 - \xi_2 \end{pmatrix} \quad \star$$

Rewrite: Matrix-vector multiplication

$$Ax = b$$

where $A = (u \ v \ w) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$x = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$; b is the linear combination

So $$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 - \xi_1 \\ \xi_3 - \xi_2 \end{pmatrix} \quad \star$$

Row - vector inner products:

$$A = (u \; v \; w) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T x \\ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}^T x \\ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}^T x \end{pmatrix} = \begin{pmatrix} 1 \cdot \xi_1 + 0 \cdot \xi_2 + 0 \cdot \xi_3 \\ -1 \cdot \xi_1 + 1 \cdot \xi_2 + 0 \cdot \xi_3 \\ 0 \cdot \xi_1 - 1 \cdot \xi_2 + 1 \cdot \xi_3 \end{pmatrix}$$

$$= \begin{pmatrix} \xi_1 \\ \xi_2 - \xi_1 \\ \xi_3 - \xi_2 \end{pmatrix}$$

Linear systems (again):

$$\begin{cases} x_1 & = b_1 \\ -x_1 + x_2 & = b_2 \\ \quad - x_2 + x_3 & = b_3 \end{cases} \Rightarrow \begin{cases} x_1 = b_1 \\ x_2 = b_1 + b_2 \\ x_3 = b_1 + b_2 + b_3 \end{cases}$$

$$Ax = b \qquad\qquad Sb = x$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad\qquad S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Matrix notation

$\underset{m \times n}{A} = (\alpha_{ij})$; m is the number of rows
n is the number of columns

$i \rightarrow$ row index
$j \rightarrow$ column index

$\underset{m \times n}{A} = (a_1 \; a_2 \; \dots \; a_n)$; $a_i \in \mathbb{R}^m$

Matrix-vector product : $Ax = c$

$$c = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$$

(i) linear combination : $\underset{m \times 1}{c} = \sum_{i=1}^{n} \xi_i a_i$

(ii) inner product : $\gamma_i = \sum_{j=1}^{n} \alpha_{ij} \xi_j$

Matrix-Matrix product:

$$\underset{m\times n}{A}\;\underset{n\times p}{B} = \left(Ab_1 \;\; Ab_2 \;\; \cdots \;\; Ab_p \right) = \underset{m\times p}{C}$$

Convenient observation: $\underset{m\times \cancel{n}\,\cancel{n}\times p}{A\quad B} = \underset{m\times p}{C}$

$$\underset{m\times \cancel{n}\,\cancel{n}\times \cancel{p}\,\cancel{p}\times q}{A\quad B\quad C} = \underset{m\times q}{D}$$

Inner products: $\underset{m\times p}{C} = (\gamma_{ij})$

$$\gamma_{ij} = \sum_{k=1}^{n} \alpha_{ik}\beta_{kj}$$

Properties and Laws:

Theorem Associativity: $A(BC) = (AB)C$
Not commutativity: $AB \neq BA$

In particular: $(A+B)^2 = (A+B)(A+B)$

$$= A^2 + \underbrace{AB + BA}_{\neq 2AB} + B^2$$

Square matrix $\underset{m\times m}{A}$:

$$A^p = \underbrace{A \cdot A \cdot \ldots \cdot A}_{p \text{ times}}; \quad A^pA^q = A^{p+q}; \quad (A^p)^q = A^{pq}$$

Addition:

$$A + B = B + A$$
$$\xi(A+B) = \xi A + \xi B$$
$$A + (B+C) = (A+B) + C$$

Product:

$$C(A+B) = CA + CB$$
$$(A+B)C = AC + BC$$
$$A(BC) = (AB)C$$

On the order of operations:

$$\left.\begin{array}{l} A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ B : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \end{array}\right\} \quad B \circ A : \mathbb{R}^2 \to \mathbb{R}^2$$

$$x \in \mathbb{R}^2 : \quad BAx = B(Ax) = (BA)x$$

Linear systems (yet again): $Ax = b$

$$\begin{array}{cccc} 2 & 4 & -2 & 2 \\ \underline{4} & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array} \quad \swarrow -2 \quad \searrow 1$$

$\uparrow$ first column : $\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} \xrightarrow{\text{row ops}} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

Define $\underline{E_{21}} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and compute

$$E_{21} \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

Let $E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and compute

$$E_{31} \underbrace{E_{21} a_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

E_{ij} is an elimination matrix which encodes the corresponding row operation in the Gaussian elimination.

Ultimate goal: $\underset{m \times m}{A} x = b$

$A = LU \longrightarrow$ decomposition

$$LUx = b \iff \begin{cases} Ly = b \\ Ux = y \end{cases}$$