LU - Decomposition (or Factorisation)

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$I = (e_1 \ e_2 \ e_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{diag}(1,1,1)$$

$I_{n\times n}$ is the identity matrix : $Ix = x$, $x \in \mathbb{R}^n$

Consider elimination matrices : E_{ij}

$$E_{ij} = I + l_{ij}\, e_i e_j^T, \quad i > j$$

For example:

$$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = I + 1 \cdot \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{e_3} \underbrace{(1 \ 0 \ 0)}_{e_1^T}$$

Gaussian elimination : $\underset{3\times 3}{A}$

$$E_{32} E_{31} E_{21} A = U = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

u_{12}

u_{21}

$\rightarrow$ upper triangular

$\rightarrow$ $j \geq i$

$$\Longleftrightarrow \quad u_{ij} = 0, \quad i > j$$

Similarly, every E_{ij} is a lower triangular matrix.

So,

$$(E_{32}E_{31}E_{21})A = U$$

NOTICE: THERE IS NO DIVISION HERE!

<u>Definition</u> $\underset{n\times n}{A}$ is invertible, if there exists a matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I.$$

A^{-1} is the inverse of A.

WARNING: $Ax = b \implies x = A^{-1}b$

For a product: $(AB)^{-1} = B^{-1}A^{-1}$

We claim: $E_{ij}^{-1} = I - l_{ij}e_ie_j^T$

$$E_{ij}E_{ij}^{-1} = (I + l_{ij}e_ie_j^T)(I - l_{ij}e_ie_j^T)$$

$$= I + l_{ij}e_ie_j^T - l_{ij}e_ie_j^T$$

$$- l_{ij}^2 e_i\underbrace{e_j^Te_i}_{=0}e_j^T, \quad i > j$$

$$= I$$

$$A = (E_{32} E_{31} E_{21})^{-1} U$$

$$= E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} U$$

$= LU$, where L is lower triangular

Why?

$$E_{31}^{-1} E_{32}^{-1} = (I - l_{31} e_3 \underline{e_1^T})(I - l_{32} \underline{e_3} e_2^T)$$

$$= I - l_{31} e_3 e_1^T - l_{32} e_3 e_2^T$$

$$E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = I - l_{21} e_2 e_1^T - l_{31} e_3 e_1^T$$
$$- l_{32} e_3 e_2^T$$

<u>Example</u>

$$\begin{matrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{matrix} \quad \swarrow -\tfrac{1}{2} \qquad\qquad \begin{matrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 1 & 2 \end{matrix} \quad \swarrow -\tfrac{2}{3}$$

$$\underbrace{\begin{matrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{matrix}}_{U} \qquad\qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix}$$

$$LU = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Every U can be decomposed further:

$$U = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \begin{pmatrix} 1 & u_{12}/d_1 & \cdots & \\ & 1 & u_{23}/d_2 & \cdots \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$= D \hat{U}$$

Standard notation: $A = LDU$

Symmetric case: $\alpha_{ij} = \alpha_{ji}$

$$A = LDU \; ; \; \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} =$$

$$\underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix}} \underset{D}{\begin{pmatrix} 2 & & \\ & 3/2 & \\ & & 4/3 \end{pmatrix}} \underset{U = L^T}{\begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$= \underbrace{L D^{1/2}} \underbrace{D^{1/2} L^T} = \hat{L} \hat{L}^T \; ; \; D^{1/2} = \begin{pmatrix} \sqrt{2} & & 0 \\ & \sqrt{3/2} & \\ 0 & & \sqrt{4/3} \end{pmatrix}$$

Cholesky decomposition

Thus, solution of $Ax = b$ becomes

(1) $A = LU$ (factorisation)

(2) $LUx = b \iff \begin{cases} Ly = b & \text{" forward"} \\ Ux = y & \text{" backward"} \end{cases}$

Computational complexity : $\underset{n\times n}{A} = LU$

1. elimination step : n^2 multiplications
2. " : $(n-1)^2$ "
3. " : $(n-2)^2$ "

Together:

$$n^2 + (n-1)^2 + (n-2)^2 + \dots + 2^2 + 1^2$$

$$= \frac{1}{6} n(n+1)(2n+1) \longrightarrow \frac{1}{3} n^3 \text{ is the leading term}$$

Triangular systems : n^2

NOTICE: Decomposition is the expensive part!

But, finding the inverse is even more expensive!

$\rightarrow$ the memory requirements for inverses are higher

PA = LU

$$
\begin{matrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{matrix}
\qquad
\begin{matrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{matrix} \; -2
\qquad
\begin{matrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{matrix} \; -3
$$

$$
LU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} = PA,
$$

where $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a permutation matrix.

$$
P \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}
$$

Theorem For every invertible matrix A there exists a decomposition

$$
PA = LU.
$$

(P is not unique.)