Inverses and Transposes

Synthesis: $\underset{n\times n}{A}$; A is invertible

(1) The Gaussian elimination produces n pivots.

(2) A^{-1} is unique.

<u>Proof</u>

Suppose $BA = I$ and $AC = I$, then $B = C$:

$$B(AC) = (BA)C$$
$$\Rightarrow \quad BI = IC \quad \Rightarrow B = C \quad \square$$

(3) $Ax = b$ has one and only one solution: $x = A^{-1}b$

(4) If $Ax = 0$, $x \neq 0$, then A <u>is not invertible</u>.

Useful inverses:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Notice: $ad - bc \neq 0$ is required

$$\mathrm{diag}(d_1, d_2, \dots, d_n)^{-1} = \mathrm{diag}(1/d_1, 1/d_2, \dots, 1/d_n)$$

$$d_i \neq 0$$

Gauss - Jordan Algorithm

The idea: Find X such that $\underset{n\times n}{A} X = I$.

Columnwise: $A(x_1\ x_2\ \dots\ x_n) = (e_1\ e_2\ \dots\ e_n)$

Eliminate all RHSs simultaneously!

$$\left(\begin{array}{ccc|ccc} \underline{2} & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \quad \downarrow \tfrac{1}{2}$$

$$\begin{array}{cccccc} \underline{2} & -1 & 0 & 1 & 0 & 0 \\ 0 & \underline{3/2} & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \quad \downarrow \tfrac{2}{3}$$

$$\begin{array}{cccccc} \underline{2} & -1 & 0 & 1 & 0 & 0 \\ 0 & \underline{3/2} & -1 & 1/2 & 1 & 0 \\ 0 & 0 & \underline{\underline{4/3}} & 1/3 & 2/3 & 1 \end{array} \quad \begin{array}{l} \uparrow \tfrac{2}{3} \\ \uparrow \tfrac{3}{4} \end{array}$$

three pivots!

this stage: $UX = B \xrightarrow{?} IX = A^{-1}$

Elimination upwards means substituting the value!

$$\begin{array}{cccccc|c} \underline{2} & 0 & 0 & 3/2 & 1 & 1/2 & : 2 \\ 0 & \underline{3/2} & 0 & 3/4 & 3/2 & 3/4 & : 3/2 \\ 0 & 0 & \underline{4/3} & 1/3 & 2/3 & 1 & : 4/3 \end{array}$$

$$\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \quad \underbrace{\begin{matrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{matrix}}_{A^{-1}}$$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} ; \quad AA^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Operation count $\sim n^3$

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{pmatrix}$$

A is a tridiagonal matrix :

$$\underset{4\times 4}{A} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} ; \quad A^{-1} \text{ is full.}$$

Storage requirements : $\underset{n\times n}{A}$: $LU \sim 3n$

$A^{-1} \sim n^2$

Transpose

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} ; \quad A^T = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$$

In short : $A = (\alpha_{ij})$, $A^T = (\alpha_{ji})$

Two formulae :

(a) $(AB)^T = B^T A^T$

(b) $(A^{-1})^T = (A^T)^{-1}$

(b) assuming that (a) is true :

$$\begin{cases} AA^{-1} = I \iff (AA^{-1})^T = I^T \\ A^{-1}A = I \iff (A^{-1}A)^T = I^T \end{cases}$$

$$(AA^{-1})^T = (A^{-1}A)^T$$

$$\Rightarrow \quad (A^{-1})^T A^T = A^T (A^{-1})^T = I$$

$$\Rightarrow \quad (A^{-1})^T = (A^T)^{-1} \qquad \square$$

__Definition__ Matrix A is symmetric, if $A = A^T$.

Identity: $R^T R$ is symmetric

$$(R^T R)^T = R^T R \quad , \quad \underset{m\times n}{R}$$

For symmetric matrices:

$$A = LDU = LDL^T \quad \text{is symmetric!}$$

__Special case__: A is a complex matrix?

$z \in \mathbb{C}$; $\operatorname{mod} z = \sqrt{\bar{z} z}$

$x \in \mathbb{C}^n$; $\|x\| = \sqrt{x^H x}$; H = Hermite

$$x = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}, \quad x^H = (1-i \quad 2)$$

$A \in \mathbb{C}^{n\times n}$; $A^H = (\bar{\alpha}_{ji})$

$\rightarrow$ universal notation A^*

Definition Permutation Matrix

Rows of P are the rows of I but in some order.

Any inverse permutation is also a permutation. Moreover,

$$P^{-1} = P^T.$$

Definition If $A^{-1} = A^T$, A is orthogonal.

Example

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P^T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$PP^T = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} = I$$

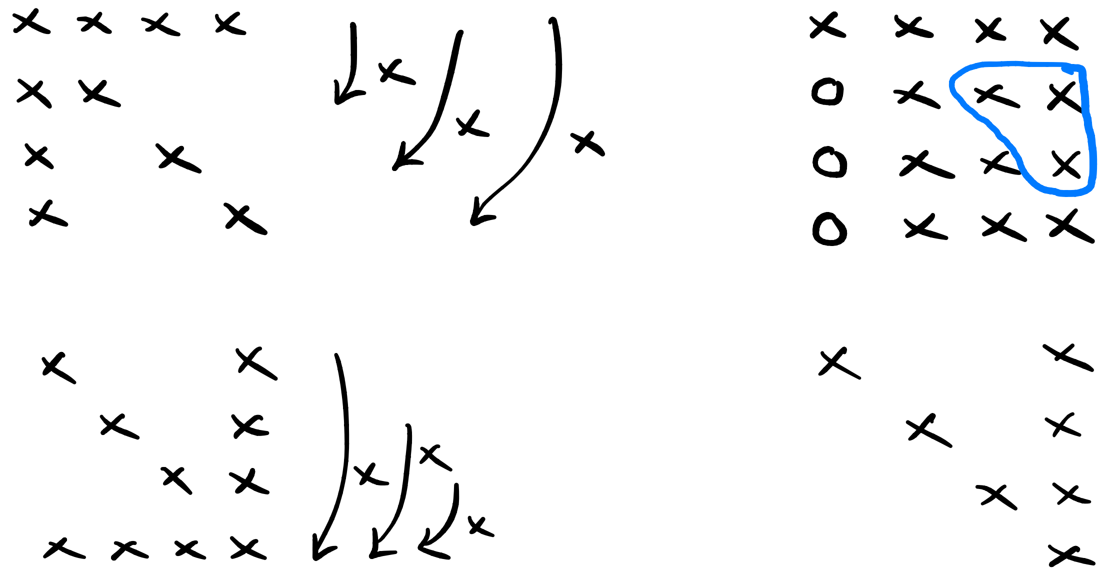
$$P \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} P^T = \begin{pmatrix} \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{11} & \alpha_{12} & \alpha_{13} \end{pmatrix} P^T$$

$$= \begin{pmatrix} \alpha_{22} & \alpha_{23} & \alpha_{21} \\ \alpha_{32} & \alpha_{33} & \alpha_{31} \\ \alpha_{12} & \alpha_{13} & \alpha_{11} \end{pmatrix}$$

P from left : permutes rows

P^T from right : permutes columns

$$Ax = b \iff (PA\underbrace{P^T)(P}_{I}x) = Pb$$



$\longrightarrow$ perfect elimination order