$Ax = b$ and $Ax = 0$

Let $\underset{3\times4}{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ 3 & 6 & 11 & 14 \end{pmatrix}$

$x_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

Now : $\underset{3\times4}{A} \underset{4\times1}{x_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $Ax_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Hence, $A(\xi_1 x_1 + \xi_2 x_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

<u>Definition</u> Nullspace

$$N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

$$(= \ker(A))$$

In other words, $x \in \operatorname{span}(\{x_1, x_2\})$

$$\Rightarrow Ax = 0$$

Why?

$$A x_p = b \quad \text{and} \quad A x_H = 0$$

$$\Rightarrow \quad A(x_p + x_H) = b$$

We have already seen that $AB = 0 \not\Rightarrow \begin{matrix} A = 0 \\ B = 0 \end{matrix}$

Reinterpretation: $A(b_1\ b_2 \ldots b_n) = 0$, $b_i \in N(A)$

But also:

$$B^T A^T = 0 \quad \Rightarrow \text{every column of } A^T \in N(B^T)$$

<u>Example</u>

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 1 \\ 2x_1 + 4x_2 + 8x_3 + 10x_4 = 6 \\ 3x_1 + 6x_2 + 11x_3 + 14x_4 = 7 \end{cases}$$

$$\Leftrightarrow \quad Ax = b$$

First $Ax = 0$

$$\begin{matrix} \underline{1} & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ 3 & 6 & 11 & 14 \end{matrix} \quad -2 \quad -3$$

$$\begin{matrix} \underline{1} & 2 & 3 & 4 \\ 0 & 0 & \underline{2} & 2 \\ 0 & 0 & 2 & 2 \end{matrix} \quad -1 \qquad\qquad \begin{matrix} \underline{1} & 2 & 3 & 4 \\ 0 & 0 & \underline{2} & 2 \\ 0 & 0 & 0 & 0 \end{matrix} \quad :2 \quad -3$$

$$\begin{matrix} \underline{1} & 2 & 0 & 1 \\ 0 & 0 & \underline{1} & 1 \\ 0 & 0 & 0 & 0 \\ \uparrow & & \uparrow & \end{matrix}$$

$Ax = 0 \Rightarrow Rx = 0$

Reduced echelon form

Variables : Two sets :

(a) pivot variables $\{x_1, x_3\}$

(b) free variables $\{x_2, x_4\}$

Let $x_2 = \underline{\sigma}$, $x_4 = \underline{\tau}$, $\sigma, \tau \in \mathbb{R}$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \sigma \\ x_3 \\ \tau \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 = -2\sigma - \tau & \leftarrow \\ x_3 = - \tau & \leftarrow \end{cases}$$

$$\Leftrightarrow \quad x = \sigma \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \tau \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\begin{matrix} \underline{1} & 2 & 0 & 1 \\ 0 & 0 & \underline{1} & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Shortcut :

1) set $x_2 = 1$, $x_4 = 0$
$\Rightarrow x_3 = 0$, $x_1 = -2$

2) set $x_2 = 0$, $x_4 = 1$
$\Rightarrow x_3 = -1$, $x_1 = -1$

$\rightarrow$ $N(A)$ is non-trivial

$$\begin{matrix} 1 & 2 & 3 & 4 & b_1 \\ 2 & 4 & 8 & 10 & b_2 \\ 3 & 6 & 11 & 14 & b_3 \end{matrix} \quad \swarrow -2 \quad \swarrow -3 \quad \swarrow -1$$

$$\begin{matrix} \underline{1} & 2 & 3 & 4 & b_1 \\ 0 & 0 & \underline{2} & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{matrix} \leftarrow \text{consistence}$$

A solution exists only if $b_3 - b_2 - b_1 = 0$;

Here $b = \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix}$, i.e., $7 - 6 - 1 = 0$

Now we know that the number of solutions is infinite.

$$\begin{matrix} \underline{1} & 2 & 3 & 4 & 1 \\ 0 & 0 & \underline{2} & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \quad \begin{matrix} \\ :2 \\ \\ \end{matrix} \quad -3$$

$$\begin{matrix} \underline{1} & 2 & 0 & 1 & -5 \\ 0 & 0 & \underline{1} & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$Rx = d$

Set $x_2 = 0$, $x_4 = 0$

$\Rightarrow x_1 = -5$, $x_3 = 2$

$$x = \begin{pmatrix} -5 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \sigma \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \tau \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix},$$

$\sigma, \tau \in \mathbb{R}$

$$Ax = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ 3 & 6 & 11 & 14 \end{pmatrix} \begin{pmatrix} -5 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix}$$

Block matrix representation:

$$\left(\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} \;\middle|\; \begin{matrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{matrix}\right) \quad \begin{pmatrix} -5 \\ 2 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} \underset{2\times 2}{I} & \underset{2\times 2}{F} \\ \underset{1\times 2}{0} & \underset{1\times 2}{0} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} \underset{2\times 1}{b_1} \\ \underset{1\times 1}{b_2} \end{pmatrix}$$

$\underset{2\times 2}{I}$: The number of pivots $= 2$
This is the <u>rank</u> of A.

If $\underset{n\times n}{A}$ with rank $= n$, then we know that A is invertible.

<u>Theorem</u> Any $\underset{p\times n}{A}\,x = b$ can be transformed to

$$\left[\begin{array}{cc|c} \underset{r\times r}{I} & \underset{r\times(n-r)}{F} & \underset{r\times 1}{b_1} \\ \underset{(p-r)\times r}{0} & \underset{(p-r)\times(n-r)}{0} & \underset{(p-r)\times 1}{b_2} \end{array}\right]$$

	# of solutions
If $r < p$ and $b_2 \neq 0$	0
$(r = p$ or $b_2 = 0)$ and $r = n$	1
$(r = p$ or $b_2 = 0)$ and $r < n$	∞

Block matrices:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 32 \end{pmatrix}$$

$$\left(\begin{array}{cc:c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right) \left(\begin{array}{c} 1 \\ 2 \\ \hdashline 3 \end{array}\right) =$$

$$= \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \end{pmatrix} (3)$$

$$= \begin{pmatrix} 5 \\ 14 \end{pmatrix} + \begin{pmatrix} 9 \\ 18 \end{pmatrix} = \begin{pmatrix} 14 \\ 32 \end{pmatrix}$$