Linear Independence, Basis and Dimension

$$\underset{p\times n}{A}x = b \quad \Longrightarrow \quad \left(\begin{array}{cc|c} I & F & b_1 \\ 0 & 0 & b_2 \end{array}\right)$$

$\underset{r\times r}{I}$; r is the rank of A ;
↳ number of pivots

$$\begin{pmatrix} I \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & 0 & \ddots & \vdots \\ & & & 1 \\ 0 & \vdots & & 0 \end{pmatrix}}_{(e_1\, e_2 \dots e_r)}$$

$e_i \in \mathbb{R}^p$

Question:

Are there genuinely independent rows or columns in a given matrix?

<u>Definition</u> Linear independence

Let $a_1, a_2, \dots, a_p \in \mathbb{R}^n$, and $\xi_1, \xi_2, \dots, \xi_p$ unknown scalars. The vectors a_i are linearly independent, if the only solution of

$$\sum_{i=1}^{p} \xi_i a_i = 0 \quad \text{is} \quad \xi_1 = \xi_2 = \dots = \xi_p = 0 .$$

Otherwise, the vectors are linearly dependent.

<u>Example</u> $a_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $a_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\sum_{i=1}^{3} \xi_i a_i = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$$

$\Longrightarrow$ for instance $\xi_1 = -2$, $\xi_2 = 1$, $\xi_3 = 0$

Thus, $\{a_1, a_2, a_3\}$ are linearly dependent.

Theorem In $\mathbb{R}^n$ there can always be n linearly independent vectors, but any collection with more than n vectors is always linearly dependent.

Dimension is the largest possible number of linearly independent vectors;

$$\dim \mathbb{R}^n = n .$$

Definition Any linearly independent collection of n vectors in $\mathbb{R}^n$ is a basis.

Theorem Let $\{b_1, b_2, \dots, b_n\}$ be a basis of $\mathbb{R}^n$ and $y \in \mathbb{R}^n$. Then y is a unique linear combination of the basis vectors:

$$y = \xi_1 b_1 + \xi_2 b_2 + \dots + \xi_n b_n = \sum_{k=1}^{n} \xi_k b_k$$

Proof Establish: $\{b_1, b_2, \dots, b_n, y\}$ is linearly dependent.

Let us choose the scalars $\{\xi_1, \xi_2, \dots, \xi_n, \eta\}$ such that

$$\sum_{k=1}^{n} \xi_k b_k + \eta y = 0 .$$

If $\eta = 0$, then $\xi_1 = \dots = \xi_n = 0$ since $\{b_1, b_2, \dots, b_n\}$ is a basis!

$\longrightarrow$ Contradiction

So, $\eta \neq 0$ ($y \neq 0$ by construction).

The linear combination becomes

$$y = \sum_{k=1}^{n} \left(-\frac{\xi_k}{\eta} \right) b_k .$$

Is it unique?

$$y = \sum_{k=1}^{n} \xi_k b_k = \sum_{k=1}^{n} \xi_k' b_k$$

$$\Rightarrow \sum_{k=1}^{n} (\xi_k - \xi_k') b_k = 0$$

Since $\{ b_1, b_2, \dots, b_n \}$ is a basis, it follows that

$$\xi_k = \xi_k' , \quad k = 1, 2, \dots, n .$$

$\square$

<u>Notice:</u> $\underset{n \times n}{A} x = b$, A is invertible

$x = A^{-1} b$ unique : $N(A) = 0$

The rank of A is n.

$b \in R(A)$

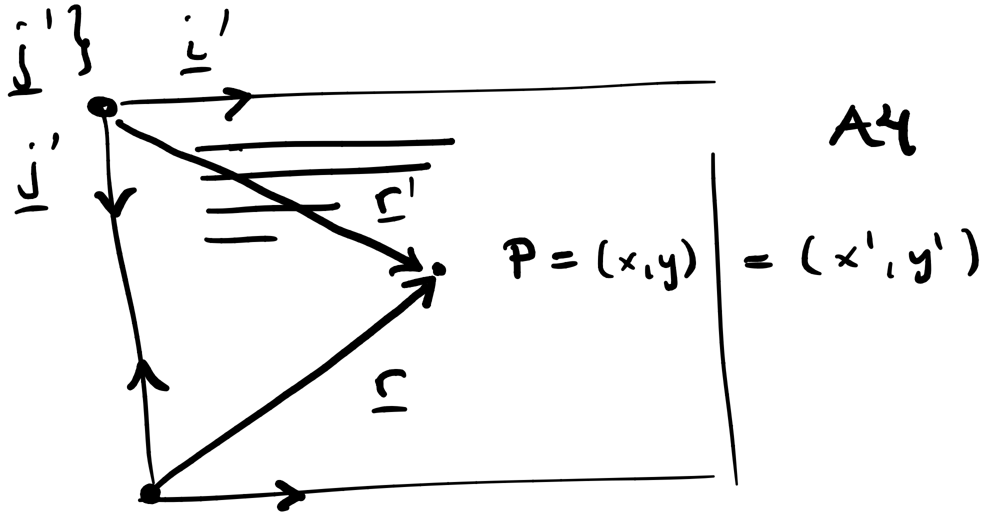
Definition The coefficients of the linear combination are the coordinates of the vector in a given basis.

Natural basis: $x = \sum_{k=1}^{n} \xi_k e_k$,

The components of x are its coordinates in the natural basis.

Example of multiple bases

$\{O', \underline{i}', \underline{j}'\}$



$\{O, \underline{i}, \underline{j}\}$

Change of basis: Two systems:

$$\{O, \underline{b}_1, \underline{b}_2, \underline{b}_3\}, \quad \{O', \underline{b}_1', \underline{b}_2', \underline{b}_3'\}$$

Origin O: $\underline{r}_0 = \sum_{k=1}^{3} p_k \underline{b}_k'$

$$\underline{b}_j = \sum_{k=1}^{3} \tau_{kj} \, \underline{b}'_k \, , \quad j = 1, 2, 3$$

$\hookrightarrow$ transpose (an educated guess)

Let $P \in \mathbb{R}^3$:

$$\underline{r} = \sum_{k=1}^{3} \xi_k \underline{b}_k \, , \quad \underline{r}' = \sum_{k=1}^{3} \xi'_k \underline{b}'_k \, .$$

Connect the systems : $\underline{r}' = \underline{r} + \underline{r}_0$

$$\sum_{k=1}^{3} \xi'_k \underline{b}'_k = \underline{r}_0 + \sum_{j=1}^{3} \xi_j \underline{b}_j$$

$$= \sum_{k=1}^{3} \rho_k \underline{b}'_k + \sum_{j=1}^{3} \xi_j \sum_{k=1}^{3} \tau_{kj} \underline{b}'_k$$

$$= \sum_{k=1}^{3} \rho_k \underline{b}'_k + \sum_{k=1}^{3} \sum_{j=1}^{3} \tau_{kj} \xi_j \, \underline{b}'_k$$

$$= \sum_{k=1}^{3} \left(\rho_k + \sum_{j=1}^{3} \tau_{kj} \xi_j \right) \underline{b}'_k$$

$$\Longrightarrow \xi'_k = \rho_k + \sum_{j=1}^{3} \tau_{kj} \xi_j \, , \quad k = 1, 2, 3$$

or $\quad x' = x_0 + Tx \iff x = -T^{-1} x_0 + T^{-1} x'$

Choose $S = T^{-1}$: $\quad x = -S x_0 + S x'$

Example $\{\underline{b}_1, \underline{b}_2, \underline{b}_3\}$ is a basis.

$$\begin{cases} \underline{b}_1' = 2\,\underline{b}_1 + 2\underline{b}_2 + 7\underline{b}_3 \\ \underline{b}_2' = \underline{b}_2 + 9\,\underline{b}_3 \\ \underline{b}_3' = 6\,\underline{b}_1 + 8\underline{b}_2 \end{cases}$$

Is $\{\underline{b}_1', \underline{b}_2', \underline{b}_3'\}$ a basis?

$$x = Sx' = \underbrace{\begin{pmatrix} 2 & 0 & 6 \\ 2 & 1 & 8 \\ 7 & 9 & 0 \end{pmatrix}}_{\text{invertible or not ?}} \begin{pmatrix} \xi_1' \\ \xi_2' \\ \xi_3' \end{pmatrix}$$

A: Yes, it is invertible!

Find α and β such that $\underline{v} = \alpha\,\underline{b}_1' + \beta\,\underline{b}_2' + \underline{b}_3'$ has constant coordinates:

$$\underline{v} = \gamma\,\underline{b}_1 + \gamma\,\underline{b}_2 + \gamma\,\underline{b}_3$$

$$\underline{x} = Sx' = S\begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix} = \underline{\begin{pmatrix} 2\alpha + 6 \\ 2\alpha + \beta + 8 \\ 7\alpha + 9\beta \end{pmatrix}}$$

$$\Rightarrow \quad \alpha = \frac{24}{5}, \quad \beta = -2$$