Linear Transformations

Definition Let $F: V \to W$. F is a linear transform, if

(1) $F(x+y) = F(x) + F(y) \quad \forall x, y \in V$

(2) $F(\lambda x) = \lambda F(x) \quad \forall x \in V, \lambda \in \mathbb{R}$

Here: $F: \mathbb{R}^n \to \mathbb{R}^p$

→ F could well be matrix-vector multiplication

Example $p_1(x) = x^2 + x + 1$, $p_2(x) = 2x^2 - 1$

Let F be differentiation D:

$$D(p_1(x) + p_2(x)) = D(p_1(x)) + D(p_2(x))$$
$$D(\lambda p_1(x)) = \lambda D(p_1(x))$$

→ Conclusion: D is a linear transform!

Every linear transform has a matrix representation.

Theorem Let $F: \mathbb{R}^n \to \mathbb{R}^p$ be a linear transform, which maps the natural basis vectors $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ onto the vectors $a_1, a_2, \dots, a_n \in \mathbb{R}^p$:

$$F(e_k) = a_k, \quad k = 1, \dots, n.$$

Let $\underset{p \times n}{A} = (a_1 \; a_2 \; \dots \; a_n)$, then $F(x) = Ax \;\; \forall x \in \mathbb{R}^n$

Proof

Let $x \in \mathbb{R}^n$: $x = \sum_{k=1}^{n} \xi_k e_k$

F is a linear transform :

$$F(x) = F\left(\sum_{k=1}^{n} \xi_k e_k\right)$$

$$= \sum_{k=1}^{n} F(\xi_k e_k) = \sum_{k=1}^{n} \xi_k F(e_k)$$

$$= \sum_{k=1}^{n} \xi_k a_k = Ax \qquad \square$$

SIDE STEP : $z_1, z_2 \in \mathbb{C}$: $z_1 = x_1 + i y_1$

$z_2 = x_2 + i y_2$

$$z_2 z_1 = (x_2 + i y_2)(x_1 + i y_1)$$

$$= x_2 x_1 + i(x_2 y_1) + i(y_2 x_1) - y_2 y_1$$

$$= x_2 x_1 - y_2 y_1 + i(x_2 y_1 + y_2 x_1) \in \mathbb{C}$$

$$\underbrace{\begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix}}_{\text{"} z_2 \times \text{"}} \underbrace{\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}}_{\text{"} z_1 \text{"}} = \begin{pmatrix} x_2 x_1 - y_2 y_1 \\ x_2 y_1 + y_2 x_1 \end{pmatrix}$$

$\rightarrow$ "z_2 times" "z_1"

Geometric Transforms   Euclidean transforms

The Euclidean transforms preserve the shape of the geometric object.

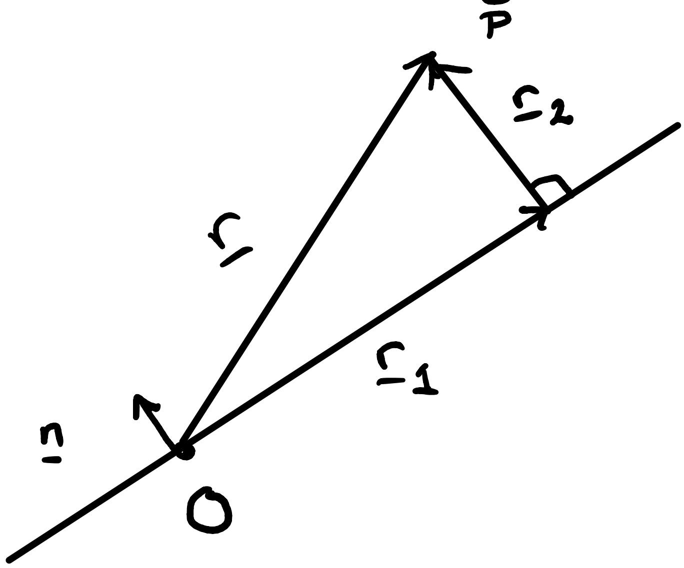
Four : Translation, reflection, rotation, scaling

(1) Translation : $T_{\underline{a}}(\underline{r}) = \underline{r}' = \underline{r} + \underline{a}$

This is not a linear transform!

(2) Reflection

→ Origin is a fixed point, i.e., The symmetry axis (plane) goes through the origin



$$\underline{r} = \underline{r}_1 + \underline{r}_2$$

$$\underline{r}_2 = (\underline{n} \cdot \underline{r})\underline{n}$$

$$\underline{r}_1 = \underline{r} - \underline{r}_2$$
$$= \underline{r} - (\underline{n} \cdot \underline{r})\underline{n}$$

The image : $\underline{r}' = \underline{r}_1 - \underline{r}_2 = \underline{r} - 2(\underline{n} \cdot \underline{r})\underline{n}$

$$H_{\underline{n}}(\underline{r}) = \underline{r}' = \underline{r} - 2(\underline{n} \cdot \underline{r})\underline{n}$$

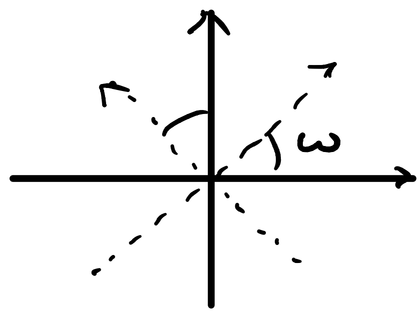
$$H_{\underline{n}}(\underline{r}) = \underline{r} - 2(\underline{n}\cdot\underline{r})\underline{n} = \underline{r}'$$

$\mathbb{R}^n$:

$$\begin{aligned} x' &= x - 2(n^T x)n \\ &= x - 2n(n^T x) \\ &= x - 2(nn^T)x \\ &= (I - 2nn^T)x \\ &= H_n x \end{aligned}$$

$$\begin{aligned} H_n H_n &= (I - 2nn^T)(I - 2nn^T) \\ &= I - 2nn^T - 2nn^T + 4n\underbrace{(n^T n)}_{=1}n^T \\ &= I \end{aligned}$$

(3) Rotation



Images of the axes:

$(1,0) \longrightarrow (\cos\omega, \sin\omega)$

$(0,1) \longrightarrow (-\sin\omega, \cos\omega)$

$$U_\omega = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$$

(4) Scaling $\quad S_\lambda(\underline{r}) = \underline{r}' = \lambda\underline{r}$

(5) General fixed point : $P_0 \mathrel{\hat{=}} \underline{r}_0$

$$F_0(\underline{r}) = F(\underline{r} - \underline{r}_0) + \underline{r}_0$$

that is

$$
\begin{aligned}
x' &= A(x - x_0) + x_0 \\
&= Ax + (x_0 - Ax_0) \\
&= Ax + b \longrightarrow \text{affine transform}
\end{aligned}
$$

(See change of basis!)