DETERMINANT

Today we shall cover many definitions.

<u>Hand Waving</u> Determinant is a number which represents either some area, volume, or generalised volume.

$$\mathrm{Det} : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R} \; ; \quad \mathrm{Det}(A) = |A| = \det A$$

<u>Definition</u> $|A| =$ the product of pivots

$\longrightarrow$ computational complexity : Gaussian elimination

Properties: (I) – (III) are sufficient

(I) $\det I = 1$

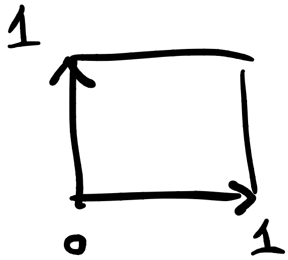
(II) Changing two rows changes the sign:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = - \begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

(III) Linearity on rows:

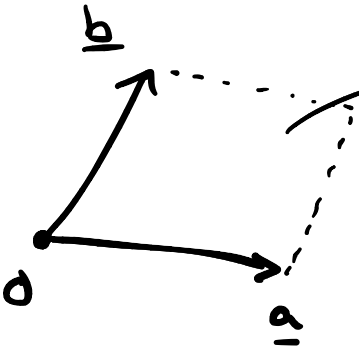
$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$



Area = 1

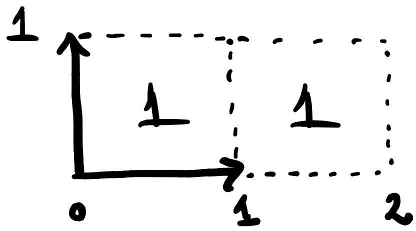
$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (\mathrm{I})$$



Area $= |\det B|$ $\qquad (\mathrm{II})$

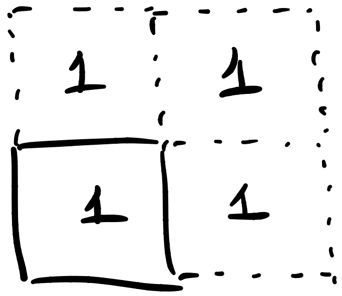
$$B = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

$$\tilde{B} = \begin{pmatrix} \alpha_2 & \beta_2 \\ \alpha_1 & \beta_1 \end{pmatrix}$$



$$B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad (\mathrm{III})$$

$$|B| = 2\left|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 2$$



Area = 4 $\qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$$|B| = 2 \cdot 2 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 4$$

Seven additional ones: $\underset{n\times n}{A}$

4) If there are two equal rows, then $\det A = 0$.

5) Row operation does not change the value of the determinant.

$$\begin{vmatrix} a+\lambda c & b+\lambda d \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} \lambda c & \lambda d \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \lambda \underbrace{\begin{vmatrix} c & d \\ c & d \end{vmatrix}}_{=0} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

6) If there is a row of zeros, then $\det A = 0$

7) For triangular matrices the determinant is the product of the diagonal elements.

$$\begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} = 2 \cdot 3 \cdot \underbrace{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}}_{=1} = 6$$

8) For non-invertible (i.e., singular) matrices determinant is zero.

9) $|AB| = |A||B|$ $\longrightarrow$ theorem

10) $\det A^T = \det A$

Practical rules for pen & paper:

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \qquad 2!$$

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = \alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32} - \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32} \qquad 3!$$

$$= \begin{vmatrix} \alpha_{11} & & \\ & \alpha_{22} & \alpha_{23} \\ & \alpha_{32} & \alpha_{33} \end{vmatrix} + \begin{vmatrix} & \alpha_{12} & \\ \alpha_{21} & & \alpha_{23} \\ \alpha_{31} & & \alpha_{33} \end{vmatrix} + \begin{vmatrix} & & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \\ \alpha_{31} & \alpha_{32} & \end{vmatrix}$$

<u>Definition</u> Let M_{ij} be a $(n-1)\times(n-1)$ matrix (A with the i^{th} row and j^{th} column removed). Let $C_{ij} = (-1)^{i+j} \det M_{ij}$.

Then $\det A = \alpha_{i1}C_{i1} + \alpha_{i2}C_{i2} + \dots + \alpha_{in}C_{in}$.

Combinatorial definition:

$$\det A = \sum_{P \in \text{Permutation matrices}} \det(P)\, \alpha_{1\alpha}\, \alpha_{2\beta} \cdots \alpha_{n\omega}$$

where $P\begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \vdots \\ \omega \end{pmatrix}$; Greek alphabet!

Application : Vector algebra : Cross Product

Definition Let $\underline{a}$ and $\underline{b}$ be two vectors in space. $\underline{a} \times \underline{b}$ is a vector :

(i) $\| \underline{a} \times \underline{b} \| = \| \underline{a} \| \| \underline{b} \| \sin \measuredangle (\underline{a}, \underline{b})$

(ii) $\underline{a} \times \underline{b} \perp \underline{a}$, $\underline{a} \times \underline{b} \perp \underline{b}$

(iii) $\{ \underset{\substack{\uparrow \\ \text{thumb}}}{\underline{a}}, \underset{\substack{\uparrow \\ \text{first}}}{\underline{b}}, \underset{\substack{\uparrow \\ \text{middle}}}{\underline{a} \times \underline{b}} \}$ is a right-handed system

Theorem $\underline{a} = \alpha_1 \underline{i} + \alpha_2 \underline{j} + \alpha_3 \underline{k}$
$\underline{b} = \beta_1 \underline{i} + \beta_2 \underline{j} + \beta_3 \underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}$$

Definition Scalar Triple Product

$$[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c}$$

$$= \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$$

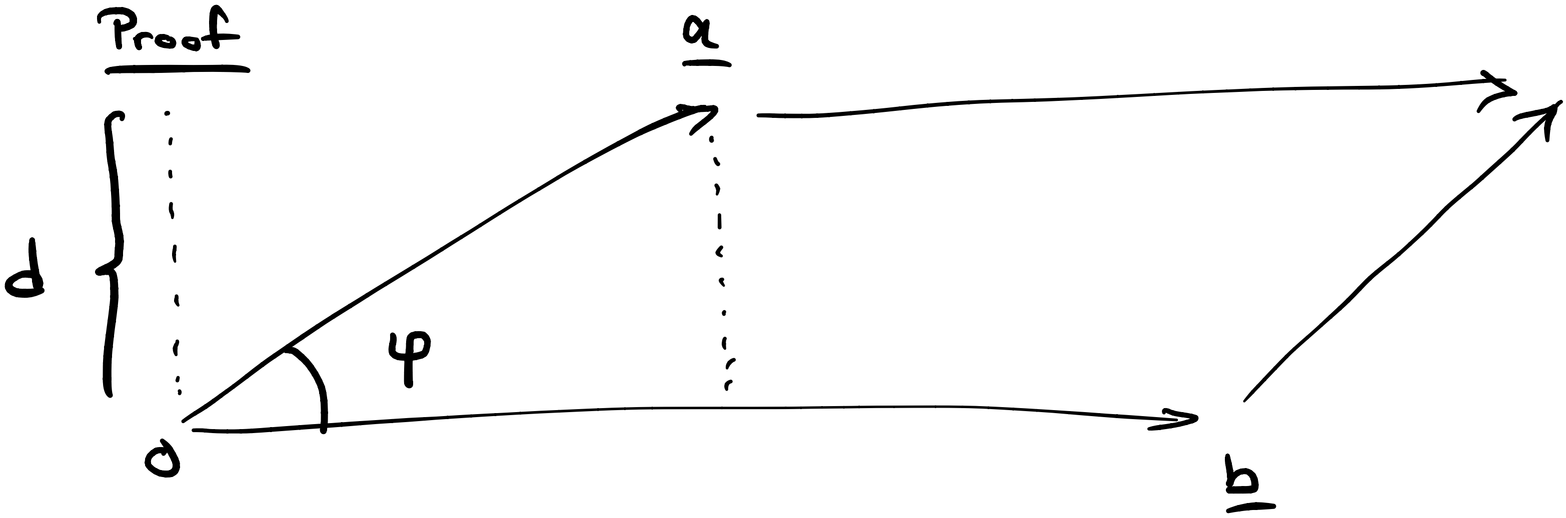
Notice: $[\underline{a}, \underline{b}, \underline{c}] = [\underline{c}, \underline{a}, \underline{b}] = [\underline{b}, \underline{c}, \underline{a}]$.

Theorem

Area of a parallelogram spanned by $\underline{a}$ and $\underline{b}$ is $\|\underline{a} \times \underline{b}\|$.

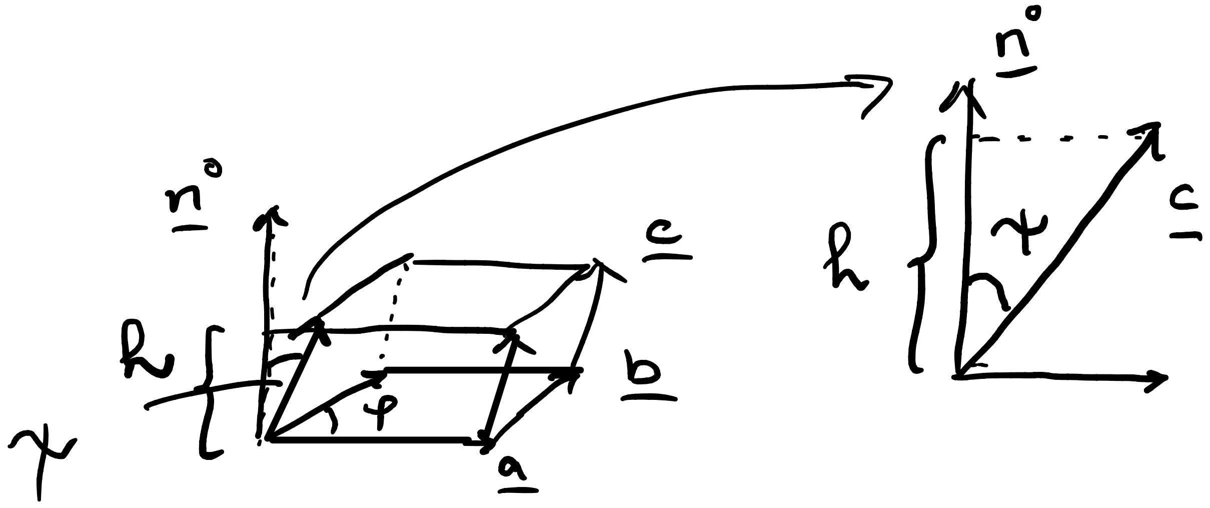
Volume of an object spanned by $\{\underline{a}, \underline{b}, \underline{c}\}$ is $|[\underline{a}, \underline{b}, \underline{c}]|$.

Proof



$d = \|\underline{a}\| \sin\varphi \implies \text{Area} = d \cdot \|\underline{b}\|$

$\implies \text{Area} = \|\underline{a}\| \|\underline{b}\| \sin\varphi = \|\underline{a} \times \underline{b}\|$



$$
\begin{aligned}
\text{Volume} &= \text{height} \cdot \text{area of base} \\
&= \|\underline{a} \times \underline{b}\| \, \|\underline{c}\| \cos\psi \\
&= \|\underline{a} \times \underline{b}\| \, \left| \underline{n}^\circ \cdot \underline{c} \right| \\
&= \|\underline{a} \times \underline{b}\| \, \left| \frac{\underline{a} \times \underline{b}}{\|\underline{a} \times \underline{b}\|} \cdot \underline{c} \right| \\
&= \left| \underline{a} \times \underline{b} \cdot \underline{c} \right| = \left| [\underline{a}, \underline{b}, \underline{c}] \right| \qquad \square
\end{aligned}
$$