EIGENVALUES AND EIGENVECTORS

<u>Definition</u> Real or complex valued scalar λ is an eigenvalue of a matrix $\underset{n\times n}{A}$, if there exist a vector $x \neq 0$ such that

$$Ax = \lambda x.$$

The eigenvectors x are the solutions of $Ax = \lambda x$.

We are interested in eigenpairs (λ, x).

<u>Example</u> $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

$$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \quad Rx_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \lambda_1 = 1$$

$$x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} : \quad Rx_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \lambda_2 = -1$$

$$= -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda_2 x_2$$

<u>Example</u> $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$; $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

A rotation, $\varphi = \frac{\pi}{4} \Rightarrow \lambda \in \mathbb{C}$

<u>Theorem</u> λ is an eigenvalue, if and only if

$$\det(A - \lambda I) = 0.$$

<u>Proof</u> $\quad Ax = \lambda x \iff (A - \lambda I)x = 0$

If $\det(A - \lambda I) \neq 0$, then $A - \lambda I$ is invertible, and $x = 0$ is the unique solution.
Otherwise $x \neq 0$ and by definition λ is an eigenvalue.

$\square$

<u>Definition</u> $\quad p(\lambda) = \det(A - \lambda I)$
is the characteristic polynomial.

Notice: The roots are the same for
$$\det(A - \lambda I) = 0 = \det(\lambda I - A)$$

<u>Example</u> $\quad A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

$$\iff (2-\lambda)(-1-\lambda) - (-1)1 = 0$$

$$\iff \lambda^2 - \lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2} \in \mathbb{R}$$

<u>Example</u> $\quad A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$p(\lambda) = \left(\frac{1}{\sqrt{2}} - \lambda\right)^2 + \frac{1}{2} = 0, \quad \lambda_{1,2} = \underbrace{\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}}_{\in\, \mathbb{C}}$$

Example $\quad A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$; $\lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}$

Linear system: 2x2

$$\longrightarrow \begin{array}{cc|c} 2 - \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) & 1 & 0 \\ -1 & -1 - \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) & 0 \end{array}$$

$$\Rightarrow \xi_1 = \left(-\frac{3}{2} - \frac{\sqrt{5}}{2}\right) \xi_2 \; ; \; \xi_2 \text{ is free}$$

We get

$$x_1 = \sigma \begin{pmatrix} -\frac{3}{2} - \frac{\sqrt{5}}{2} \\ 1 \end{pmatrix}, \; \sigma \in \mathbb{R}$$

→ IMPORTANT: It's the direction that counts!

$$\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} - \frac{\sqrt{5}}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -3 - \sqrt{5} + 1 \\ 3/2 + \sqrt{5}/2 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 - \sqrt{5} \\ 1/2 + \sqrt{5}/2 \end{pmatrix} = \lambda_1 \begin{pmatrix} -3/2 - \frac{\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

Here: $x_1 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$; $\longrightarrow (-1)\xi_1 + (-1-\lambda_1)\xi_2 = 0$

$$\Rightarrow \xi_1 = (-1-\lambda_1)\xi_2$$

Power Iteration

$\underset{n\times n}{A}$; Eigenpairs (λ_i, v_i)

$\{v_1, v_2, \ldots, v_n\}$ are linearly independent.

$x \in \mathbb{R}^n$; $x = \sum_{k=1}^{n} \xi_k v_k$ $\quad \{v_i\}$ is a basis

Then $Ax = A\left(\sum \xi_k v_k\right)$

$$= \sum_{k=1}^{n} \xi_k A v_k = \sum_{k=1}^{n} \xi_k \lambda_k v_k$$

and $A^k x = \xi_1 \lambda_1^k v_1 + \xi_2 \lambda_2^k v_2 + \ldots + \xi_n \lambda_n^k v_n$

Assuming, that $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$

$$|\lambda_1|^k \gg |\lambda_j|^k \,, \quad j > 1, \; k \gg 0 ,$$

If $\xi_1 \neq 0$, then in the sequence

$x^{(k)} = A^k x$ eventually

$$\Longrightarrow \quad x^{(k)} \simeq \underbrace{\xi_1 \lambda_1^k}_{\text{scalar}} v_1$$

λ_1 can be recovered : $\lambda_1 \simeq x_i^{(k)} / x_i^{(k-1)}$, $x_i^{(k-1)} \neq 0$

Summary

(1) Form $p(\lambda) = \det(A - \lambda I)$

(2) Find the roots : $p(\lambda) = 0$

(3) Solve $Ax_i = \lambda_i x_i$ for all λ_i.

Notice: If $\lambda = 0$, then A is singular.

$$|AB| = |A||B|$$

Assume that $|B| = 1$. "Volume" of AB is scaled by $|A|$.

Two useful identities:

(i) $\det A = \prod_{i=1}^{n} \lambda_i$

(ii)
$$\begin{aligned} \operatorname{tr} A &= \alpha_{11} + \alpha_{22} + \dots + \alpha_{nn} \\ &= \lambda_1 + \lambda_2 + \dots + \lambda_n \\ &= \sum_{i=1}^{n} \alpha_{ii} = \sum_{i=1}^{n} \lambda_i \end{aligned}$$

$\operatorname{tr} A$ is the trace of A.