DIAGONALISATION

<u>Eigenvalue problem:</u> $\underset{n\times n}{A}$

(1) Find $p(\lambda) = \det(A - \lambda I) = 0$
(2) Find the roots of $p(\lambda)$.
(3) Solve $Ax_i = \lambda x_i$ for all λ_i.

If λ is an eigenvalue, then $\det(A - \lambda I) = 0$.
But also the rank of $A - \lambda I < n$, i.e., the columns of $A - \lambda I$ are linearly dependent.

<u>Theorem</u> Let $\underset{n\times n}{S} = (x_1\ x_2 \dots x_n)$, where x_i are the linearly independent eigenvectors of $\underset{n\times n}{A}$. Then

$$S^{-1} A S = \Lambda = \mathrm{diag}(\lambda_1\ \lambda_2 \dots \lambda_n)$$

Naturally, $AS = S\Lambda$ ← and

$$A = S \Lambda S^{-1} \leftarrow$$

→ Notice : $Ax_1 = \lambda_1 x_1$

$$A(x_1\ x_2) = (x_1\ x_2)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

→ $Ax = S\Lambda S^{-1} x$

(i) S^{-1} changes the basis
(ii) Λ is "pure" scaling
(iii) S changes the basis back to the original

Notice:

$$A^k = \underbrace{A \cdot A \cdot \ \ldots \ \cdot A}_{k \text{ times}} = S\Lambda \underbrace{S^{-1} S}_{=I} \Lambda \underbrace{S^{-1} S}_{=I} \ldots \underbrace{S}_{=I} \Lambda S^{-1}$$

$$= S \Lambda^k S^{-1}$$

<u>Theorem</u> If (λ_i, v_i) are the eigenpairs of A and $\lambda_i \neq \lambda_j$, $i \neq j$, then $\{v_i\}$ are linearly independent.

If $\underset{n\times n}{A}$ has n such eigenvalues, it is diagonalisable.

<u>Proof</u> (i) $c_1 v_1 + c_2 v_2 = 0$

$$\begin{array}{ll} A\cdot & \left\{ \begin{array}{l} c_1 A v_1 + c_2 A v_2 = 0 \\ c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 = 0 \end{array} \right. \\ \lambda_2 \cdot & \end{array}$$

$$\Leftrightarrow \quad \left\{ \begin{array}{l} c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0 \\ c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 = 0 \end{array} \right.$$

$$\Rightarrow \quad c_1 \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} v_1 = 0 \quad \Rightarrow c_1 = 0$$

Similarly $c_2 = 0$ $\Rightarrow \{v_1, v_2\}$ are lin. independent.

(ii) $\sum_{i=1}^{j} c_i v_i = 0$

for instance: $c_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\ldots(\lambda_1 - \lambda_j) v_1 = 0$

...

$\square$

Example $A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$, $\lambda_1 = 1$, $\lambda_2 = 0.5$

$$\lim_{k\to\infty} A^k = ?$$

$$A = S \Lambda S^{-1} = \begin{pmatrix} 0.6 & 1 \\ 0.4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0.4 & -0.6 \end{pmatrix}$$

Remember the order of eigenvalues and eigenvectors!

$$A^k = S \Lambda^k S^{-1}$$

$$\Rightarrow \lim_{k\to\infty} A^k = S \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$$

Side note: $A^k \xrightarrow{k\to\infty} 0$, if $|\lambda_i| < 1$ for all $i = 1, \dots, n$.

Example $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; $\lambda_{1,2} = 1$, $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\dim N(A - 1 \cdot I) = 1$ (geometric order)

$\lambda = 1$ is a double eigenvalue (algebraic order)

$\Rightarrow$ Orders do not match, A is defective

Example

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{pmatrix}$$

Same eigenvalues, i.e., the same spectra.

However, $\dim N(A - 2\cdot I) = 1 \Rightarrow A$ defective

$\dim N(B - 2\cdot I) = 2 \Rightarrow$

B is diagonalisable

DIGRESSION:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \quad ; \quad e^A = S\left(\sum_{k=0}^{\infty} \frac{\Lambda^k}{k!}\right)S^{-1}$$

SYMMETRIC MATRICES

Theorem Spectral Theorem

Every symmetric matrix is diagonalisable:

$$A = Q \Lambda Q^T, \quad \lambda_i \in \mathbb{R}, \quad Q \text{ orthogonal.}$$

Theorems:

(A) The eigenvalues of a real symmetric matrix are real.

(B) If $\lambda_i \neq \lambda_j$, $i \neq j$, then corresponding eigenvectors are orthogonal.

Not (c) The algebraic and geometric orders are equal for all λ_is.

Proof (A) $\lambda \in \mathbb{C}$. $\quad Ax = \lambda x$ •

then $\quad A\bar{x} = \bar{\lambda}\bar{x}$ or tranpose $\bar{x}^T A = \bar{x}^T \bar{\lambda}$ •

Inner products
$$\begin{cases} \bar{x}^T A x = \bar{x}^T \lambda x & \bullet \\ \bar{x}^T A x = \bar{x}^T \bar{\lambda} x & \bullet \end{cases}$$

$$\Rightarrow \lambda \bar{x}^T x = \bar{\lambda} \underbrace{\bar{x}^T x}_{\|x\|^2} \Rightarrow \operatorname{Im} \lambda = 0 \qquad \square$$

(B) Let $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$,

$A = A^T$, $\lambda_1 \neq \lambda_2$.

$$(\underline{\lambda_1 x})^T y = (Ax)^T y = x^T A^T y$$

$$= x^T A y = \underline{x^T(\lambda_2 y)}$$

$$\Rightarrow x^T y = 0$$

$\square$

$$\lambda_1 (x^T y) = \lambda_2 (x^T y)$$

$$\lambda_1 \neq \lambda_2 \neq 0 \quad \Rightarrow x^T y = 0$$