REVISION

MARGINAL MATTERS AND ECHOS FROM THE FUTURE

Determinants : Two applications

$$\det A = \alpha_{i1} C_{i1} + \alpha_{i2} C_{i2} + \ldots + \alpha_{in} C_{in}$$

$$C_{ij} = (-1)^{i+j} \det M_{ij} \qquad \text{(cofactors)}$$

$M_{ij} \longrightarrow$ A with i^{th} row and j^{th} column removed ; $(n-1) \times (n-1)$

(1) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix} = \frac{C^T}{\det A}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

For the general case : $$(A^{-1})_{ij} = \frac{C_{ji}}{\det A}$$

Why $$AC^T = (\det A) I \quad ?$$

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$C_{11} = (-1)^{1+1} \det M_{11} = 1 \cdot d$$

$$C_{12} = (-1)^{1+2} \det M_{12} = -1 \cdot c$$

$$\Rightarrow \quad c \cdot C_{11} + d \cdot C_{12} = 0 = \alpha_{21} C_{11} + \alpha_{22} C_{12}$$

$\Rightarrow$ There is implicit duplication of rows!

$\Rightarrow$ This observations holds for $A_{n \times n}$!

(2) Cramer's Rule : $x^{-1} = A^{-1} b = \dfrac{C^T b}{\det A}$

$$x_j = \frac{\det B_j}{\det A},$$

where $B_j = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \beta_1 & \cdots & \alpha_{1n} \\ \vdots & & & & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \beta_n & \cdots & \alpha_{nn} \end{pmatrix}$

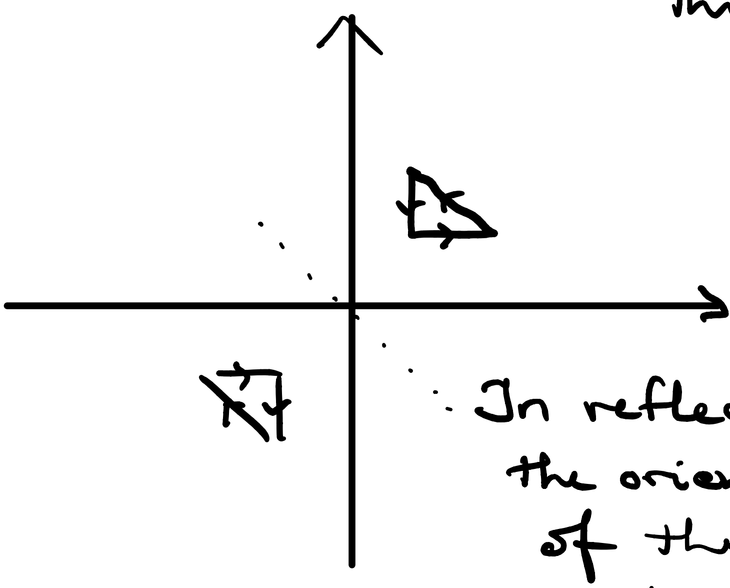
Example

$$\underbrace{\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}}_{A}\underline{\underline{\begin{pmatrix}1\\ 1\end{pmatrix}}} = \underbrace{\begin{pmatrix}3\\ 3\end{pmatrix}}_{b}$$

$$\underline{\underline{x_1}} = \frac{\begin{vmatrix}3 & 2\\ 3 & 1\end{vmatrix}}{\begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix}} = \frac{3 - 3\cdot 2}{1-4} = \underline{\underline{1}}$$

$$\underline{\underline{x_2}} = \frac{\begin{vmatrix}1 & 3\\ 2 & 3\end{vmatrix}}{\begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix}} = \frac{-3}{-3} = \underline{\underline{1}}$$

Think in 3D!



In reflection, the orientation of the system changes.

SINGULAR VALUE DECOMPOSITION (SVD)

$$\underset{m \times n}{A} = U \Sigma V^T \; ; \; \Sigma \text{ diagonal}, U, V \text{ orthogonal}$$

<u>Left</u> singularvectors:

$$AA^T = U\Sigma V^T V \Sigma U^T = U \Sigma^2 U^T$$

$$\Rightarrow (AA^T) U = U \Sigma^2$$

<u>Right</u> singularvectors:

$$A^T A = V \Sigma U^T U \Sigma V^T = V \Sigma^2 V^T$$

$$\Rightarrow (A^T A) V = V \Sigma^2$$

$$(AA^T)(Av_j) = \sigma_j^2 (Av_j)$$

eigenvectors of AA^T

Scaling? $\quad v_j^T A^T A v_j = \sigma_j^2 v_j^T v_j$

$$\Rightarrow \| Av_j \|^2 = \sigma_j^2$$

$\Rightarrow$ We get a unit eigenvector

$$Av_j / \sigma_j = u_j$$

We conclude :

$$AV = U\Sigma$$

Compression :

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$$

$$= \underset{m\times r}{U}\,\underset{r\times r}{\Sigma}\,\underset{r\times n}{V^T} \quad , \quad r \text{ is the rank}$$

$$= \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

If $\sigma_1 \geq \sigma_2 \geq \dots$ decreases rapidly,

then the sum can be a reasonable approximation even with a small number of terms.

Example : $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 10\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ; \quad \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

...

$$U = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \; ; \; \Sigma = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{pmatrix}$$

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$A = U \Sigma V = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{10} & \\ & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

$$= \frac{1}{\cancel{\sqrt{10}}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} (\cancel{\sqrt{10}}) \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

Extended Synthesis : $Ax = b$

$\underset{n\times n}{A}$: The following conditions are equivalent :

1) A^{-1} exist, i.e., A is invertible

2) $\det A \neq 0$

3) The columns are linearly independent

4) The rows are linearly independent

5) $Ax = b$ has a unique solution for all b

6) $Ax = 0$ has a unique solution $x = 0$

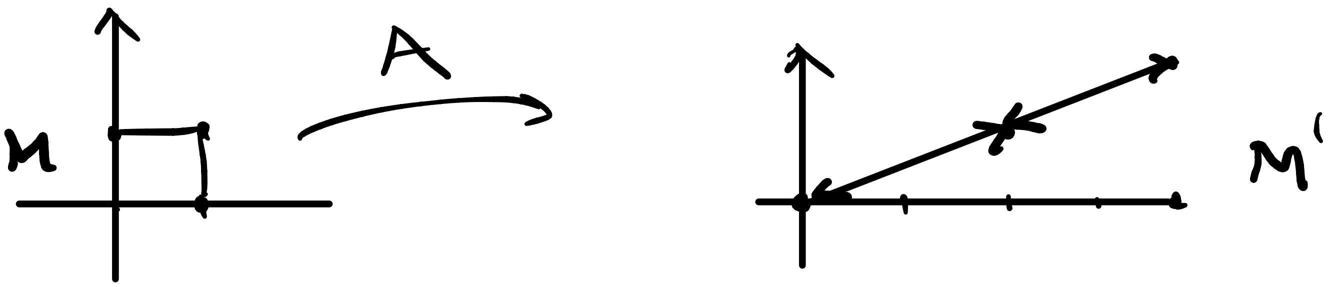
7) Zero cannot be an eigenvalue.

Geometric Interpretation

Unit square: $M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$; singular

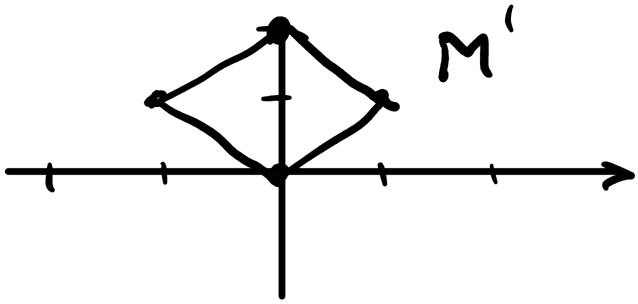
$M' = AM = \begin{pmatrix} 0 & 2 & 4 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$



$\det A = 0$

$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $\Rightarrow$ the area of M' is $\det A = 2$

$M' = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{pmatrix}$



Central Topics :

(1) Computations ; definitions

$$A+B \; ; \; AB \; ; \; (AB)^T$$

(2) Linear systems

→ classification of solutions
→ counting them
→ rank

(3) $PA = LU$

(4) $Ax = \lambda x$; $A = S \Lambda S^{-1}$;

$$A = Q \Lambda Q^T$$

$$A = Q \Lambda Q^T \; ; \; Q = (v_1 \; v_2 \dots v_n)$$

$$= \sum_{k=1}^{n} \lambda_k v_k v_k^T$$