MATRIX ALGEBRA

VECTORS

Consider a set $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n \text{ times}}$

$$= \{ (\xi_1, \ldots, \xi_n) \mid \xi_1, \xi_2, \ldots, \xi_n \in \mathbb{R} \}$$

The elements of $\mathbb{R}^n$ are vectors ; column vectors :

$$x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^2 .$$

Let us distinguish between "physical" vectors and vectors in $\mathbb{R}^n$.

We denote physical vectors $\underline{a}$, but use no decoration for column vectors.

For our purposes it is sufficient to recognise that in $\mathbb{R}^n$ the origin is fixed : $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, whereas in physical system the origin can be chosen for every coordinate system separately.

$\underline{\mathbb{R}^n}$: Let us define two operations :

(i) addition : $x, y \in \mathbb{R}$; $x + y \in \mathbb{R}^n$

(ii) multiplication by a scalar :

$$\alpha \in \mathbb{R} , \; x \in \mathbb{R}^n \; ; \quad \alpha x \in \mathbb{R}^n$$

$$x = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}, \; y = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_3 \end{pmatrix}, \quad x + y = \begin{pmatrix} \xi_1 + \eta_1 \\ \xi_2 + \eta_2 \\ \vdots \\ \xi_n + \eta_n \end{pmatrix} ; \; \alpha x = \begin{pmatrix} \alpha \xi_1 \\ \alpha \xi_2 \\ \vdots \\ \alpha \xi_n \end{pmatrix}$$

Let us shamelessly abuse the different systems and agree that

$$\underline{r} = x_1 \underline{i} + x_2 \underline{j} + x_3 \underline{k} \;\hat{=}\; x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where $\{ 0, \underline{i}, \underline{j}, \underline{k} \}$ coincides with the vectors

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \in \mathbb{R}^3 .$$

Scalar product (dot product, inner product)

$$\underline{a} \cdot \underline{b} = \begin{cases} \| \underline{a} \| \, \| \underline{b} \| \cos \measuredangle (\underline{a}, \underline{b}) , & \text{if } \underline{a} \neq \underline{0} , \underline{b} \neq \underline{0} \\ 0 , & \text{if } \underline{a} = \underline{0} \text{ or } \underline{b} = \underline{0} \end{cases}$$

Orthogonality : $\underline{a} \cdot \underline{b} = 0$ i.e. $\underline{a} \perp \underline{b}$

The scalar component of the vector $\underline{b}$ in the direction of $\underline{a}$:

$$\| \underline{b} \| \cos \measuredangle (\underline{a}, \underline{b}) = \frac{\underline{a}}{\| \underline{a} \|} \cdot \underline{b} = \underline{a}^0 \cdot \underline{b}$$

The vector component : $(\underline{a}^0 \cdot \underline{b}) \, \underline{a}^0$

$$\left. \begin{aligned} \text{If } \quad \underline{a} &= \alpha_1 \underline{i} + \alpha_2 \underline{j} + \alpha_3 \underline{k} \\ \underline{b} &= \beta_1 \underline{i} + \beta_2 \underline{j} + \beta_3 \underline{k} \end{aligned} \right\} \; \underline{a} \cdot \underline{b} = \alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3$$

Using our agreement above : $\underline{a} \;\hat{=}\; a$, $\underline{b} \;\hat{=}\; b$,

define $\quad a^T b = \sum_{i=1}^{3} \alpha_i \beta_i$, where $a^T = (\alpha_1 \; \alpha_2 \; \alpha_3)$

or row vector.

Note: Using the scalar product we can define angles between vectors in $\mathbb{R}^n$.

Lines and planes

Line: $\underline{r} = \underline{r}_0 + \tau \underline{t}$, $\underline{t} \neq \underline{0}$, $\tau \in \mathbb{R}$

Plane: $\underline{r} = \underline{r}_0 + \sigma \underline{s} + \tau \underline{t}$, $\underline{s}, \underline{t} \neq \underline{0}$, $\sigma, \tau \in \mathbb{R}$

A line on a plane is defined by a point and the normal:

$$\underline{n} \cdot (\underline{r} - \underline{r}_0) = 0$$

Exactly the same equation defines a plane in $\mathbb{R}^3$!

Let $\underline{n} = n_1 \underline{i} + n_2 \underline{j} + n_3 \underline{k}$;

$$n_1 x + n_2 y + n_3 z = d, \quad \text{where } d = \underline{n} \cdot \underline{r}_0,$$

is the coordinate form of the plane.

Hence, a line in space cannot have a coordinate form!

Linear combination of vectors

Consider two vectors $a = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$, $b = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$, and assume that they are not parallel.

Every vector $\xi a + \eta b$ lies on the plane spanned by a and b, $\xi, \eta \in \mathbb{R}$.

The vector $\xi a + \eta b$ is a linear combination of a and b.

Formally: $\xi a + \eta b \in \mathrm{span}(\{a, b\})$.

Linear Equations

$$\begin{cases} 2x - y = 1 \\ x + y = 5 \end{cases} \quad \text{or} \quad x\begin{pmatrix} 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

So, solving a linear system is finding the scalars in the linear combination of the column vectors.

That is, for the solution to exist here : $\begin{pmatrix} 1 \\ 5 \end{pmatrix} \in \text{span}\left(\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}\right)$

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix} \in \text{span}\left(\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}\right)$$

In general, any linear system has either

(A) 1 unique solution

(B) no solutions

(C) infinite number of solutions

Consider three planes in $\mathbb{R}^3$:

(B) two or more parallel planes ;
every pair of planes intersects in a line, and those lines are parallel

(C) all planes have a line in common