Eigenvalues and Eigenvectors

Definition Real or complex valued scalar λ is an eigenvalue of a matrix $\underset{n\times n}{A}$, if there exists a vector $x \neq 0$ such that

$$Ax = \lambda x.$$

The eigenvectors x are the solutions of $Ax = \lambda x$.

We are interested in eigenpairs $(\lambda, x\}$.

Example $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$: $Rx_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \lambda_1 = 1$

$x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$: $Rx_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \lambda_2 = -1$

Example $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$; $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

A is a rotation matrix ; $\varphi = \pi/4$

It is clear that $\lambda \in \mathbb{R}$ cannot exist ! Thus, $\lambda \in \mathbb{C}$.

How to find the eigenvalues ? For our purposes the following is sufficient :

Theorem λ is an eigenvalue, if and only if

$$\det(A - \lambda I) = 0.$$

Proof $Ax = \lambda x \iff (A - \lambda I)x = 0$

If $\det(A - \lambda I) \neq 0$, then $A - \lambda I$ is invertible and $x = 0$ is the unique solution. Otherwise $x \neq 0$ and by definition λ is an eigenvalue.

$\square$

$\det(A-\lambda I)$ is a polynomial and its roots are the eigenvalues. Of course $\det(\lambda I - A)$ has the same roots.

<u>Definition</u> $\quad p(\lambda) = \det(A-\lambda I)$ is the characteristic polynomial.

<u>Example</u> $\quad A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$; $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0$

$$\Leftrightarrow (2-\lambda)(-1-\lambda) - (-1)1 = 0$$

$$\Leftrightarrow \lambda^2 - \lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2} \in \mathbb{R}$$

<u>Example</u> $\quad A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$; $p(\lambda) = \left(\frac{1}{\sqrt{2}} - \lambda\right)^2 + \frac{1}{2} = 0$

$$\lambda_{1,2} = \frac{1}{\sqrt{2}} \pm i \frac{1}{\sqrt{2}} \in \mathbb{C}$$

What about eigenvectors?

<u>Example</u> $\quad A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$; $\lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}$;

$$\begin{pmatrix} 2 - \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) & 1 \\ -1 & -1 - \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) \end{pmatrix} x_1 = 0$$

No need to use elimination : $\xi_1 = \left(-\frac{3}{2} - \frac{\sqrt{5}}{2}\right) \xi_2$

ξ_2 is free : $\quad x_1 = \sigma \begin{pmatrix} -\frac{3}{2} - \frac{\sqrt{5}}{2} \\ 1 \end{pmatrix}$, $\sigma \in \mathbb{R}$

<u>Important</u> : The direction is important, any scaling can be chosen!

Power Iteration

Let $\underset{n\times n}{A}$ such that its eigenvectors $v_1, v_2, \dots, v_n$ are linearly independent and the corresponding eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_n$.
The vector x has its unique coordinates:

$$x = \sum_{k=1}^{n} \xi_k v_k \qquad \left(\{v_i\} \text{ is a basis} \right)$$

Then $$Ax = A\left(\sum_{k=1}^{n} \xi_k v_k\right) = \sum_{k=1}^{n} \xi_k A v_k = \sum_{k=1}^{n} \xi_k \lambda_k v_k$$

and $$A^k x = \xi_1 \lambda_1^k v_1 + \xi_2 \lambda_2^k v_2 + \dots + \xi_n \lambda_n^k v_n .$$

Let $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$. Assuming $|\lambda_1|^k \gg |\lambda_j|^k$, $j > 1$, $k \gg 0$:

If $\xi_1 \neq 0$, then in the sequence $x^{(k)} = A^k x$ eventually

$$x^{(k)} \simeq \xi_1 \lambda_1^k v_1 .$$

λ_1 can be recovered: $\lambda_1 \simeq x_i^{(k)} / x_i^{(k-1)}$, if $x_i^{(k-1)} \neq 0$.

As you can see the order have been reversed, first we have found the eigenvector and only then the eigenvalue.
This is our first example of iterative methods.

Summary

(1) Find $p(\lambda) = \det(A - \lambda I) = 0$.

(2) Find the roots of $p(\lambda)$.

(3) Solve $Ax_i = \lambda_i x_i$ for all λ_i.

Notice: If $\lambda = 0$, then A is singular.

Two useful identities:

(i) $\det A = \prod_{i=1}^{n} \lambda_i$

(ii) $\operatorname{tr} A = \alpha_{11} + \alpha_{22} + \dots + \alpha_{nn} = \lambda_1 + \lambda_2 + \dots + \lambda_n$

$$= \sum_{i=1}^{n} \alpha_{ii} = \sum_{i=1}^{n} \lambda_i$$

$\operatorname{tr} A$ is the trace of A.