Diagonalisation

Theorem Let $\underset{n\times n}{S} = (x_1\ x_2 \dots x_n)$, where x_i are the linearly independent eigenvectors of $\underset{n\times n}{A}$. Then

$$S^{-1} A S = \Lambda = \mathrm{diag}(\lambda_1\ \lambda_2 \dots \lambda_n).$$

Naturally, $AS = S\Lambda$ and $A = S\Lambda S^{-1}$.

This has a remarkable consequence:

$$A^k = \underbrace{A \cdot A \cdots\cdots A}_{k \text{ times}} = S\Lambda S^{-1} S \Lambda S^{-1} \dots S\Lambda S^{-1}$$

$$= S\Lambda^k S^{-1}$$

But, when exactly are the eigenvectors linearly independent?

Theorem If (λ_i, v_i) are the eigenpairs of A and $\lambda_i \neq \lambda_j$, $i \neq j$, then $\{v_i\}$ are linearly independent.

If $\underset{n\times n}{A}$ has n such eigenvalues, it is diagonalisable.

Proof (i) $c_1 v_1 + c_2 v_2 = 0$

$$\begin{cases} c_1 A v_1 + c_2 A v_2 = 0 \\ c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 = 0 \end{cases} \iff \begin{cases} c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0 \\ c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 = 0 \end{cases}$$

$\implies c_1(\lambda_1 - \lambda_2) v_1 = 0 \implies c_1 = 0$

Similarly $c_2 = 0$. Hence $\{v_1, v_2\}$ are linearly independent.

(ii) $\sum_{i=1}^{j} c_i v_i = 0$; Using the same trick as above

$$c_1 \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \underbrace{(\lambda_1 - \lambda_3)}_{\neq 0} \dots \underbrace{(\lambda_1 - \lambda_j)}_{\neq 0} v_1 = 0$$

That is, $S = (v_1\ v_2 \dots v_n)$ can be constructed. □

<u>Example</u> $A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$; $\lambda_1 = 1$, $\lambda_2 = 0.5$

$$A = S \Lambda S^{-1} = \begin{pmatrix} 0.6 & 1 \\ 0.4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0.4 & -0.6 \end{pmatrix}$$

Remember to maintain the order of λ_is and x_is !

$$A^k = S \Lambda^k S^{-1} \Rightarrow \lim_{k \to \infty} A^k = S \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{pmatrix}$$

Side note: $A^k \xrightarrow{k \to \infty} 0$, if $|\lambda_i| < 1$ for all $i = 1, \dots, n$.

<u>Example</u> $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; $\lambda_{1,2} = 1$, $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\dim N(A - 1 \cdot I) = 1$ (geometric order)

$\lambda = 1$ is a double eigenvalue (algebraic order)

$\Rightarrow$ Orders do not match, A is defective.

<u>Example</u> $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{pmatrix}$

Same eigenvalues : i.e. the same spectra.

However, $\dim N(A - 2I) = 1 \Rightarrow A$ defective
$\dim N(B - 2I) = 2 \Rightarrow B$ diagonalisable

Symmetric Matrices

Theorem Spectral theorem

Every symmetric matrix is diagonalisable:

$$A = Q \Lambda Q^T, \quad \lambda_i \in \mathbb{R}, \quad Q \text{ orthogonal}.$$

Theorems:

(A) The eigenvalues of a real symmetric matrix are real.

(B) If $\lambda_i \neq \lambda_j$, $i \neq j$, then corresponding eigenvectors are orthogonal.

Not (C) The algeraic and geometric orders are equal for all λ_i.

Proof (A) $\lambda \in \mathbb{C}$. $Ax = \lambda x$ or $A\bar{x} = \bar{\lambda}\bar{x}$ or transposed $\bar{x}^T A = \bar{x}^T \bar{\lambda}$.

Inner products: $\left\{ \begin{array}{l} \bar{x}^T A x = \bar{x}^T \lambda x \\ \bar{x}^T A x = \bar{x}^T \bar{\lambda} x \end{array} \right. \Rightarrow \lambda \bar{x}^T x = \bar{\lambda} \underbrace{\bar{x}^T x}_{\|x\|^2}$

$$\Rightarrow \operatorname{Im} \lambda = 0 \qquad \square$$

(B) Let $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$; $A = A^T$, $\lambda_1 \neq \lambda_2$.

$$(\lambda_1 x)^T y = (Ax)^T y = x^T A^T y = x^T A y = x^T (\lambda_2 y)$$

$$\Rightarrow x^T y = 0$$

$\square$

Singular Value Decomposition (SVD)

$$\underset{m\times n}{A} = U \Sigma V^T \quad ; \quad \Sigma \text{ diagonal}, \; U, V \text{ orthogonal}$$

Left singularvectors: $AA^T = U\Sigma V^T V \Sigma U^T = U\Sigma^2 U^T$

$$\Rightarrow (AA^T)U = U\Sigma^2$$

Right singularvectors: $A^TA = V\Sigma U^T U \Sigma V^T = V\Sigma^2 V^T$

$$\Rightarrow (A^TA)V = V\Sigma^2$$

Multiply by A: $A\underbrace{A^TAv_j} = \sigma_j^2 \underbrace{Av_j}$

eigenvector of AA^T

$$v_j^T A^T A v_j = \sigma_j^2 v_j^T v_j \quad \Rightarrow \quad \|Av_j\|^2 = \sigma_j^2$$

Therefore we get a unit eigenvector $Av_j / \sigma_j = u_j$.

We conclude with a remarkable identity:

$$AV = U\Sigma .$$

Compression: $\underset{m\times n}{A} = \underset{m\times m}{U} \; \underset{m\times n}{\Sigma} \; \underset{n\times n}{V^T} = \underset{m\times r}{U} \; \underset{r\times r}{\Sigma} \; \underset{r\times n}{V^T}$, where r is the rank.

$$= \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

If $\sigma_1 \geq \sigma_2 \geq \ldots$ decreases rapidly, then the sum can be a reasonable approximation even with a small number of terms.