Gaussian elimination

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$$; Two straight lines intersect at one point.

$$\begin{cases} x + 2y + 3z = 6 \\ 2x + 5y + 2z = 4 \\ 6x - 3y + z = 2 \end{cases}$$; Three planes intersecting at exactly one point.

Remember the corresponding problem of finding the coefficients for the linear combination of the column vectors.

The simplest possible problem:

$$x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The second most simple problem:

$$x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longrightarrow$$ FORWARD SUBSTITUTION

or

$$x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longrightarrow$$ BACKWARD SUBSTITUTION

OUR GOAL:

An algorithm that transforms the original linear system to one of the simple ones.

We want a systematic approach (algorithm!) that does not rely on the structure of the problem, in other words, on human heuristics.

Two observations : The solution of a linear system is not affected by

(a) the order of the equations

(b) if one of the equations is multiplied by a scalar on both sides or two equations are added together

We want :

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & 11 \end{array}\right) \xrightarrow{?} \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & ? & ?? \end{array}\right)$$

Row operation :

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & 11 \end{array}\right) \big\downarrow -3 \qquad \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 8 & 8 \end{array}\right)$$

Notation : $\big\downarrow -3$

$$-3(x-2y) + 3x + 2y = -3 \cdot 1 + 11$$

$$\Leftrightarrow \quad 8y = 8$$

The first unknown x is not present in the transformed equation, i.e., it has been eliminated.

How to choose the operation :

$$\left(\begin{array}{cc|c} \underline{1} & 2 & 1 \\ 3 & 2 & 11 \end{array}\right)$$

$\underline{1}$ is the so-called pivot.
If we want to replace 3 with 0, the scalar should be chosen as

$$-\left(\frac{3}{1}\right) = -3.$$

$\hookrightarrow$ Hence, pivot.

Next:

$$\left[\begin{array}{ccc|c} \underline{2} & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right] \quad -2, \; 1$$

$$\left[\begin{array}{ccc|c} \underline{2} & 4 & -2 & 2 \\ 0 & \underline{1} & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right] \quad -1$$

$$\left[\begin{array}{ccc|c} \underline{2} & 4 & -2 & 2 \\ 0 & \underline{1} & 1 & 4 \\ 0 & 0 & \underline{4} & 8 \end{array}\right] \quad \text{BACKWARD SUBSTITUTION} \Rightarrow \begin{cases} x = -1 \\ y = 2 \\ z = 2 \end{cases}$$

What about the other cases: 0 or ∞ solutions, pivot element 0

1) Parallel straight lines

$$\left[\begin{array}{cc|c} \underline{1} & -2 & 1 \\ 3 & -6 & 11 \end{array}\right] \quad -3 \qquad \left[\begin{array}{cc|c} \underline{1} & -2 & 1 \\ 0 & \underline{0} & 8 \end{array}\right]$$

0 is not an admissible pivot! Contradiction, no solutions.

2) Overlapping straight lines

$$\left[\begin{array}{cc|c} \underline{1} & -2 & 1 \\ 3 & -6 & 3 \end{array}\right] \quad -3 \qquad \left[\begin{array}{cc|c} \underline{1} & -2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

The second equation is identically true.
The second unknown can be chosen freely!

3) Permuting equations

$$\left[\begin{array}{cc|c} 0 & 2 & 4 \\ 3 & -2 & 5 \end{array}\right] \qquad \left[\begin{array}{cc|c} \underline{3} & -2 & 5 \\ 0 & \underline{2} & 4 \end{array}\right]$$