MATRIX

Let $u = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

All linear combinations of the given three vectors are of the form

$$\xi_1 u + \xi_2 v + \xi_3 w = \xi_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \xi_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \xi_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 - \xi_1 \\ \xi_3 - \xi_2 \end{pmatrix}$$

Let us rewrite this as a matrix-vector product:

$$Ax = b,$$

where $A = (u \ v \ w) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$$x = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

and b is the linear combination.

So $$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 - \xi_1 \\ \xi_3 - \xi_2 \end{pmatrix}.$$

However, there is an alternative formulation:

Take the inner products of the rows of A with x.

$$\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T x \\ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}^T x \\ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}^T x \end{pmatrix} = \begin{pmatrix} 1 \cdot \xi_1 + 0 \cdot \xi_2 + 0 \cdot \xi_3 \\ -1 \cdot \xi_1 + 1 \cdot \xi_2 + 0 \cdot \xi_3 \\ 0 \cdot \xi_1 - 1 \cdot \xi_2 + 1 \cdot \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 - \xi_1 \\ \xi_3 - \xi_2 \end{pmatrix}$$

Digression: Linear systems again

$$\begin{cases} x_1 & = b_1 \\ -x_1 + x_2 & = b_2 \\ \quad - x_2 + x_3 & = b_3 \end{cases} \Rightarrow \begin{cases} x_1 = b_1 \\ x_2 = b_1 + b_2 \\ x_3 = b_1 + b_2 + b_3 \end{cases}$$

$$Ax = b \qquad\qquad Sb = x$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad\qquad S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

It would seem that S is an inverse of A.

Matrix notation

$\underset{m\times n}{A} = (\alpha_{ij})$; m is the number of rows, n is the number of columns

$\underset{m\times n}{A} = (a_1 \; a_2 \; \dots \; a_n)$; a_i are the columns $\in \mathbb{R}^m$

Matrix-vector product: $Ax = c$; $c = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$

(i) linear combination : $\underset{m\times 1}{c} = \sum_{i=1}^{n} \xi_i a_i$

(ii) inner products : $\gamma_i = \sum_{j=1}^{n} \alpha_{ij} \xi_j$

But, if a matrix is just a collection of columns, matrix-matrix product becomes natural.

Matrix-matrix product :

$$\underset{m\times n}{A}\ \underset{n\times p}{B} = \begin{pmatrix} Ab_1 & Ab_2 & \dots & Ab_p \end{pmatrix} = \underset{m\times p}{C}$$

With inner products : $\underset{m\times p}{C} = (\gamma_{ij})\,,\quad \gamma_{ij} = \sum_{k=1}^{n} \alpha_{ik}\beta_{kj}$

Properties and laws :

Theorem Matrix-matrix product is associative :

$$A(BC) = (AB)C \qquad \text{(if defined)}$$

but not commutative :

$$AB \neq BA \qquad \text{in general.}$$

In particular : $(A+B)^2 \neq A^2 + 2AB + B^2$

For square matrices $\underset{m\times m}{A}$: $A^p = \underbrace{A\cdot A\cdot \dots \cdot A}_{p \text{ times}}$

$$A^p A^q = A^{p+q}$$

$$(A^p)^q = A^{pq}$$

Addition:

$$A + B = B + A$$

$$\xi(A+B) = \xi A + \xi B$$

$$A + (B+C) = (A+B) + C$$

Product:

$$C(A+B) = CA + CB$$

$$(A+B)C = AC + BC$$

$$A(BC) = (AB)C$$

Consider $Ax = b$:

$$\begin{array}{cccc} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array} \qquad \begin{array}{c} -2 \\ 1 \end{array}$$

First elimination step: Take the first column $\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}$ and matrix $E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and compute $E_{21} a_1$:

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \quad ; \quad \text{Let } E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and compute $E_{31} E_{21} a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.

E_{ij} is an elimination matrix which encodes the corresponding row operation in the Gaussian elimination.