LU

Let us denote the vectors $\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\\vdots\\0\end{pmatrix}, \ldots$

with e_i, where

i is the index of the element 1.

These vectors are called natural basis vectors.

We have already seen the elimination matrices E_{ij}.

Now, using matrix notation we can write

$$E_{ij} = I + l_{ij} e_i e_j^T , \quad i > j .$$

$$\underset{n\times n}{I} = (e_1 \; e_2 \ldots e_n) = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} = \mathrm{diag}(1, 1, \ldots, 1)$$

is the identity matrix.

$$\underset{n\times n}{I} x = x \quad \forall\, x \in \mathbb{R}^n$$

<u>More terminology</u> : E_{ij} is a lower triangular matrix, its elements at positions $j > i$ are identically zero.

I is also a diagonal matrix.

<u>Definition</u> $\underset{n\times n}{A}$ is invertible, if there exists a matrix A^{-1} such that $A^{-1}A = AA^{-1} = I$.

For a product : $(AB)^{-1} = B^{-1}A^{-1}$

A^{-1} is the inverse of A.

We claim: $E_{ij}^{-1} = I - l_{ij} e_i e_j^T$, $i > j$.

How to verify this claim? Let us use the definition directly:

$$E_{ij} E_{ij}^{-1} = (I + l_{ij} e_i e_j^T)(I - l_{ij} e_i e_j^T)$$

$$= I + l_{ij} e_i e_j^T - l_{ij} e_i e_j^T - l_{ij}^2 e_i \underbrace{e_j^T e_i}_{=0} e_j^T = I$$

This means that given $\underset{3\times 3}{A}$:

$(E_{32} E_{31} E_{21}) A = U$, where U is upper triangular.

$$\Leftrightarrow \quad A = (E_{32} E_{31} E_{21})^{-1} U = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} U$$

$= LU$, where L is lower triangular.

Why?

Consider $E_{31}^{-1} E_{32}^{-1} = (I - l_{31} e_3 e_1^T)(I - l_{32} e_3 e_2^T)$

$$= I - l_{31} e_3 e_1^T - l_{32} e_3 e_2^T$$

In fact the product simply puts the row operation scalars in their natural locations, however, with the opposite sign.

<u>Example</u>

$$\begin{matrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{matrix} \quad \downarrow -\tfrac{1}{2} \qquad \begin{matrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 1 & 2 \end{matrix} \quad \downarrow -\tfrac{2}{3}$$

$$\underbrace{\begin{matrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{matrix}}_{U} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix}$$

Every U can be further decomposed:

$$U = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \begin{pmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdots \\ & 1 & u_{23}/d_2 & \cdots \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$= D\hat{U}$$

Often this is denoted by $A = LDU$, where it is implicitly assumed that every diagonal element of U is one.

Thus, solution of $Ax = b$ becomes

(1) $A = LU$ "factorisation"

(2) $LUx = b \iff \begin{cases} Ly = b & \text{"forward"} \\ Ux = y & \text{"backward"} \end{cases}$

Computational complexity: $\underset{n\times n}{A} = LU$ counting multiplications

1. elimination : n^2
2. " : $(n-1)^2$
3. " : $(n-2)^2$

...

Together: $n^2 + (n-1)^2 + (n-2)^2 + \ldots + 2^2 + 1^2 = \frac{1}{6} n(n+1)(2n+1)$

As n increases, the leading term is $\frac{1}{3}n^3$.

Solutions of triangular systems require n^2 operations.

Notice: Decomposition is the expensive part!

But, finding the inverse is even more expensive!

And, in the general case memory requirements can be significantly lower for decompositions.

PA = LU

What about the case where we must permute rows?

$$\begin{matrix}0&1&1\\1&2&1\\2&7&9\end{matrix} \qquad \begin{matrix}1&2&1\\0&1&1\\2&7&9\end{matrix}\ \ -2 \qquad \begin{matrix}1&2&1\\0&1&1\\0&3&7\end{matrix}\ \ -3$$

$$LU = \begin{pmatrix}1&0&0\\0&1&0\\2&3&1\end{pmatrix}\begin{pmatrix}1&2&1\\0&1&1\\0&0&4\end{pmatrix} = PA,$$

where $P = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$ is a permutation matrix.

Theorem For every invertible matrix A there exists a decomposition $PA = LU$.

(P is not unique.)