Inverses and Transposes

First synthesis:

Assume that $\underset{n\times n}{A}$ is invertible.

(1) The elimination produces n pivots.

(2) A^{-1} is unique.

<u>Proof</u> Suppose $BA = I$ and $AC = I$, then $B = C$:

$B(AC) = (BA)C \Rightarrow BI = IC \Rightarrow B = C$ $\square$

(3) $Ax = b$ has one and only one solution: $x = A^{-1}b$.

(4) If $Ax = 0$, $x \neq 0$, then A <u>is not</u> invertible.

Useful inverses:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\mathrm{diag}(d_1, d_2 \dots, d_n)^{-1} = \mathrm{diag}(1/d_1, 1/d_2, \dots, 1/d_n), \quad d_i \neq 0$$

<u>Gauss-Jordan</u>: The idea: Find X such that $\underset{n\times n}{A}X = I$.

Columnwise: $A(x_1\, x_2 \dots x_n) = (e_1\, e_2 \dots e_n)$

Eliminate all RHSs simultaneously!

$$\left(\begin{array}{ccc|ccc} \underline{2} & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \downarrow \tfrac{1}{2}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \downarrow \tfrac{2}{3}$$

$$\begin{array}{ccc|ccc} \underline{2} & -1 & 0 & 1 & 0 & 0 \\ 0 & \underline{3/2} & -1 & 1/2 & 1 & 0 \\ 0 & 0 & \underline{4/3} & 1/3 & 2/3 & 1 \end{array} \quad \uparrow 3/4 \quad \uparrow 2/3$$

This stage is $U\underline{X} = B$.
We want $I\underline{X} = A^{-1}$.
Solution : Eliminate upwards!

$$\begin{array}{ccc|ccc|l} \underline{2} & 0 & 0 & 3/2 & 1 & 1/2 & :2 \\ 0 & \underline{3/2} & 0 & 3/4 & 3/2 & 3/4 & :3/2 \\ 0 & 0 & \underline{4/3} & 1/3 & 2/3 & 1 & :4/3 \end{array}$$

$$\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \quad \underbrace{\begin{array}{ccc} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{array}}_{A^{-1}}$$

$\rightarrow$ Operation count : n^3
$\rightarrow$ no zeros in A^{-1}
High cost in storage!

Transpose : $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$; $A^T = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$

In short : $A = (\alpha_{ij})$, $A^T = (\alpha_{ji})$.

Two essential formulas :

(a) $(AB)^T = B^T A^T$

(b) $(A^{-1})^T = (A^T)^{-1}$

(b) assuming (a) is true: $\left\{ \begin{array}{lll} AA^{-1} = I & \Leftrightarrow & (AA^{-1})^T = I^T \\ A^{-1}A = I & \Leftrightarrow & (A^{-1}A)^T = I^T \end{array} \right.$

$$(A^{-1})^T A^T = A^T (A^{-1})^T = I \Rightarrow (A^{-1})^T = (A^T)^{-1}$$

Definition Matrix A is symmetric, if $A = A^T$.

Important identity : $R^T R$ is symmetric

$$(R^T R)^T = R^T R$$

For symmetric matrices : $A = LDU = LDL^T$ is symmetric!

Definition Permutation Matrix

The rows of any permutation matrices are rows of the corresponding identity matrix in some order.

Any inverse permutation is also a permutation.

Moreover, $P^{-1} = P^T$.

Definition If $A^{-1} = A^T$, A is orthogonal.

Example
$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow P^{-1} = P^T \; (= P \,!)$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow P^T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$P \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} P^T = \begin{pmatrix} \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{11} & \alpha_{12} & \alpha_{13} \end{pmatrix} P^T$$

$$= \begin{pmatrix} \alpha_{22} & \alpha_{23} & \alpha_{21} \\ \alpha_{32} & \alpha_{33} & \alpha_{31} \\ \alpha_{12} & \alpha_{13} & \alpha_{11} \end{pmatrix}$$

P from left permutes rows ; P^T from right permutes columns

$$Ax = b \iff PA\underbrace{P^T P}_{I}x = Pb$$