$Ax = b$ and $Ax = 0$

Let $\underset{3\times 4}{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ 3 & 6 & 11 & 14 \end{pmatrix}$, $x_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$.

Interestingly, $\underset{3\times4\,4\times1}{A\,x_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $Ax_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Hence, $A(\xi_1 x_1 + \xi_2 x_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

In other words, $x \in \text{span}(\{x_1, x_2\}) \Rightarrow Ax = 0$.

<u>Definition</u> Nullspace $N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$

$(= \ker(A))$

Why is this important?

If $Ax_p = b$ and $Ax_H = 0$, then $A(x_p + x_H) = b$.

If the nullspace is non-trivial, i.e., there are other vectors beside $x = 0$, then the solution cannot be unique if it exists.

We have already seen that $AB = 0 \not\Rightarrow A = 0$ or $B = 0$.

Now we can reinterpret this : $B = (b_1\ b_2 \dots b_n)$, $b_i \in N(A)$

and : $A^T = (\tilde{a}_1\ \tilde{a}_2 \dots \tilde{a}_k)$, $\tilde{a}_k \in N(B^T)$

In order to determine all solutions of $Ax = b$, it is necessary to find the nullspace of A, $N(A)$. This can either be done separately or during the standard elimination process.

Let us solve one example:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 1 \\ 2x_1 + 4x_2 + 8x_3 + 10x_4 = 6 \\ 3x_1 + 6x_2 + 11x_3 + 14x_4 = 7 \end{cases} \iff Ax = b$$

For illustrative purposes only, we solve first $Ax = 0$.

$$\begin{matrix} \underline{1} & 2 & 3 & 4 \\ 2 & 4 & 8 & 10 \\ 3 & 6 & 11 & 14 \end{matrix} \quad \downarrow -2 \quad \downarrow -3 \qquad \begin{matrix} \underline{1} & 2 & 3 & 4 \\ 0 & 0 & \underline{2} & 2 \\ 0 & 0 & 2 & 2 \end{matrix} \quad \downarrow -1$$

$$\begin{matrix} \underline{1} & 2 & 3 & 4 \\ 0 & 0 & \underline{2} & 2 \\ 0 & 0 & 0 & 0 \end{matrix} \quad \begin{matrix} \\ :2 \\ \end{matrix} \quad \uparrow -3 \qquad \begin{matrix} \underline{1} & 2 & 0 & 1 \\ 0 & 0 & \underline{1} & 1 \\ 0 & 0 & 0 & 0 \end{matrix} \quad \begin{matrix} \text{Reduced} \\ \text{echelon} \\ \text{form} \end{matrix}$$

the original system $Ax = 0$ has been reduced to $Rx = 0$.

Let us next divide the variables into two sets:

(a) pivot variables $\{x_1, x_3\}$

(b) free variables $\{x_2, x_4\}$

Let $x_2 = \sigma$, and $x_4 = \tau$, $\sigma, \tau \in \mathbb{R}$.

So:
$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \sigma \\ x_3 \\ \tau \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 = -2\sigma - \tau \\ x_3 = - \tau \end{cases} \iff x = \sigma \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \tau \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

That is,

$$N(A) = \text{span}\left(\left\{\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\-1\\1\end{pmatrix}\right\}\right), \quad \dim N(A) = 2.$$

Notice: There is a powerful method for findind $N(A)$ from R:

1) set $x_2 = 1$, $x_4 = 0$ $\Rightarrow$ $x_3 = 0$, $x_1 = -2$
2) set $x_2 = 0$, $x_4 = 1$ $\Rightarrow$ $x_3 = -1$, $x_1 = -1$

Sanity check: We have now determined that if there is a particular solution for $Ax = b$, then the number of solutions is infinite.

Let us first consider a general $b = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & b_1 \\ 2 & 4 & 8 & 10 & b_2 \\ 3 & 6 & 11 & 14 & b_3 \end{array} \quad -2, \ -3, \ -1$$

$$\begin{array}{ccccc} \underline{1} & 2 & 3 & 4 & b_1 \\ 0 & 0 & \underline{2} & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array} \quad \leftarrow \text{ consistence}$$

There can be a solution only if $b_3 - b_2 - b_1 = 0$.

Here, $b = \begin{pmatrix}1\\6\\7\end{pmatrix}$, i.e., $7 - 6 - 1 = 0$.

Now we know that the number of solutions is infinite.

$$\begin{matrix} \underline{1} & 2 & 3 & 4 & 1 \\ 0 & 0 & \underline{2} & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \quad \begin{matrix} \\ :2 \quad \uparrow -3 \\ \\ \end{matrix}$$

$$\begin{matrix} \underline{1} & 2 & 0 & 1 & -5 \\ 0 & 0 & \underline{1} & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Set $x_2 = 0$, $x_4 = 0$ $\Rightarrow \begin{cases} x_1 = -5 \\ x_3 = 2 \end{cases}$

The solution

$$x = \begin{pmatrix} -5 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \sigma \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \tau \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad \sigma, \tau \in \mathbb{R}.$$

Block matrix representation :

$$\left(\begin{array}{cc:cc:c} 1 & 0 & 2 & 1 & -5 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \text{or} \quad \begin{pmatrix} \underset{2\times 2}{I} & \underset{2\times 2}{F} \\ \underset{1\times 2}{0} & \underset{1\times 2}{0} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} \underset{2\times 1}{b_1} \\ \underset{1\times 1}{b_2} \end{pmatrix}$$

$\underset{2\times 2}{I}$: The number of pivots $= 2$, is the rank of A.

<u>Theorem</u> Any $\underset{p\times n}{A}\, x = b$ can be transformed to

$$\left(\begin{array}{cc:c} \underset{r\times r}{I} & F & \underset{r\times 1}{b_1} \\ 0 & 0 & b_2 \end{array}\right)$$

If $r < p$ and $b_2 \neq 0$	No solutions	; 0
$(r = p$ or $b_2 = 0)$ and $r = n$	Unique solution	; 1
$(r = p$ or $b_2 = 0)$ and $r < n$	Infinitely many	; ∞