Linear Independence, Basis and Dimension

The rank of a matrix is the number of pivots in the elimination process. As we have seen, the reduced echelon form has a block of the size of the rank and thus columns corresponding to the natural basis vectors.

The question we pose now is the following:

"Are there genuinely independent rows or columns in a given matrix?"

Definition Linear independence

Let $a_1, a_2, \dots, a_p \in \mathbb{R}^n$ and $\xi_1, \xi_2, \dots, \xi_p$ unknown scalars. The vectors a_i are linearly independent, if the only solution of $\sum_{i=1}^{p} \xi_i a_i = 0$ is $\xi_1 = \xi_2 = \dots = \xi_p = 0$.

If other solutions exist the vectors a_i are linearly dependent.

Example $a_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $a_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $a_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$\sum_{i=1}^{3} \xi_i a_i = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3 = 0,$$

for instance with $\xi_1 = -2$, $\xi_2 = 1$, $\xi_3 = 0$.

Thus $\{a_1, a_2, a_3\}$ are linearly dependent.

Theorem In $\mathbb{R}^n$ there can always be n linearly independent vectors, but any collection with more than n vectors is always linearly dependent.

Dimension is the largest possible number of linearly independent vectors ; $\dim \mathbb{R}^n = n$.

Definition Any linearly independent collection of n vectors in $\mathbb{R}^n$ is a basis.

Theorem Let $\{b_1, b_2, \dots, b_n\}$ be a basis of $\mathbb{R}^n$ and $y \in \mathbb{R}^n$. Then y is a unique linear combination of the basis vectors :

$$y = \xi_1 b_1 + \xi_2 b_2 + \dots + \xi_n b_n = \sum_{k=1}^{n} \xi_k b_k .$$

Proof

First we must establish that $\{b_1, b_2, \dots, b_n, y\}$ is linearly dependent.

Let us choose the scalars $\{\xi_1, \xi_2, \dots, \xi_n, \eta\}$ such that $\sum_{k=1}^{n} \xi_k b_k + \eta y = 0$. If $\eta = 0$, then $\xi_1 = \dots = \xi_n = 0$ since $\{b_1, \dots, b_n\}$ is a basis. This is a contradiction, since there cannot be $n+1$ linearly independent vectors. So, $\eta \neq 0$ ($y \neq 0$ by construction).

The linear combination becomes

$$y = \sum_{k=1}^{n} \left(-\frac{\xi_k}{\eta} \right) b_k .$$

Is it unique?

$$y = \sum_{k=1}^{n} \xi_k b_k = \sum_{k=1}^{n} \xi_k' b_k \Rightarrow \sum_{k=1}^{n} (\xi_k - \xi_k') b_k = 0$$

Since $\{b_1, \dots, b_n\}$ is a basis, it follows that

$$\xi_k = \xi_k' , \quad k = 1, \dots, n.$$

$\square$

<u>Definition</u> the coefficients of the linear combination are the coordinates of the vector in a given basis.

Natural basis: $x = \sum_{k=1}^{n} \xi_k e_k$, components of x are its coordinates in the natural basis.

<u>Change of basis</u>: Let us have two coordinate systems:

$$\{0, \underline{b}_1, \underline{b}_2, \underline{b}_3\}, \{0', \underline{b}'_1, \underline{b}'_2, \underline{b}'_3\}$$

The origin O must have some coordinates in the primed system:

$$\underline{r}_0 = \sum_{k=1}^{3} p_k \underline{b}'_k, \quad \text{and of course} \quad \underline{b}_j = \sum_{k=1}^{3} \tau_{kj} \underline{b}'_k, \; j=1,2,3$$

Notice the implicit transpose: τ_{kj}. (This is an educated guess.)

Let P be a point in $\mathbb{R}^3$. It must have unique coordinates in both systems:

$$\underline{r} = \sum_{k=1}^{3} \xi_k \underline{b}_k, \quad \underline{r}' = \sum_{k=1}^{3} \xi'_k \underline{b}'_k.$$

The two systems can now be connected, since $\underline{r}' = \underline{r} + \underline{r}_0$:

$$\begin{aligned}
\sum_{k=1}^{3} \xi'_k \underline{b}'_k &= \underline{r}_0 + \sum_{j=1}^{3} \xi_j \underline{b}_j \\
&= \sum_{k=1}^{3} p_k \underline{b}'_k + \sum_{j=1}^{3} \xi_j \sum_{k=1}^{3} \tau_{kj} \underline{b}'_k \\
&= \sum_{k=1}^{3} p_k \underline{b}'_k + \sum_{k=1}^{3} \sum_{j=1}^{3} \tau_{kj} \xi_j \underline{b}'_k \\
&= \sum_{k=1}^{3} \left(p_k + \sum_{j=1}^{3} \tau_{kj} \xi_j \right) \underline{b}'_k
\end{aligned}$$

The coordinates are unique: For $\{0', \underline{b}'_1, \underline{b}'_2, \underline{b}'_3\}$

$$\xi'_k = p_k + \sum_{j=1}^{3} \tau_{kj} \xi_j, \quad k = 1, 2, 3, \quad \text{or} \quad x' = x_0 + Tx.$$

Thus,

$$x' = x_0 + Tx \iff x = -T^{-1}x_0 + T^{-1}x'$$

transformation in the reverse direction : $S = T^{-1}$

$$x = -Sx_0 + Sx'$$

<u>Example</u> $\{\underline{b}_1, \underline{b}_2, \underline{b}_3\}$ is a basis.

$$\begin{cases} \underline{b}_1' = 2\underline{b}_1 + 2\underline{b}_2 + 7\underline{b}_3 \\ \underline{b}_2' = \underline{b}_2 + 9\underline{b}_3 \\ \underline{b}_3' = 6\underline{b}_1 + 8\underline{b}_2 \end{cases}$$

Find α and β such that the vector $\underline{v} = \alpha\underline{b}_1' + \beta\underline{b}_2' + \underline{b}_3'$ has constant coordinates in the other system: $\underline{v} = \gamma\underline{b}_1 + \gamma\underline{b}_2 + \gamma\underline{b}_3$

$$x = Sx' = \begin{pmatrix} 2 & 0 & 6 \\ 2 & 1 & 8 \\ 7 & 9 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix} = \begin{pmatrix} 2\alpha + 6 \\ 2\alpha + \beta + 8 \\ 7\alpha + 9\beta \end{pmatrix}$$

Equal (constant) coordinates:

$$\begin{cases} 2\alpha + 6 = 2\alpha + \beta + 8 \\ 2\alpha + 6 = 7\alpha + 9\beta \end{cases} \Rightarrow \alpha = \frac{24}{5}, \quad \beta = -2$$