Linear Transformations

Definition Let $F: V \to W$. F is a linear transform, if

(1) $F(x+y) = F(x) + F(y) \quad \forall\, x, y \in V$

(2) $F(\lambda x) = \lambda F(x) \quad \forall\, x \in V,\ \lambda \in \mathbb{R}$

Here: $F: \mathbb{R}^n \to \mathbb{R}^p$.

We identify immediately that matrix-vector-product is a linear transform.

This is much more general, however.

Let $p_1(x) = x^2 + x + 1$, $p_2(x) = 2x^2 - 1$, and let F be the derivative operator D:

(1) $D((x^2+x+1) + (2x^2-1)) = D(x^2+x+1) + D(2x^2-1)$

(2) $D(\lambda(2x^2-1)) = \lambda D(2x^2-1)$

It is a remarkable fact that every linear transform has a matrix representation.

Theorem Let $F: \mathbb{R}^n \to \mathbb{R}^p$ be a linear transform, which maps the natural basis vectors $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ onto the vectors $a_1, a_2, \dots, a_n \in \mathbb{R}^p$: $F(e_k) = a_k$, $k = 1, \dots, n$.

Let $A = (a_1\ a_2 \dots a_n)$, then

$$F(x) = Ax \quad \forall\, x \in \mathbb{R}^n \ ; \ \underset{p \times n}{A}.$$

Proof Let $x \in \mathbb{R}^n$: $x = \sum_{k=1}^{n} \xi_k e_k$. F is a linear transform:

$$F(x) = F\left(\sum_{k=1}^{n} \xi_k e_k\right) = \sum_{k=1}^{n} \xi_k F(e_k) = \sum_{k=1}^{n} \xi_k a_k = Ax \qquad \square$$

Geometric Transforms : Euclidean transforms

The Euclidean transforms preserve the shape of the geometric object.

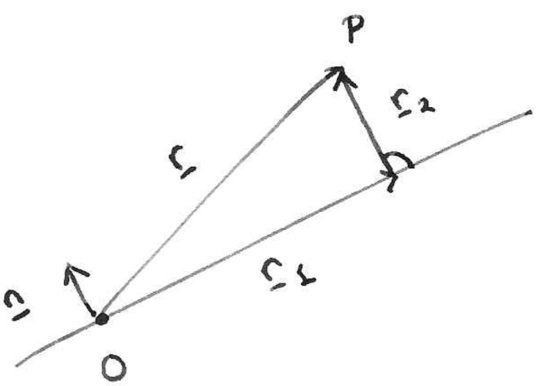
There are four of them: Translation, reflection, rotation, and scaling.

Now, let us argue with physical vectors, but compute with vectors in $\mathbb{R}^n$.

1) Translation: $T_{\underline{a}}(\underline{r}) = \underline{r}' = \underline{r} + \underline{a}$

This is not a linear transform!

2) Reflection: Let us assume that the symmetry axis goes through the origin (similarly for the symmetry plane)



$$\underline{r} = \underline{r}_1 + \underline{r}_2 ,$$

$$\underline{r}_2 = (\underline{n} \cdot \underline{r})\,\underline{n} ,$$

$$\underline{r}_1 = \underline{r} - \underline{r}_2$$
$$= \underline{r} - (\underline{n} \cdot \underline{r})\,\underline{n}$$

Thus, for the image $\underline{r}' = \underline{r}_1 - \underline{r}_2 = \underline{r} - 2(\underline{n} \cdot \underline{r})\,\underline{n}$.

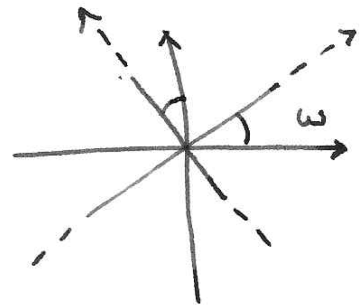
In other words: $H_{\underline{n}}(\underline{r}) = \underline{r}' = \underline{r} - 2(\underline{n} \cdot \underline{r})\,\underline{n}$

What about $\mathbb{R}^n$?

$$x' = x - 2(n^T x)\,n = x - 2n(n^T x)$$
$$= x - 2(nn^T)\,x = (I - 2nn^T)\,x$$

or $x' = H_n x$. Linear transform! Notice: $H_n H_n = I$.

3) Rotation



The origin is a fixed point.

Images of the axes:

$(1, 0) \rightarrow (\cos\omega, \sin\omega)$

$(0, 1) \rightarrow (-\sin\omega, \cos\omega)$

$$U_\omega = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$$

Rotation in $\mathbb{R}^3$ about an arbitrary axis can be derived, but it doesn't have a simple form.

4) Scaling

$$S_\lambda(\underline{r}) = \underline{r}' = \lambda \underline{r}$$

5) General reflection, rotation, and scaling

Let the point $P_0 \mathrel{\hat{=}} \underline{r}_0$ be fixed.

$$F_0(\underline{r}) = F(\underline{r} - \underline{r}_0) + \underline{r}_0$$

that is

$$x' = A(x - x_0) + x_0 = Ax + (x_0 - Ax_0)$$

$$= Ax + b$$

Any transform that has the form $Ax + b$ is affine.

For instance, the change of basis!