Determinant

Definition (Hand waving) Determinant is a number which represents either some area, volume, or generalised volume.

$$\mathrm{Det} : \mathbb{R}^{n\times n} \to \mathbb{R} \; ; \qquad \mathrm{Det}(A) = |A| = \det A$$

Definition $|A| =$ the product of pivots

For computing $|A|$ the definition above is sufficient and establishes computational complexity to be the same as that of Gaussian elimination.

However, there is more! Let us define the determinant via its properties and keeping both definitions in mind.

(I) $\det I = 1$

(II) Swap of two rows changes the sign:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = - \begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

(III) Linearity on rows:

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

The properties (I) – (III) are sufficient for yet another definition.

Let us nevertheless add seven more:

4) If there are two equal rows, then $\det A = 0$.

5) Row operation (elimination step) does not change the value of the determinant.

6) If there is a row of zeros, then $\det A = 0$.

7) For triangular matrices the determinant is the product of the diagonal elements.

8) For non-invertible (i.e., singular) matrices, determinant is zero.

9) $|AB| = |A||B|$

10) $\det A^T = \det A$

<u>Theorem</u> $|AB| = |A||B|$

<u>Proof</u>

(i) Let us assume that $|B| \neq 0$. If the number $D(A) = \dfrac{|AB|}{|B|}$ has the properties (I) – (III), then $D(A) = \mathrm{Det}(A)$.

(I) $A = I \Rightarrow D(A) = \dfrac{|B|}{|B|} = 1$

(II) Let us swap two rows of A: PA
The same rows are swapped: $P(AB)$
$\Rightarrow D(A)$ changes sign always when $\det A$ does.

(III) Let $A = (\alpha_{ij})$, $B = (\beta_{ij})$

If $\alpha_{1j} = \tilde{\alpha}_{1j} + \hat{\alpha}_{1j}$, then $\alpha_{1j}\beta_{j1} = \tilde{\alpha}_{1j}\beta_{j1} + \hat{\alpha}_{1j}\beta_{jl}$,

and $D(A) = D(\tilde{A} + \hat{A}) = \dfrac{|\tilde{A}B| + |\hat{A}B|}{|B|}$.

(ii) $|B| = 0$; AB is singular, if B is.
$|AB| = 0 = |A||B|$ ◻

For 3×3 - matrices there exists a highly useful identity.

Rule of Sarrus :

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = \begin{array}{l} \alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{12}\alpha_{23}\alpha_{31} + \alpha_{13}\alpha_{21}\alpha_{32} \\ - \alpha_{13}\alpha_{22}\alpha_{31} - \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32} \end{array}$$

$$= \begin{vmatrix} \alpha_{11} & & \\ & \alpha_{22} & \alpha_{23} \\ & \alpha_{32} & \alpha_{33} \end{vmatrix} + \begin{vmatrix} & \alpha_{12} & \\ \alpha_{21} & & \alpha_{23} \\ \alpha_{31} & & \alpha_{33} \end{vmatrix} + \begin{vmatrix} & & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \\ \alpha_{31} & \alpha_{32} & \end{vmatrix}$$

The first identity holds only for 3×3 - matrices.

Definition Let M_{ij} be a $(n-1) \times (n-1)$ - matrix which is constructed by removing the i^{th} row and j^{th} column of A. Let further $C_{ij} = (-1)^{i+j} \det M_{ij}$.

Then $\det A = \alpha_{i1} C_{i1} + \alpha_{i2} C_{i2} + \dots + \alpha_{in} C_{in}$.

The C_{ij} are the so-called cofactors.

Finally,

The Definition to Rule Them All

$$\det A = \sum_{\substack{P \in \text{Permutation} \\ \text{matrices}}} \det(P)\, \alpha_{1\alpha}\alpha_{2\beta} \cdots \alpha_{n\omega},$$

where $P \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \vdots \\ \omega \end{pmatrix}$.

Here ω refers to the last letter of the Greek alphabet.

Applications

1.
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{\det A}\begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix}$$

$$= \frac{C^T}{\det A} = A^{-1}$$

In other words : $\displaystyle (A^{-1})_{ij} = \frac{C_{ji}}{\det A}$

Why is this true for $n \times n$ - systems : $AC^T = (\det A)\, I$?

→ the off-diagonals are zero because the cofactors introduce copies of rows $\Rightarrow$ "determinants" $= 0$

2. Cramer's Rule : $\displaystyle x = A^{-1} b = \frac{C^T b}{\det A}$

$$x_j = \frac{\det B_j}{\det A}, \quad \text{where } B_j = \begin{array}{c} \begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{pmatrix} \\ \uparrow j^{th} \text{ column} \end{array}$$

So, there exists a formula for the solution of a linear system. It is practical only for small n and for symbolic solutions.

3. Vector algebra : Cross Product (Vector Product)

Definition Let $\underline{a}$, $\underline{\ \ }$ be two vectors in space. Their cross product is a vector $\underline{a} \times \underline{b}$:

(i) $\| \underline{a} \times \underline{b} \| = \| \underline{a} \| \| \underline{b} \| \sin \measuredangle (\underline{a}, \underline{b})$

(ii) $\underline{a} \times \underline{b} \perp \underline{a}$, $\underline{a} \times \underline{b} \perp \underline{b}$

(iii) $\{ \underline{a}, \underline{b}, \underline{a} \times \underline{b} \}$ is a right-handed system

Theorem

$$\underline{a} = \alpha_1 \underline{i} + \alpha_2 \underline{j} + \alpha_3 \underline{k}$$

$$\underline{b} = \beta_1 \underline{i} + \beta_2 \underline{j} + \beta_3 \underline{k}$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}$$

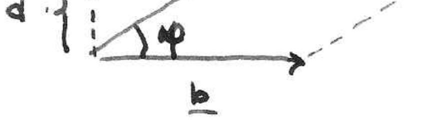
Definition Scalar Triple Product

$$[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c} = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$$

Notice: $[\underline{a}, \underline{b}, \underline{c}] = [\underline{c}, \underline{a}, \underline{b}] = [\underline{b}, \underline{c}, \underline{a}]$.

Theorem Area of a parallelogram spanned by $\underline{a}$ and $\underline{b}$ is $\| \underline{a} \times \underline{b} \|$. Volume of an object spanned by $\{\underline{a}, \underline{b}, \underline{c}\}$ is $| [\underline{a}, \underline{b}, \underline{c}] |$.

Proof



$d = \| \underline{a} \| \sin \varphi \quad \Rightarrow \quad \| \underline{a} \times \underline{b} \| = \| \underline{a} \| \| \underline{b} \| \sin \varphi$

Add the third dimension : Volume = area of the base · height

Let ψ be the angle between $\underline{c}$ and the base spanned by $\{\underline{a}, \underline{b}\}$.

Volume is $\| \underline{a} \times \underline{b} \| \| \underline{c} \| \cos \psi = \| \underline{a} \times \underline{b} \| \, | \underline{n}^{\circ} \cdot \underline{c} |$

$$= \| \underline{a} \times \underline{b} \| \left| \frac{\underline{a} \times \underline{b}}{\| \underline{a} \times \underline{b} \|} \cdot \underline{c} \right| = | \underline{a} \times \underline{b} \cdot \underline{c} | = | [\underline{a}, \underline{b}, \underline{c}] | \quad \square$$