# PHYS-C0252 - Quantum Mechanics

## Exercise set 1

**Due date : October 30, 2020 before 10.00**

1. Consider states $|\psi\rangle = 3i|\phi_1\rangle - 7i|\phi_2\rangle$ and $|\chi\rangle = -|\phi_1\rangle + 2i|\phi_2\rangle$, where $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal states, i.e. $\langle\phi_k|\phi_m\rangle = \delta_{km}$, where $\delta_{km} = 1$ for $k = m$ and $\delta_{km} = 0$ for $k \neq m$.

   (a) Express $|\psi\rangle + |\chi\rangle$ and $\langle\psi| + \langle\chi|$ in their simplest form using $|\phi_1\rangle$ and $|\phi_2\rangle$.

   (b) Calculate the inner products $\langle\psi|\chi\rangle$ and $\langle\chi|\psi\rangle$. Are they equal?

   (c) Show that $|\psi\rangle$ and $|\chi\rangle$ satisfy the Cauchy–Schwarz inequality.

   (d) Show that $|\psi\rangle$ and $|\chi\rangle$ satisfy the triangle inequality.

2. (a) Prove the Cauchy–Schwarz inequality $|\langle\psi|\phi\rangle| \leq ||\psi||\,||\phi||$. Here we use the short-hand notation $||\psi||$ $(= ||\,|\psi\rangle||)$ for the norm of $|\psi\rangle$ as on lectures. Hint: Start from $0 \leq ||\,|\psi\rangle + \lambda|\phi\rangle\,||$ and choose the scalar $\lambda$ in a clever way.
   (b) Prove the triangle inequality $||\,|\psi\rangle + |\phi\rangle|| \leq ||\psi|| + ||\phi||$. Hint: Calculate $||\,|\psi\rangle + |\phi\rangle\,||^2$ and use (a).
   (c) Demonstrate the necessary and sufficient conditions for these inequalities to become equalities.

3. (a) Consider the so-called Pauli operators $\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\hat{\sigma}_y = \mathrm{i}|0\rangle\langle 1| - \mathrm{i}|1\rangle\langle 0|$ and $\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, where $\{|0\rangle, |1\rangle\}$ form an orthonormal basis of the considered Hilbert space.

   i. Show that each Pauli operator is Hermitian.

   ii. Give the eigenvalues and the corresponding eigenstates of each Pauli operator.

   iii. For each Pauli operator, show that the eigenstates are orthogonal.

   (b) Show that if the inverse of a linear operator $\hat{A} : \mathcal{H} \to \mathcal{H}$ exists, the eigenvalues of the inverse $\hat{A}^{-1}$ are just the inverses of those of $\hat{A}$. The inverse is defined by $\hat{A}\hat{A}^{-1} = \hat{A}^{-1}\hat{A} = \hat{I}$, where $\hat{I}$ is the identity operator.

4. (a) Show that for a Hermitian linear bounded operator $\hat{H} : \mathcal{H} \to \mathcal{H}$, all of its eigenvalues are real and the eigenvectors corresponding to different eigenvalues are orthogonal.
   (b) Show that the eigenvalues of an anti-Hermitian linear bounded operator $\hat{A} : \mathcal{H} \to \mathcal{H}$ are either purely imaginary or equal to zero. Note that for anti-Hermition operators $\hat{A}$, we have $\hat{A}^\dagger = -\hat{A}$.
   (c) Consider any linear bounded operator $\hat{B} : \mathcal{H} \to \mathcal{H}$

i. Show that $\hat{B} - \hat{B}^\dagger$ is anti-Hermitian and $\hat{B} + \hat{B}^\dagger$ is Hermitian.

ii. Show that $\hat{B}$ can be expressed as a linear combination of a Hermitian and anti-Hermitian operator.