ELEC-E8116 Model-based control systems
/ exercises with solutions 7

1. Let the weight of the sensitivity function be given as

$$\frac{1}{W_s} = A\frac{\dfrac{s}{A\omega_0}+1}{\dfrac{s}{B\omega_0}+1}, \quad 0 < A << 1,\ B >> 1$$

Sketch a schema for the magnitude plot of the frequency response and investigate its characteristics. What is the slope in the increasing part of the curve? What is the magnitude at frequency ω_0?

Generate a second order model, where the slope is twice as large as in the previous case. Investigate again the characteristics. What is the magnitude at frequency ω_0?

Solution:

$$\frac{1}{W_s(j\omega)} = A\frac{\dfrac{j\omega}{A\omega_0}+1}{\dfrac{j\omega}{B\omega_0}+1} = A\frac{\dfrac{1}{A\omega_0}+\dfrac{1}{j\omega}}{\dfrac{1}{B\omega_0}+\dfrac{1}{j\omega}} \quad \text{Clearly } \frac{1}{W_s(j0)} = A, \quad \frac{1}{W_s(j\infty)} = B$$

For $\omega \to \omega_0$ $$\left|\frac{1}{W_s(j\omega)}\right| = A\sqrt{\frac{1+\left(\dfrac{\omega}{A\omega_0}\right)^2}{1+\left(\dfrac{\omega}{B\omega_0}\right)^2}} = \underbrace{A\sqrt{\frac{1+\left(\dfrac{1}{A}\right)^2}{1+\left(\dfrac{1}{B}\right)^2}}}_{\omega=\omega_0} = \sqrt{\frac{1+A^2}{1+\dfrac{1}{B^2}}} \approx 1,$$

because B is "large" and A is "small".

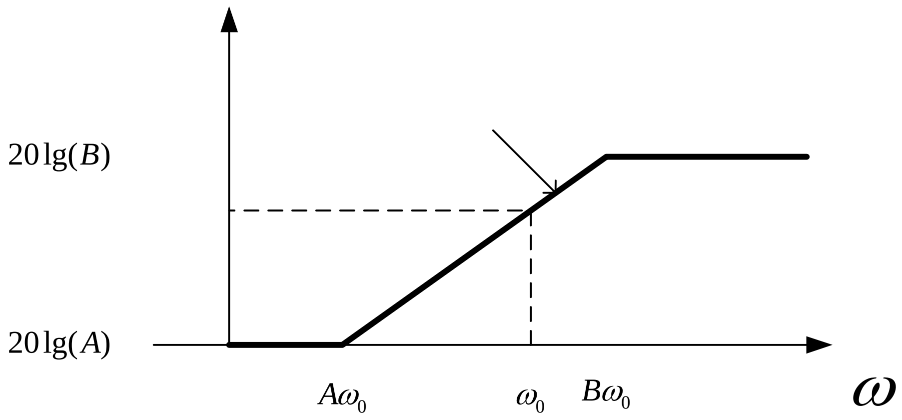
The Bode diagram (amplitude) is shown below:

Note that for the absolute value of the term $1+j\omega T$ in the frequency response it holds

$$\sqrt{1+(\omega T)^2} \underbrace{=}_{\omega=1/T} \sqrt{2} \approx 3\text{dB}$$ which can be approximated as 0 dB. For higher frequencies

$$\sqrt{1+(\omega T)^2} \approx \sqrt{(\omega T)^2} = \omega T \Rightarrow 20\lg(\omega T) = 20\lg(\omega) + 20\lg(T)$$

increases 20dB/decade (slope = 1) from zero decibels at $\omega = 1/T$.



Note that in the lecture slides an example of *Mixed Sensitivity Design* was shown with the desired sensitivity weight

$$\frac{1}{W_s(s)} = \frac{s+\omega_B^* A}{\frac{s}{M}+\omega_B^*}.$$ This is the same parameterization as in the problem, by $M = B, \omega_B^* = \omega_0$.

The second order model is

$$\frac{1}{W_s} = A\frac{(\frac{j\omega}{A^{1/2}\omega_0}+1)^2}{(\frac{j\omega}{B^{1/2}\omega_0}+1)^2}$$

Similar calculus as above shows that the amplitude curve is as in the above figure but with the angular frequencies $\left(A^{1/2}\omega_0, \omega_0, B^{1/2}\omega_0\right)$ instead of $\left(A\omega_0, \omega_0, B\omega_0\right)$. The curve increases 40 dB/decade, slope is 2. Note that this is again the same as

$$\frac{1}{W_s(s)} = \frac{(s+\omega_B^* A^{1/2})^2}{(\frac{s}{M^{1/2}}+\omega_B^*)^2}.$$

2. Consider the angular frequencies $\omega_B, \omega_c, \omega_{BT}$ which are used to define the bandwidth of a controlled system. State the definitions. Prove that when the phase margin is less than 90 degrees ($PM < \pi/2$) it holds $\omega_B < \omega_c < \omega_{BT}$. Interpretations?

Solution: Definitions:

ω_B : where S crosses $1/\sqrt{2} \approx -3\text{dB}$ from below.

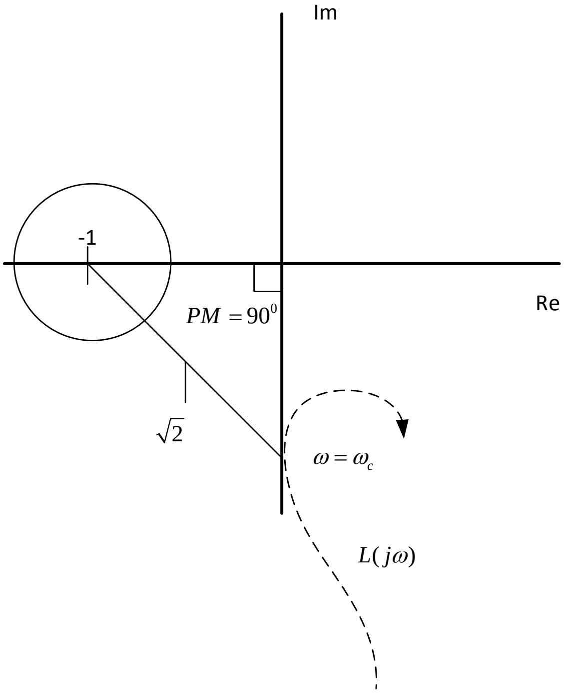
ω_c : where L crosses 1 = 0 dB (gain crossover (angular) frequency)

ω_{BT} : where T crosses $1/\sqrt{2} \approx -3\text{dB}$ from above.

At the gain crossover frequency it holds

$$|L(j\omega_c)| = 1 \Rightarrow |T(j\omega_c)| = \left|\frac{L(j\omega_c)}{1+L(j\omega_c)}\right| = \frac{|L(j\omega_c)|}{|1+L(j\omega_c)|} = \frac{1}{|1+L(j\omega_c)|} = \left|\frac{1}{1+L(j\omega_c)}\right| = |S(j\omega_c|$$

(Note that $L(j\omega_c)$ is a complex number and so $|1+L(j\omega_c)| \neq 1+|L(j\omega_c)|$.
$|1+x+jy| = \sqrt{(1+x)^2+y^2} \neq 1+\sqrt{x^2+y^2}$, except in some rare exceptional cases (when?)).



The figure shows the Nyquist diagram of L where the phase margin PM = 90 degrees. In the gain crossover frequency then

$|S(j\omega_c| = |T(j\omega_c| = 1/\sqrt{2} \approx -3\,\text{dB}$ (The distance from the point (-1,0) is inversely proportional to the absolute value of *S*. See lecture slides, Chapter 3).

So, at ω_c all the bandwidths would coincide.

But when *PM* < 90 degrees $|S(j\omega_c| = |T(j\omega_c| > 1/\sqrt{2}$, which implies directly that

S approaches from below $\Rightarrow \omega_B < \omega_c$
T approaches from above $\Rightarrow \omega_{BT} > \omega_c$.

We can conclude that roughly all the frequencies described can be used to discuss bandwidth, describing the behaviour of the closed-loop system.

3. In solving the discrete-time LQ problem an essential step is to find a "first control step" by minimizing the cont

$$J_{N-1} = \frac{1}{2} x_{N-1}^T Q x_{N-1} + \frac{1}{2} u_{N-1}^T R u_{N-1} + \frac{1}{2}\left(A x_{N-1} + B u_{N-1}\right)^T S_N \left(A x_{N-1} + B u_{N-1}\right)$$

Do it.

Solution:

Note: *Q*, *R*, *S* are symmetric $Q = Q^T$ etc.

$$\frac{\partial}{\partial x}(Ax) = A, \quad \frac{\partial}{\partial x}(x^T A x) = x^T(A + A^T) \underbrace{=}_{A\text{symmetric}} 2x^T A$$

$$J_{N-1} = \frac{1}{2} x_{N-1}^T Q x_{N-1} + \frac{1}{2} u_{N-1}^T R u_{N-1} + \frac{1}{2}\left(A x_{N-1} + B u_{N-1}\right)^T S_N \left(A x_{N-1} + B u_{N-1}\right)$$

$$J = \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + \frac{1}{2} x^T A^T S A x + \frac{1}{2} x^T A^T S B u + \underbrace{\frac{1}{2} u^T B^T S A x}_{\text{scalar, can be transposed}} + \frac{1}{2} \underbrace{u^T B^T S B u}_{\text{symmetric}}$$

To solve the extreme value the derivative with respect to *u* must be zero.

$$\begin{aligned}\frac{\partial J}{\partial u} &= u^T R + \frac{1}{2} x^T A^T S B + \frac{1}{2} x^T A^T S B + u^T B^T S B \\ &= u^T R + x^T A^T S B + u^T B^T S B = 0\end{aligned}$$

Taking the transpose does not change the equation

$$
\begin{aligned}
& Ru + B^T SAx + B^T SBu = 0 \\
& \Rightarrow \left(R + B^T SB\right) u = -B^T SAx \\
& \Rightarrow u^* = -\left(R + B^T SB\right)^{-1} B^T SAx
\end{aligned}
$$

Note that the inverse exists, because *S* is positive semidefinite and *R* is positive definite. Also, the *Hessian*

$$
\frac{\partial^2 J}{\partial u^2} = \frac{\partial}{\partial u}\left(u^T R + u^T B^T SB\right)^T = \frac{\partial}{\partial u}\left(Ru + B^T SBu\right) = R + B^T SB > 0 \quad \text{(pos. def.)}
$$

shows that the extreme value is a minimum.

4. The discrete time LQ problem and its solution can be given as

$$
\begin{aligned}
& x_{k+1} = A_k x_k + B_k u_k, \quad k > i \\
& J_i = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=i}^{N-1} \left(x_k^T Q_k x_k + u_k^T R_k u_k \right) \\
& S_N \geq 0, \quad Q_k \geq 0, \quad R_k > 0
\end{aligned}
$$

(final state free)

$$
\begin{aligned}
& S_k = \left(A_k - B_k K_k\right)^T S_{k+1} \left(A_k - B_k K_k\right) + K_k^T R_k K_k + Q_k \\
& K_k = \left(B_k^T S_{k+1} B_k + R_k\right)^{-1} B_k^T S_{k+1} A_k, \quad k < N \\
& u_k^* = -K_k x_k, \quad k < N \\
& J_i^* = \frac{1}{2} x_i^T S_i x_i
\end{aligned}
$$

Show that the Riccati equation can also be written in the form

$$
S_k = A_k^T \left[S_{k+1} - S_{k+1} B_k \left(B_k^T S_{k+1} B_k + R_k \right)^{-1} B_k^T S_{k+1} \right] A_k + Q_k, \; k < N, \; S_N \text{ given}
$$

(The "Joseph-stabilized form " of the Riccati equation)

Solution: Start from the equations

$$
\begin{aligned}
& S_k = \left(A_k - B_k K_k\right)^T S_{k+1} \left(A_k - B_k K_k\right) + K_k^T R_k K_k + Q_k \\
& K_k = (B_k^T S_{k+1} B_k + R_k)^{-1} B_k^T S_{k+1} A_k
\end{aligned} \tag{1}
$$

and try to reach

$$S_k = A_k^T \left[S_{k+1} - S_{k+1} B_k \left(B_k^T S_{k+1} B_k + R_k \right)^{-1} B_k^T S_{k+1} \right] A_k + Q_k \qquad (2)$$

First note in equation (1) that when Q and R have been chosen to be symmetric and S_N is symmetric, then S_i is symmetric for all i (verification by taking the transpose of S_k in equation (1); remember the calculation rules of transposition).

Start from (1) and use the short notation $S_{k+1} = S$, $K_k = K$ etc.

$$\begin{aligned}
& A^T SA - A^T SBK - K^T B^T SA + K^T B^T SBK + K^T RK + Q \\
&= A^T SA - A^T SBK - K^T B^T SA + K^T \left[B^T SB + R \right] K + Q \\
&= A^T SA - A^T SB\left(B^T SB + R\right)^{-1} B^T SA - A^T SB\left(B^T SB + R\right)^{-1} B^T SA \\
&\quad + A^T SB\left(B^T SB + R\right)^{-1} \underbrace{(B^T SB + R)(B^T SB + R)^{-1}}_{I} B^T SA + Q \\
&= A^T \left\{ S - SB\left(B^T SB + R\right)^{-1} B^T S - SB\left(B^T SB + R\right)^{-1} B^T S + SB\left(B^T SB + R\right)^{-1} B^T S \right\} A + Q \\
&= A^T \left\{ S - SB\left(B^T SB + R\right)^{-1} B^T S \right\} A + Q
\end{aligned}$$

which is the same as (2).

Note that especially in the calculation of the transpose of K the fact that Q, R and S are symmetric, has been utilized.