4. Representations of $\mathfrak{sl}_2(\mathbb{C})$

We start by analyzing an easy but fundamental case, namely the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. It is a three-dimensional complex Lie algebra.

The importance of focusing on this particular case stems for example from the following:

- The complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to the complexification of the real Lie algebras $\mathfrak{so}_3$ and $\mathfrak{su}_2$, i.e., $\mathfrak{so}_3 \otimes \mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{su}_2 \otimes \mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$. As such, the complex representations of $\mathfrak{so}_3$ and $\mathfrak{su}_2$ are exactly the same as those of $\mathfrak{sl}_2(\mathbb{C})$. In particular, by understanding the representations of $\mathfrak{sl}_2(\mathbb{C})$, we will ultimately understand the representations of the very important Lie groups $SO_3$ and $SU_2$, whose Lie algebras are $\mathfrak{so}_3$ and $\mathfrak{su}_2$.

- The complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, viewed as a six-dimensional real Lie algebra, is isomorphic to the Lie algebra of the Lorentz group, i.e. the group of linear transformations of the Minkowski space-time.

- The analysis of all semisimple Lie algebras $\mathfrak{g}$ and their representations will be achieved by finding subalgebras in $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, and applying our knowledge of the representation theory of $\mathfrak{sl}_2(\mathbb{C})$. Despite the importance of $\mathfrak{sl}_2(\mathbb{C})$ for its own sake (witnessed, e.g., by the previous examples), this is really the fundamental reason for studying it!

4.1. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

Recall that $\mathfrak{sl}_2(\mathbb{C})$ is the set

$$\mathfrak{sl}_2(\mathbb{C}) = \{ M \in \mathbb{C}^{2 \times 2} \mid \text{tr}(M) = 0 \}$$

of traceless (complex) two-by-two matrices, equipped with the Lie bracket $[M_1, M_2] = M_1 M_2 - M_2 M_1$. As a (complex) vector space, it is three dimensional (cf. Exercise [??]), and we will use the basis

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

(II.7)

for it. The brackets of these basis elements are


(II.8)

The chosen basis elements are quite simple matrices, but more importantly this basis choice is a fundamental instance of a canonical basis that can be chosen for any semisimple Lie algebra. This should become clear gradually, and at least by the time we treat the general structure of semisimple Lie algebras.

We can immediately give two examples of representations of $\mathfrak{sl}_2(\mathbb{C})$.

**Example II.20.** The space $V = \mathbb{C}^2$ is naturally a representation of $\mathfrak{sl}_2(\mathbb{C})$: any element $X \in \mathfrak{sl}_2(\mathbb{C})$ is a $2 \times 2$-matrix, which we let act on any vector $v \in V = \mathbb{C}^2$ by matrix multiplication $Xv$. This two-dimensional representation is called the standard representation of $\mathfrak{sl}_2(\mathbb{C})$.

**Example II.21.** The adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$ is the vector space $V = \mathfrak{sl}_2(\mathbb{C})$ equipped with the adjoint action: for $X \in \mathfrak{sl}_2(\mathbb{C})$ and $Y \in V = \mathfrak{sl}_2(\mathbb{C})$, we set

$$X(Y) = \text{ad}_X(Y) = [X, Y].$$

This is a three-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. 
Concretely, in the basis $E, H, F$ of $\mathfrak{sl}_2(\mathbb{C})$, the adjoint representation $\rho: \mathfrak{sl}_2(\mathbb{C}) \rightarrow \text{End}(\mathfrak{sl}_2(\mathbb{C}))$ becomes, in view of (II.8),

$$
\rho(E) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \rho(H) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \rho(E) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.
$$

4.2. The irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$

Let $V$ be a finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. We will use:

**Fact II.10.** The action of $H$ on $V$ is diagonalizable.

This fact follows from the preservation of Jordan form (see [FH91]), but it is also not particularly difficult to verify directly either.

By Fact II.18, we have an eigenspace decomposition

$$
V = \bigoplus_{\mu} V_{\mu},
$$

where $\mu$ runs over the eigenvalues of $H$ on $V$, a priori some finite collection of complex numbers, and $V_{\mu}$ are the corresponding eigenspaces for $H$

$$
V_{\mu} = \{ v \in V \mid Hv = \mu v \},
$$

The decomposition (II.9) completely describes the action of $H$ on $V$, and the remaining task is to describe the action of $E$ and $F$ — in particular, to see what $E$ and $F$ do to the $H$-eigenspaces $V_{\mu}$. Suppose that $v \in V_{\mu}$. Consider the vector $Ev \in V$. We can figure out the action of $H$ on it by an easy but important calculation which uses the commutator of $H$ and $E$ given by the bracket (II.8).

**Fundamental calculation (first time):**

$$
H(Ev) = E(Hv) + [H, E]v \\
= E(\mu v) + 2Ev \\
= (\mu + 2)Ev.
$$

This calculation shows that if $v$ is an eigenvector of $H$ with eigenvalue $\mu$, then $Ev$ is an eigenvector of $H$ with eigenvalue $\mu + 2$ (although not necessarily a non-zero vector). In other words, for any $\mu$ we have

$$
E: V_{\mu} \rightarrow V_{\mu+2}.
$$

By an entirely similar calculation we see that $F: V_{\mu} \rightarrow V_{\mu-2}$.

If we assume that $V$ is an irreducible representation, then it follows that the eigenvalues $\mu$ of $H$ differ from each other by integer multiples of two. Indeed, if $\mu' \in \mathbb{C}$ is one eigenvalue of $H$, then the subspace

$$
\bigoplus_{n \in \mathbb{Z}} V_{\mu'+2n}
$$

is invariant not only for $H$ but also for $E$ and $F$, and therefore actually invariant for the entire $\mathfrak{sl}_2(\mathbb{C})$. Thus the subspace is a subrepresentation, and by irreducibility it must be the entire $V$. In fact we can conclude a little more. For irreducible $V$
the $H$-eigenvalues $\mu$ must form an uninterrupted string of complex numbers, of the form

$$\zeta, \zeta + 2, \zeta + 4, \ldots, \zeta + 2(k-1), \zeta + 2k,$$

since otherwise the direct sum of only a subset of eigenspaces would be invariant for $H$, $E$, and $F$, and would thus be a proper subrepresentation of $V$.

So, assume from now on that $V$ is a finite dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. Denote by $\lambda = \zeta + 2k$ the last number in the above string of $H$-eigenvalues — a priori we have $\lambda \in \mathbb{C}$, but we will soon see that $\lambda$ must be a non-negative integer. Choose a non-zero vector $v_0 \in V_\lambda$. Note that $V_{\lambda+2} = \{0\}$, so necessarily we have $Ev_0 = 0$. We will need to understand the action of $F$ on $v_0$, and concerning that, we have the following:

**Claim II.11.** Denote $v_m = F^m v_0$, for $m \in \mathbb{Z}_{\geq 0}$. Then the vectors $v_0, v_1, v_2, \ldots$ span $V$.

*Proof.* Let $W \subset V$ be the subspace spanned by the above vectors, $W = \text{span} \{F^m v_0 | m \in \mathbb{Z}_{\geq 0}\}$. By irreducibility of $V$, it suffices to show that $W$ is invariant under $H$, $E$, and $F$. By definition $W$ is invariant under $F$. Since $F^m v_0 \in V_{\lambda-2m}$, it is also invariant under $H$. It suffices to check that $EW \subset W$. We calculate

$$E(F^m v_0) = [E, F](F^{m-1} v_0) - F(E(F^{m-1} v_0)).$$

Equation (II.10) serves as the induction step, and to complete the proof, we note that in the case $m = 0$ we have $E(v_0) = Ev_0 = 0 \in W$ by an earlier observation. In fact by this induction we can prove not only that $E(F^m v_0) \in W$, but we moreover obtain the explicit formula

$$E(F^m v_0) = (\lambda - m + 1) m F^{m-1} v_0.$$  (II.11)

The calculation above has some interesting consequences.

**Observation II.12.** All eigenspaces $V_\mu$ of $H$ are one-dimensional.

*Proof.* Indeed, $\mu = \lambda - 2m$ for some $m \in \mathbb{Z}_{\geq 0}$ and $V_{\lambda-2m} = \text{span} \{F^m v_0\}$. □

**Observation II.13.** The representation $V$ is determined by the number $\lambda$.

*Proof.* Indeed, if $d$ is the smallest power of $F$ that annihilates $v_0$, then we see that the vectors $F^m v_0$ for $m = 0, 1, 2, \ldots, d-1$ form a basis of $V$. We have described explicitly the action of $H$, $E$, and $F$ on each basis vector, and the matrix elements of $H$, $E$, and $F$ only involved $\lambda$ as a parameter. □

**Observation II.14.** The dimension of $V$ is $\lambda + 1$, and in particular $\lambda$ is a non-negative integer, $\lambda = \dim(V) - 1 \in \mathbb{Z}_{\geq 0}$.

*Proof.* Let again $d$ be the smallest power of $F$ that annihilates $v_0$. Note that $d = \dim(V)$. The calculation (II.11) is perfectly valid also for $m = d$, so we get

$$0 = E(F^d v_0) = (\lambda - d + 1) d F^{d-1} v_0.$$
But since \( F^{d-1}v_0 \neq 0 \), the prefactor on the right-hand-side must vanish, \((\lambda - d + 1) d = 0\). Also \( d > 0 \), so we must have \( \lambda - d + 1 = 0 \), that is \( d = \lambda + 1 \). \( \square \)

The final observation below follows directly from the earlier ones.

**Observation II.15.** The eigenvalues of \( H \) on \( V \) are

\[
\lambda, \lambda - 2, \lambda - 4, \ldots, -\lambda + 4, -\lambda + 2, -\lambda
\]

and the multiplicity of each eigenvalue is one. In particular, the \( H \)-eigenvalues are all integers, they all have the same parity, and they are symmetric about the origin (i.e. if \( \mu \) is an eigenvalue, then so is \(-\mu\)).

We conclude by the following complete description of all irreducible representations of \( \mathfrak{sl}_2(\mathbb{C}) \).

**Theorem II.22.** For each \( \lambda \in \mathbb{Z}_{\geq 0} \) there exists an irreducible \( \lambda + 1 \)-dimensional representation of \( \mathfrak{sl}_2(\mathbb{C}) \) with basis \( v_0, v_1, \ldots, v_\lambda \) and the actions of \( H, E, \) and \( F \) on this basis given by

\[
Fv_m = \begin{cases} 
  v_{m+1} & \text{for } 0 \leq m < \lambda \\
  0 & \text{for } m = \lambda
\end{cases}
\]

\[
Ev_m = \begin{cases} 
  0 & \text{for } m = 0 \\
  (\lambda - m + 1)m v_{m-1} & \text{for } 0 < m \leq \lambda
\end{cases}
\]

\[
Hv_m = (\lambda - 2m) v_m \quad \text{for all } m.
\]

Denote this representations by \( L(\lambda) \), Any irreducible finite-dimensional representation of \( \mathfrak{sl}_2(\mathbb{C}) \) is isomorphic to \( L(\lambda) \), for some \( \lambda \in \mathbb{Z}_{\geq 0} \).

**Proof.** We have almost proven this already: \( L(\lambda) \) is the representation we have analyzed in this section. We have shown that any finite dimensional irreducible representation of dimension \( d \in \mathbb{Z}_{>0} \) must be \( L(\lambda) \) for \( \lambda = d - 1 \). However, we have not yet strictly speaking shown that such a representation indeed exists. To show the existence, it remains to check that the formulas given above for the linear operators \( H, E, \) and \( F \) on the vector space \( L(\lambda) \) with basis \( v_0, v_1, \ldots, v_\lambda \) actually do define a representation of \( \mathfrak{sl}_2(\mathbb{C}) \). The only thing to check is that for any \( Z, W \in \mathfrak{sl}_2(\mathbb{C}) \) the action of the bracket \([Z,W]\) on \( L(\lambda) \) equals the commutator of the actions of \( Z \) and \( W \). By looking at the calculations done in this section again, you will notice that we have in fact done everything that is needed in such a check. \( \square \)

Let us make some final observations which are useful in analyzing representations of \( \mathfrak{sl}_2(\mathbb{C}) \) that we might encounter. We will use the following fact.

**Fact II.16.** Any (finite dimensional) representation of \( \mathfrak{sl}_2 \) is a direct sum of irreducible representations.

This is the general property of complete reducibility for semisimple Lie algebras, see [??]. It could also be verified more directly in the present case, see Exercise [??].

**Observation II.17.** We have:
• Any representation of $\mathfrak{sl}_2(\mathbb{C})$, in which the $H$-eigenvalues have the same parity and occur with multiplicity one, is necessarily irreducible.
• The number of irreducible subrepresentations of a (finite dimensional) representation of $\mathfrak{sl}_2(\mathbb{C})$ is the sum of multiplicities of 0 and 1 as $H$-eigenvalues.

4.3. Examples of representations of $\mathfrak{sl}_2(\mathbb{C})$

4.3.1. The standard representation

In Example II.20, we noted that the space $V = \mathbb{C}^2$ is a representation of $\mathfrak{sl}_2(\mathbb{C})$, when the elements of $\mathfrak{sl}_2(\mathbb{C})$ are understood as $2 \times 2$-matrices such as in (II.7), and the action of such a matrix on a vector in $\mathbb{C}^2$ is by the usual matrix-vector multiplication. This representation is called the standard representation of $\mathfrak{sl}_2(\mathbb{C})$. If $x = [1 0]^T$ and $y = [0 1]^T$ are the standard basis, then we have $Hx = x$ and $Hy = -y$, so that the $H$-eigenvalues are $+1$ and $-1$, and the corresponding eigenspaces are $\mathbb{C}x$ and $\mathbb{C}y$. From Observation II.17 it follows that the standard representation $V$ is irreducible, so by dimensionality in fact $V \cong L(1)$.

4.3.2. The tensor square of the standard representation

As above, denote by $V = \mathbb{C}^2 = L(1)$ the standard representation. Consider the representation $V \otimes V$. The $H$-eigenvalues on $V \otimes V$ are $+2$ with multiplicity one (eigenvector $x \otimes x$), $0$ with multiplicity two (eigenvectors $x \otimes y$ and $y \otimes x$), and $-2$ with multiplicity one (eigenvector $y \otimes y$). Note that because of the multiplicities, Observation II.17 shows that $V \otimes V$ is not irreducible, but instead decomposes into a direct sum of two irreducible subrepresentations.

Note that $V \otimes V = \text{Sym}^2 V \oplus \bigwedge^2 V$ as a vector space, and also as a representation of $\mathfrak{sl}_2(\mathbb{C})$. The two irreducible subrepresentations of the tensor square $V \otimes V$ of the standard representation are the symmetric square\(^7\) $\text{Sym}^2 V \cong L(2)$, and the alternating square\(^8\) $\bigwedge^2 V \cong L(0)$. Here, $\bigwedge^2 V \cong L(0)$ in fact coincides with the trivial representation.

4.3.3. The adjoint representation

In Example II.21, we noted that The vector space $\mathfrak{sl}_2(\mathbb{C})$ is a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ by the adjoint action. Note that $\text{ad}_H(E) = 2E$, $\text{ad}_H(H) = 0$, and $\text{ad}_H(F) = -2F$, so that the $H$-eigenvalues are $+2$, $0$, and $-2$, each with multiplicity one. The corresponding $H$-eigenspaces are $\mathbb{C}E$, $\mathbb{C}H$, and $\mathbb{C}F$. From Observation II.17 it follows that the adjoint representation is irreducible, in fact isomorphic to $L(2)$, by dimensionality again.

\(^7\)Note that $\dim(\text{Sym}^2 V) = 3$, basis $x^2, xy, y^2$.
\(^8\)Note that $\dim(\bigwedge^2 V) = 1$, basis $x \wedge y$. 
5. Lifting representations from Lie algebra to Lie group

We now illustrate how, in practice, the understanding of representations of a complex Lie algebra (such as $\mathfrak{sl}_2(\mathbb{C})$ in the previous section) allows us to study continuous symmetries that are described by a real Lie group (such as SU$_2$ or SO$_3$).

First of all, we want to note that as long as one is interested in complex representations, we are allowed to replace a real Lie algebra by its complexification.

**Lemma II.23.** Let $\mathfrak{g}$ be a real Lie algebra, and $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C} = \mathfrak{g} \oplus i \mathfrak{g}$ its complexification. Then any complex representation of $\mathfrak{g}$ has a unique structure of representation of $\mathfrak{g}_\mathbb{C}$ (which restricts back to $\mathfrak{g}$ to the original one), and $\text{Hom}_\mathbb{C}(V, W) = \text{Hom}_{\mathfrak{g}_\mathbb{C}}(V, W)$. In other words, the categories of complex representations of $\mathfrak{g}$ and $\mathfrak{g}_\mathbb{C}$ are equivalent.

**Proof.** Let $\rho: \mathfrak{g} \to \text{End}(V)$ be a representation of $\mathfrak{g}$ on a complex vector space $V$. The only $\mathbb{C}$-linear way to extend it to $\mathfrak{g}_\mathbb{C}$ is to define $\rho_\mathbb{C}: \mathfrak{g}_\mathbb{C} \to \text{End}(V)$ by setting $\rho_\mathbb{C}(X + iY) = \rho(X) + i\rho(Y)$. We leave it to the reader to check that this extension maps brackets in $\mathfrak{g}_\mathbb{C}$ to commutators in $\text{End}(V)$, and thus defines a representation of $\mathfrak{g}_\mathbb{C}$. Note that the converse direction is clear — any representation of $\mathfrak{g}_\mathbb{C}$ restricts to a representation of $\mathfrak{g} \subset \mathfrak{g}_\mathbb{C}$.

As for morphisms of representations, if $f_\mathbb{C}: V \to W$ is a morphism of $\mathfrak{g}_\mathbb{C}$-representations, then a fortiori it is a morphism of $\mathfrak{g}$-representations. We only need to show the other direction, that if $f: V \to W$ is a morphism of $\mathfrak{g}$-representations, then it is also a morphism of $\mathfrak{g}_\mathbb{C}$-representations. But this is clear by $\mathbb{C}$-linearity of $f$ and the way the representations $\rho^V_\mathbb{C}$ and $\rho^W_\mathbb{C}$ extend $\rho^V$ and $\rho^W$.

**Example II.24.** Recall that the three-dimensional real Lie algebras $\mathfrak{su}_2$ and $\mathfrak{so}_3$ are isomorphic. We next observe that the complexification of either one is the three-dimensional complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

Consider for example $\mathfrak{so}_3$ with basis $R^x, R^y, R^z$ such that $[R^x, R^y] = R^z$, $[R^y, R^z] = R^x$, and $[R^z, R^x] = R^y$, see Example II.17. The complexification $\mathfrak{so}_3(\mathbb{C}) = \mathfrak{so}_3 \otimes_\mathbb{R} \mathbb{C}$ has a corresponding basis (now over $\mathbb{C}$), which we for clarity denote here by $R^x_\mathbb{C} = R^x \otimes 1$, $R^y_\mathbb{C} = R^y \otimes 1$, $R^z_\mathbb{C} = R^z \otimes 1$. The Lie brackets of these basis elements in $\mathfrak{so}_3(\mathbb{C})$ are just

$$[R^x_{\mathbb{C}}, R^y_{\mathbb{C}}]_{\mathfrak{so}_3(\mathbb{C})} = R^z_{\mathbb{C}}, \quad [R^y_{\mathbb{C}}, R^z_{\mathbb{C}}]_{\mathfrak{so}_3(\mathbb{C})} = R^x_{\mathbb{C}}, \quad [R^z_{\mathbb{C}}, R^x_{\mathbb{C}}]_{\mathfrak{so}_3(\mathbb{C})} = R^y_{\mathbb{C}}.$$ 

We now change to another basis. Denote $R^0 = -2iR^z$ and $R^+ = R^x + iR^y$ and $R^- = R^x - iR^y$, clearly $R^0, R^+, R^-$ also forms a basis of $\mathfrak{so}_3(\mathbb{C})$. The brackets of these new basis elements are easily calculated using the $\mathbb{C}$-bilinearity of $[,]_{\mathfrak{so}_3(\mathbb{C})}$ and the brackets of $R^x_{\mathbb{C}}, R^y_{\mathbb{C}}, R^z_{\mathbb{C}}$ — we get

$$[R^0, R^+]_{\mathfrak{so}_3(\mathbb{C})} = 2R^+, \quad [R^0, R^-]_{\mathfrak{so}_3(\mathbb{C})} = 2R^-, \quad [R^+, R^-]_{\mathfrak{so}_3(\mathbb{C})} = R^0.$$ 

Comparing with the brackets of $H, E, F$ in $\mathfrak{sl}_2(\mathbb{C})$ given in Equation (II.8), we immediately see that the map $\mathfrak{so}_3(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$ defined by linear extension of $R^0 \mapsto H$, $R^+ \mapsto E$, $R^- \mapsto F$ is a Lie algebra isomorphism, $\mathfrak{so}_3(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C})$. Similarly we have $\mathfrak{su}_2(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C})$.

In particular we have equivalences

$$\{\text{complex rep'ns of } \mathfrak{su}_2\} \leftrightarrow \{\text{complex rep'ns of } \mathfrak{su}_2(\mathbb{C})\} \leftrightarrow \{\text{complex rep'ns of } \mathfrak{so}_3\}.$$ 

Recall that we found that the finite dimensional irreducible representations of the three-dimensional complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ are $L(\lambda)$, with $\lambda \in \mathbb{Z}_{\geq 0}$. By Lemma II.23, then, these are also the finite dimensional irreducible complex representations of the real Lie algebras $\mathfrak{su}_2$ and $\mathfrak{so}_3$.

The fact that allows to get from representations of Lie algebras to representations of Lie groups is the following consequence of our two principles for Lie groups.
• Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra.

(i) Every representation $\rho: G \to \text{Aut}(V)$ of the Lie group $G$ defines a representation $\rho = (d\phi)|_e: \mathfrak{g} \to \text{End}(V)$ of the Lie algebra $\mathfrak{g}$, and any intertwining map of representations of $G$ is an intertwining map of representations of $\mathfrak{g}$.

(ii) If $G$ is simply connected, then $\rho \mapsto \rho = (d\phi)|_e$ gives an equivalence of categories of representations of $G$ and representations of $\mathfrak{g}$. In particular, every representation of the Lie algebra $\mathfrak{g}$ is the derivative at $e$ of some representation of the Lie group $G$.

Example II.25. Recall that $SU_2$ is simply connected by Theorem II.12. As a special case of the theorem above we get the equivalence

$$\{\text{representations of } SU_2\} \leftrightarrow \{\text{representations of } su_2\}.$$ 

In particular, the irreducible complex representations of $SU_2$ are $L(\lambda)$, $\lambda \in \mathbb{Z}$.

The easiest way to give the explicit $SU_2$ action on $L(\lambda)$ is perhaps to realize that $L(\lambda) = \text{Sym}^\lambda \mathbb{C}^2$ is a symmetric tensor product of the standard representation $\mathbb{C}^2$. The action of $SU_2$ on the standard representation $\mathbb{C}^2$ is the obvious matrix-vector multiplication, and the action on the symmetric tensor power can be read off from here. The example of the three-dimensional irreducible $L(2)$, for example, in the basis $x^2, xy, y^2$, gives that

$$\begin{bmatrix} \xi_1 + i \xi_2 & -\xi_3 + i \xi_4 \\ \xi_3 + i \xi_4 & \xi_1 - i \xi_2 \end{bmatrix} \in SU_2$$

is represented by the matrix

$$\begin{bmatrix} \xi_1^2 + 2i \xi_2 \xi_3 - \xi_2^2 & -\xi_1 \xi_3 - i \xi_2 \xi_4 + i \xi_1 \xi_4 - \xi_2 \xi_4 & \xi_3^2 - 2i \xi_2 \xi_3 - \xi_2^2 \\ 2\xi_1 \xi_3 + 2i \xi_2 \xi_4 + 2i \xi_1 \xi_4 - 2 \xi_2 \xi_4 & \xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2 & -2\xi_1 \xi_3 + 2i \xi_2 \xi_3 + 2i \xi_1 \xi_4 + 2 \xi_2 \xi_4 \\ \xi_3^2 + 2i \xi_2 \xi_3 - \xi_2^2 & \xi_1 \xi_3 - i \xi_2 \xi_4 + i \xi_1 \xi_4 + \xi_2 \xi_4 & \xi_1^2 - 2i \xi_2 \xi_3 - \xi_2^2 \end{bmatrix}.$$ 

Although the statement of the previous fact appears to only concern simply connected Lie groups, it can in fact be used for any connected Lie groups $G$. We only need to pass through the universal cover $\tilde{G}$.

Example II.26. The group $SO_3$ of rotations of the Euclidean space $\mathbb{R}^3$ is connected but not simply connected: by Theorem ?? its universal cover is $SU_2$, and the kernel of the covering map $\phi: SU_2 \to SO_3$ is the two element subgroup $\Gamma = \{\pm I_2\}$ of the center of $SU_2$. We have $SO_3 = SU_2/\Gamma$.

By Example II.25, the irreducible complex representations of $SU_2$ are the same as the irreducible representations of $sl_2(\mathbb{C})$, i.e., $L(\lambda)$ for $\lambda \in \mathbb{Z}_{\geq 0}$. To get the irreducible representations of $SO_3$, the remaining question is: which ones among $L(\lambda)$ are trivial on $\Gamma$?

The solution is easy once we notice that

$$-I_2 = \exp(2\pi S^z) \in SU_2, \quad \text{where } S^z = -\frac{i}{2} \sigma_3 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \in su_2.$$ 

To lift a representation $\rho: su_2 \to \text{End}(V)$ to a representation of $\varrho: SU_2 \to \text{Aut}(V)$, we must set $\varrho(\exp(X)) = \exp(\rho(X))$. In particular we have $\varrho(-I_2) = \exp(2\pi \rho(S^z))$. On $L(\lambda)$, the operator $\rho(S^z) = \frac{1}{2} i \rho(H)$ is diagonalizable with eigenvalues $i^2, i (\frac{3}{2} - 1), \ldots, -i^2$. If $\lambda$ is an even integer, then these are integer multiples of $i$ and $\varrho(-I_2) = \exp(2\pi \rho(S^z))$ is the identity operator on the representation, so the representation is trivial on $\Gamma = \{\pm I_2\}$. If $\lambda$ is an odd integer, then the eigenvalues of $S^z$ are half-integer multiples of $i$, and $\varrho(-I_2) = \exp(2\pi \rho(S^z))$ is minus identity, so the representation is non-trivial on $\Gamma = \{\pm I_2\}$.

We conclude that the irreducible complex representations of $SO_3$ are $L(\lambda)$ with $\lambda \in 2 \mathbb{Z}_{\geq 0}$.