PHYS-C0252 - Quantum Mechanics

Exercise set 2

Due date : November 6, 2020 before 10.00

1. Let $\hat{A}$ be an arbitrary linear operator acting on $\mathcal{H}$. Let $a \in \mathbb{C}$ and $|\phi\rangle, |\psi\rangle \in \mathcal{H}$. Using the definition of the Hermitian conjugate, $(|\psi\rangle, \hat{A}|\phi\rangle) = (\hat{A}^\dagger|\psi\rangle, |\phi\rangle)$ denoted by the dagger, and the properties of the inner product $(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, prove the following identities:

   i. $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$

   ii. $(a\hat{A})^\dagger = a^* \hat{A}^\dagger$

   iii. $(|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi|$

   iv. $(\hat{A}^\dagger)^\dagger = \hat{A}$

2. Show that the following one-dimensional operators are Hermitian

   i. Position operator $\hat{x}$. Hint: You may use the complete eigenbasis $\{|x\rangle\}_x$ of $\hat{x}$. That is $\hat{x}|x\rangle = x|x\rangle$.

   ii. Momentum operator $\hat{p}_x = -i\hbar\frac{\partial}{\partial\hat{x}}$. Hint: Use the position representation of an arbitrary quantum state $|\psi\rangle \in \mathcal{H}$, i.e., $\langle x|\psi\rangle = \psi(x)$, and the position presentation of the identity operator.

   iii. Kinetic energy $\hat{T} = \hat{p}_x^2/2m$.

3. A quantum system described by the Hamiltonian $\hat{H}$ is initially in the state $|\psi\rangle = N[\sqrt{2}|\phi_1\rangle + \sqrt{3}|\phi_2\rangle + |\phi_3\rangle + |\phi_4\rangle]$, where $|\phi_n\rangle$ are the eigenstates of energy such that $\hat{H}|\phi_n\rangle = nE_0|\phi_n\rangle$.
   (a) Normalize $|\psi\rangle$ to find a suitable scalar N.
   (b) Let the energy of $|\psi\rangle$ be measured. Give all possible measurement results and their corresponding probabilities. Assume that the measurement is ideal, i.e., no measurement errors occur.
   (c) Consider an operator $\hat{X}$, the action of which on $|\phi_n\rangle$ $(n = 1, 2, 3, 4)$ is defined by $\hat{X}|\phi_n\rangle = (n + 2)x_0|\phi_n\rangle$, where x_0 is a real-valued scalar. Suppose that a measurement of the energy of the above-defined $|\psi\rangle$ yields $4E_0$. Assume that immediately afterwards, we ideally measure the physical quantity corresponding to $\hat{X}$. What is the value for the quantity obtained in the latter measurement?

4. Consider a system described by the Hamiltonian $\hat{H} = \epsilon(-\mathrm{i}|0\rangle\langle 1| + \mathrm{i}|1\rangle\langle 0|)$, where $\{|0\rangle, |1\rangle\}$ form an orthonormal basis of the considered Hilbert space and ϵ is a real-valued constant with the dimension of energy.

   (a) Find the eigenenergies, E_1 and E_2 , of $\hat{H}$.

   (b) If the system is initially (at $t = 0$) in the state $|\psi(0)\rangle = |0\rangle$, find the probability for a measurement of energy at $t = 0$ to yield: (i) E_1 and (ii) E_2.

   (c) Find the expectation value of the energy $\langle H \rangle$ at time $t = 0$.

   (d) Find $\sqrt{\langle H^2 \rangle - \langle H \rangle^2}$ at time $t = 0$.

   (e) Find the state $|\psi(t)\rangle$. Hint: you need to solve the Schrödinger equation.