Aalto University
School of Engineering

# GEO – E1050
# Finite Element Method in Geoengineering
# Autumn 2020

Wojciech Sołowski

# Contact

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**See My Courses for materials**

Potts D & Zdravkovic L, 1999 Finite Element Analysis in Geotechnical Engineering

**in library:**

Zienkiewicz, O. C. ; Taylor, Robert L., The finite element method, Vol 1-3
(also can be an extra author in the latest editions)

Smith IM, Griffiths DV, Margetts L. 2014. Programming the finite element method

Rao SS, 2005, The Finite Element Method in Engineering

Hughes TJR, 2000 Finite element method : linear static and dynamic finite element analysis

Munjiza A, Rougier E, Knight EE. 2014. Large Strain Finite Element Method : A Practical Course

**Feedback representative?**

Volunteers?

**2-3 persons needed...**

# Hey…

## Who are you?

What are Your expectations from this course?

What would You like to learn / focus on?

Aalto University
School of Engineering

# GEO – E1050
# Finite Element Method in Geoengineering

## Lecture 1: Introduction

Wojciech Sołowski

# Lectures: what is planned

**12 lectures in total**

**First 4 lectures: theory**

- **Refresh of continuum mechanics & elasticity**
- **Finite element theory (lectures 2-5)**
  - **derivation (simplified / generalized)**
  - **elements etc.**
  - **touching on derivation of other numerical methods**

# Lectures: what is planned

12 lectures in total

Next 3 lectures: materials modelling

- Theory of elasticity / plasticity
- Constitutive models: Mohr Coulomb, Hoek-Brown, perfectly plastic models

Necessary in the course – if you model a material, you need a constitutive model

Constitutive modelling continuation:

Advanced Soil Mechanics

Numerical Methods in Geotechnics

# Lectures: what is planned

**12 lectures in total**

**Next 3 lectures: “Advanced” Finite Element Method**

- More advanced features of FEM algorithms
- some simple issues related to Finite Element Analysis
- also summary and recap of first part of FEM

**Advanced Finite Element Analysis in:**

**Numerical Methods in Geotechnics**

# Lectures: what is planned

**12 lectures in total**

**Final 2 lectures: beyond Finite Element Method**

**About, among others:**

- **Finite Difference Method**
- **Material Point Method**
- **Discrete Element Method**

# Exercises: what is planned

**11 exercise sessions in total**

**Approximately first 3 exercises**

**Matlab & Matlab based Finite Element Method Code**

# Exercises: what is planned

**12 exercises in total**

**Approximately first 3 exercises**

**Matlab & Matlab based Finite Element Method Code**

**Approximately next 4 exercises**

**Solving simple problems in Finite Element Method codes, comparing with Matlab code**

# Exercises: what is planned

**12 exercises in total**

**Approximately first 4 exercises**

**Matlab & Matlab based Finite Element Method Code**

**Solving simple problems in Finite Element Method codes, comparing with Matlab code**

**Approximately next 6 exercises: using the Finite Element Method**

**Simple problems**, as you do not have enough background in modelling; still useful for learning how to do simulations

**Last exercise: Finite Difference Method**

# Workload

5 credits course = 5 x 27h = 135h

**6 weeks course = 22.5 h per week**

**At the university = 8h**

**Leaving 14.5 h per week of work for you**

# Workload

5 credits course = 5 x 27h = 135h

**6 weeks course = 22.5 h per week**

**At the university = 8h**

**Approximately 2h a day every day…**

# Workload

5 credits course = 5 x 27h = 135h

**6 weeks course = 22.5 h per week**

**At the university = 8h**

**→ extra 3h a day every weekdays**

**- or almost 4h on the day you have class**

# Workload

**Btw, yearly workload is 60cr = 135 x 12 = 1620h**

**- workload at best universities is higher**

**as they say at MIT: good grades, enough sleep, or a social life — pick any two.**

This course will offer the opportunity to learn a lot more than is required during the exam (note – if you do that, the workload is likely higher)

- during last years the workload has been reduced – but also the grades for the course have been lower – even though that the feedback from you has been taken into account and is not bad
  - why the discrepancy?

# Grading: we need to decide

Suggestion:

- attend all the lectures / do the lecture tests
- attend all the exercises / do the tasks

Workload: ~48h

- score from the tests from the lectures should help you pass the partial exams, and the presence during exercises should help you with passing them
- **if you (re)do the partial exam at different date (you can re-take it 1-2 times usually), lecture test score is not added anymore – just the first exam**

# Grading: we need to decide

attend all the lectures

After some lectures there will be short test in My Courses – answers should be filled based on lecture material

Automatically graded.

Based on the grade, you will get extra % score for the exams

# Grading: we need to decide

attend the exercises

Each exercise can be graded in the end, based on what has been done (let the teacher know you want grade at the end of the exercise); Standard: a report

there is some lowest amount of work required at the exercises to get grade 1… and usually when the work is graded at the end of the exercise, it will be grade 1…

For a higher grade, you need to submit a report. In case you are late with the report, the maximum grade is lowered

# Grading: we need to decide

To pass:

Pass all the exercises and do the homework , reports etc. (graded, total 50% of total grade)

Pass the 3 tests during lectures (graded, 50% of total grade)

**NO FINAL EXAM**

# Grading: we need to decide

## Workload: 135h

12 x 2h = 24h - attending lectures

11 x 2h = 22h - attending exercises

8 x 2h = 16h - preparation for lectures and lecture tests

11 x 4h = 44h - homework / finishing exercises

3 x 9h = 27 h - preparation for tests (tests held during lectures)

**the numbers add up to 133h this year…**

# Grading: we need to decide

**Workload: 135h**

All the lecture tests can be retaken if you are unhappy with the grade

If you have any problems, issues and need help you can always make an appointment with me and ask!

# Decision?

Any other questions ?

Anything unclear ?

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# Refreshing memory: continuum mechanics and elasticity

# Lecture 1 - refresh…

## After the lecture you should:

- Understand and can explain what is stress and strain (continuum mechanics course)
- Understand and can explain why stress and strain tensors are symmetric (continuum mechanics course)
- Understand the role of stress – strain relationship (continuum mechanics course)
- Have understanding about materials behaviour and limits of elasticity (materials course, continuum mechanics course)
- Understand plain strain assumption and when to use it…

# refresh…

## After the lecture **and exercises 1** you also should:

- **Be able to follow vector / matrix algebra (maths courses, continuum mechanics course)**
    - ideally with little effort… so you can concentrate on the meaning instead of the maths
- **Understand and follow various tensor notations (continuum mechanics course)**
    - ideally so the notation in the book can be followed without much thinking
- **Be able to perform vector / matrix calculus**
    - **By hand (understand the principles) (math courses)**
    - **… in Matlab (generally usually more useful)**

# refresh + a bit of Matlab…

## After the lecture **and exercises 1** you also should:

### …Be able to perform vector / matrix calculus in Matlab

- **For good grade you will need to grasp some very basic concepts on how to do programming in Matlab**
- **Matlab will be used for 3 weeks, so things will come slowly.**
- **Matlab is a useful tool – and used widely in MSc theses, research as well as in companies**
  - **GNU OCTAVE = open source Matlab**
  - **Matlab has excellent help system – you can self learn it quite easily**

**This is not a course on Matlab, so the ability to use Matlab fluently is not graded and the course will introduce what is required**

# Vector & Matrix algebra

**Resources:**

**Lecture slides & exercises material**
**- some exercises material exceeds requirements for grade 5**

**Appendix A & B in Zienkiewicz book (slightly over / around the required knowledge for grade 5; slightly different take on the subject)**

**Brannon "Functional and Structured Tensor Analysis for Engineers" (in MyCourses)**
**- gives way more above what is required in this course**
**- one of the best references on the subject**
**- advanced problems explained in an easy way**

# Matrix algebra

**Essential:**

**Matrix addition:** **A + B = C**

**A & B – same dimensions**
**We add all corresponding elements…**

$$\begin{array}{ccccc} \mathbf{A} & & \mathbf{B} & & \mathbf{C} \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} & + & \begin{bmatrix} 9 & 8 & 7 \\ 4 & 5 & 6 \\ 2 & 3 & 1 \end{bmatrix} & = & \begin{bmatrix} 10 & 10 & 10 \\ 8 & 10 & 12 \\ 10 & 10 & 10 \end{bmatrix} \end{array}$$

# Matrix algebra

**Essential:**

**Matrix transposition: $\mathbf{A}^T$**

**Transposition: we change the rows into columns and vice versa:**

$$\begin{bmatrix} 1 & 5 & 9 \\ 3 & 7 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 8 \end{bmatrix}$$

$$\begin{bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \\ \Delta\sigma_{13} \\ \Delta\sigma_{23} \end{bmatrix}^T = \begin{bmatrix} \Delta\sigma_{11} & \Delta\sigma_{22} & \Delta\sigma_{33} & \Delta\sigma_{12} & \Delta\sigma_{13} & \Delta\sigma_{23} \end{bmatrix}$$

$$\mathbf{(A^T)^T=A}$$

# Matrix algebra

## Essential:

### Matrix multiplication: Ax=B

$$[A]\{x\} = \{b\}$$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3
\end{aligned}$$

$$\mathbf{A} \equiv [A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \qquad \mathbf{x} \equiv \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} \quad \text{and} \quad \mathbf{b} \equiv \{b\} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$\mathbf{f}^T\mathbf{u} = \mathbf{u}^T\mathbf{f} \qquad (\mathbf{A}\,\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$$

# Matrix algebra

## Matrix multiplication: row * column rule

$$\begin{cases}
\Delta\sigma_{11} = \dfrac{E}{(1+v)(1-v)}\left([1-v]\Delta\varepsilon_{11} + v[\Delta\varepsilon_{22} + \Delta\varepsilon_{33}]\right) \\
\Delta\sigma_{22} = \dfrac{E}{(1+v)(1-v)}\left([1-v]\Delta\varepsilon_{22} + v[\Delta\varepsilon_{11} + \Delta\varepsilon_{33}]\right) \\
\Delta\sigma_{33} = \dfrac{E}{(1+v)(1-v)}\left([1-v]\Delta\varepsilon_{33} + v[\Delta\varepsilon_{11} + \Delta\varepsilon_{22}]\right) \\
\Delta\sigma_{12} = \dfrac{E(1-2v)}{(1+v)(1-v)}\Delta\varepsilon_{12} \\
\Delta\sigma_{13} = \dfrac{E(1-2v)}{(1+v)(1-v)}\Delta\varepsilon_{13} \\
\Delta\sigma_{23} = \dfrac{E(1-2v)}{(1+v)(1-v)}\Delta\varepsilon_{23}
\end{cases}$$

$$\overbrace{\begin{bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \\ \Delta\sigma_{13} \\ \Delta\sigma_{23} \end{bmatrix}}^{\Delta\boldsymbol{\sigma}} = \overbrace{\frac{E}{(1+v)(1-v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ & & & 1-2v & 0 & 0 \\ & & & & 1-2v & 0 \\ . & & & & & 1-2v \end{bmatrix}}^{\mathbf{D}} \overbrace{\begin{bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{33} \\ \Delta\varepsilon_{12} \\ \Delta\varepsilon_{13} \\ \Delta\varepsilon_{23} \end{bmatrix}}^{\Delta\boldsymbol{\varepsilon}}$$

# Matrix algebra

**Inverse of a matrix:**

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= b_4 \end{aligned}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\,\mathbf{A}^{-1}$$

**Symmetric matrices:**

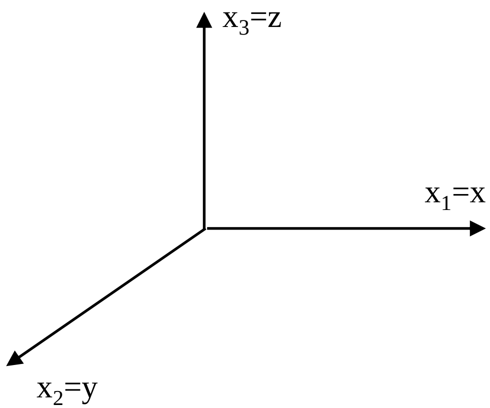
$$a_{ij} = a_{ji} \quad \text{or} \quad \mathbf{A} = \mathbf{A}^T$$

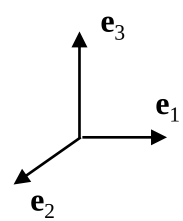
$$\mathbf{A}^{-1} = (\mathbf{A}^{-1})^T \equiv \mathbf{A}^{-T}$$

# Vectors:

**Unit vectors:** $\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3$

**correspond to:** $x_1 \ x_2 \ x_3$ or x, y z directions




**Orthogonal!**

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

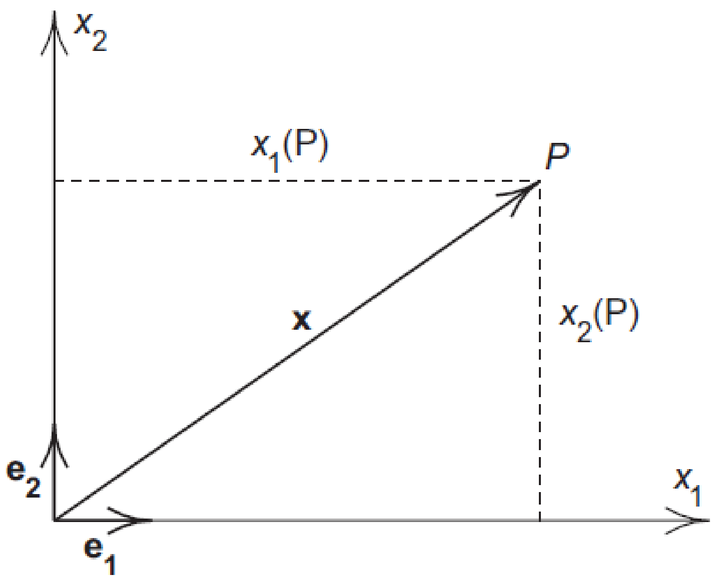
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Unit vectors are useful formality… Transforms vectors into scalar values and vice versa.**

# Vectors:

**Unit vectors: $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$**

**correspond to:** $x_1$ $x_2$ $x_3$ or x, y z directions




In 3 dimensions:

$$x_i = \mathbf{e}_i \cdot \mathbf{x}, \quad i = 1, 2, 3$$

$$\mathbf{x} = \sum_{i=1}^{3} x_i \, \mathbf{e}_i$$

$$\mathbf{x} = x_i \, \mathbf{e}_i = x_1 \, \mathbf{e}_1 + x_2 \, \mathbf{e}_2 + x_3 \, \mathbf{e}_3$$

So, displacement **u** is: $\mathbf{u} = u_i \, \mathbf{e}_i$

# Derivative:

Derivative with respect to component $x_i$ $\qquad \dfrac{\partial}{\partial x_i} \equiv (\ )_{,i}$

In particular, derivative of an array with respect to itself is:

$$y_1 = x_1 \qquad \frac{\partial y_1}{\partial x_1} = 1 \qquad \frac{\partial y_1}{\partial x_2} = 0 \qquad \frac{\partial y_1}{\partial x_3} = 0$$

$$y_2 = x_2 \qquad \frac{\partial y_2}{\partial x_1} = 0 \qquad \frac{\partial y_2}{\partial x_2} = 1 \qquad \frac{\partial y_2}{\partial x_3} = 0$$

$$y_3 = x_3 \qquad \frac{\partial y_3}{\partial x_1} = 0 \qquad \frac{\partial y_3}{\partial x_2} = 0 \qquad \frac{\partial y_3}{\partial x_3} = 1$$

$$\frac{\partial y_i}{\partial x_j} = \delta_{ij} \qquad \text{since } y_i = x_i$$

# Derivative:

Derivative with respect to component $x_i$ $\qquad \dfrac{\partial}{\partial x_i} \equiv (\,)_{,i}$

So, displacement **u** is: $\qquad \mathbf{u} = u_i\,\mathbf{e}_i$

$$\mathbf{u}_{,j} = u_{i,j}\mathbf{e}_i + u_i\mathbf{e}_{i,j} = u_{i,j}\mathbf{e}_i$$

because $\mathbf{e}_{i,j} = \mathbf{0}$

…because in Cartesian coordinate system the unit vector do not change direction / length along any direction (are direction independent)

# Tensors & tensor notation

**Resources:**

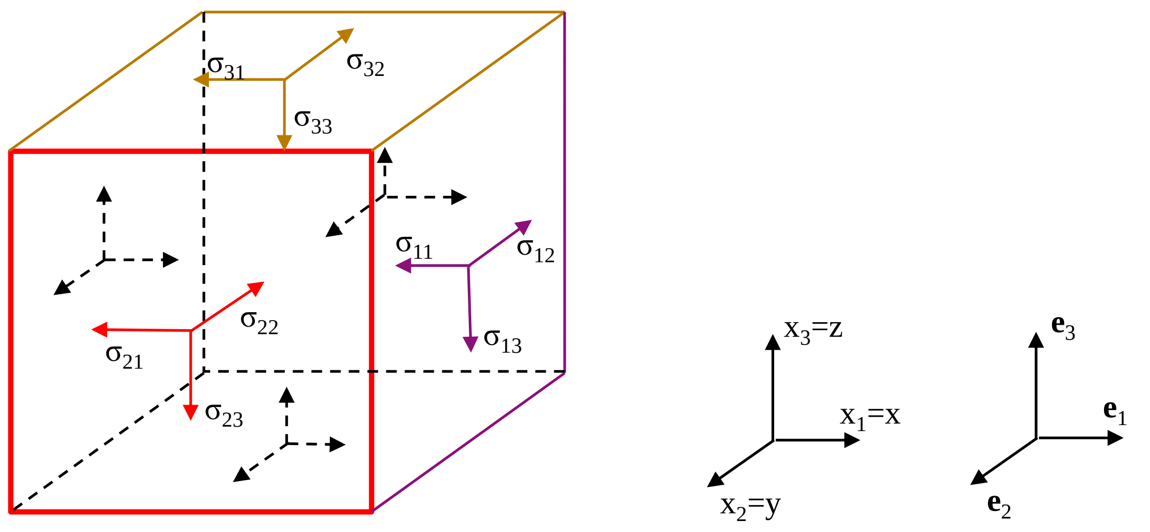
**Lecture slides, lecture & exercises material**

**Appendix C in Zienkiewicz book (somewhat exceeds the required knowledge for grade 5)**

**Brannon "Functional and Structured Tensor Analysis for Engineers" (in MyCourses)**
- **gives way more above what is required in this course**
- **one of the best references on the subject**
- **advanced problems explained in an easy way**

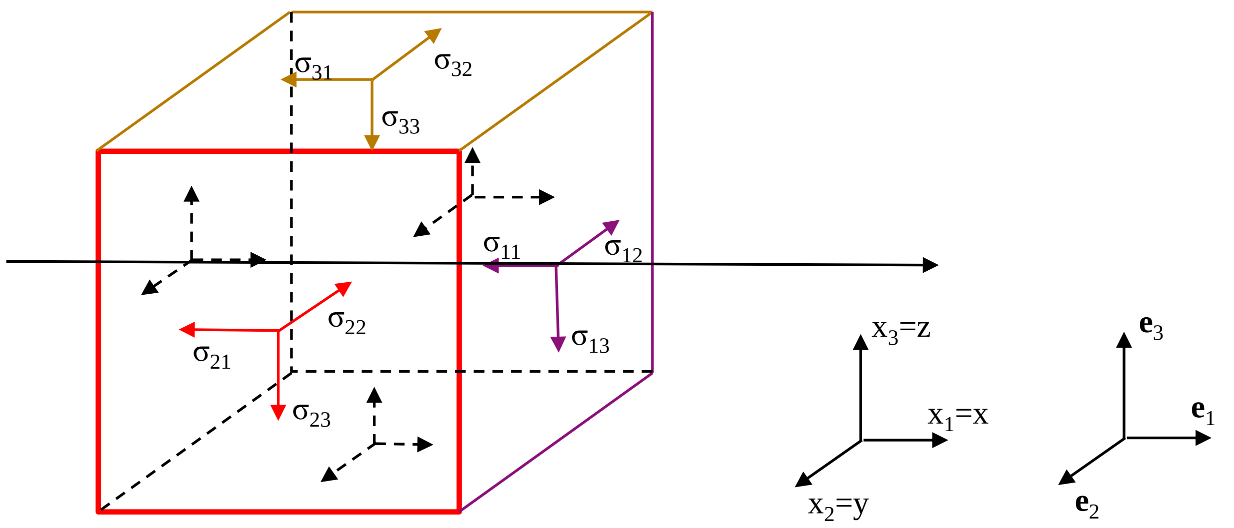
# Tensors & tensor notation

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \sigma_{ij} \quad (i, j = 1,2,3)$$



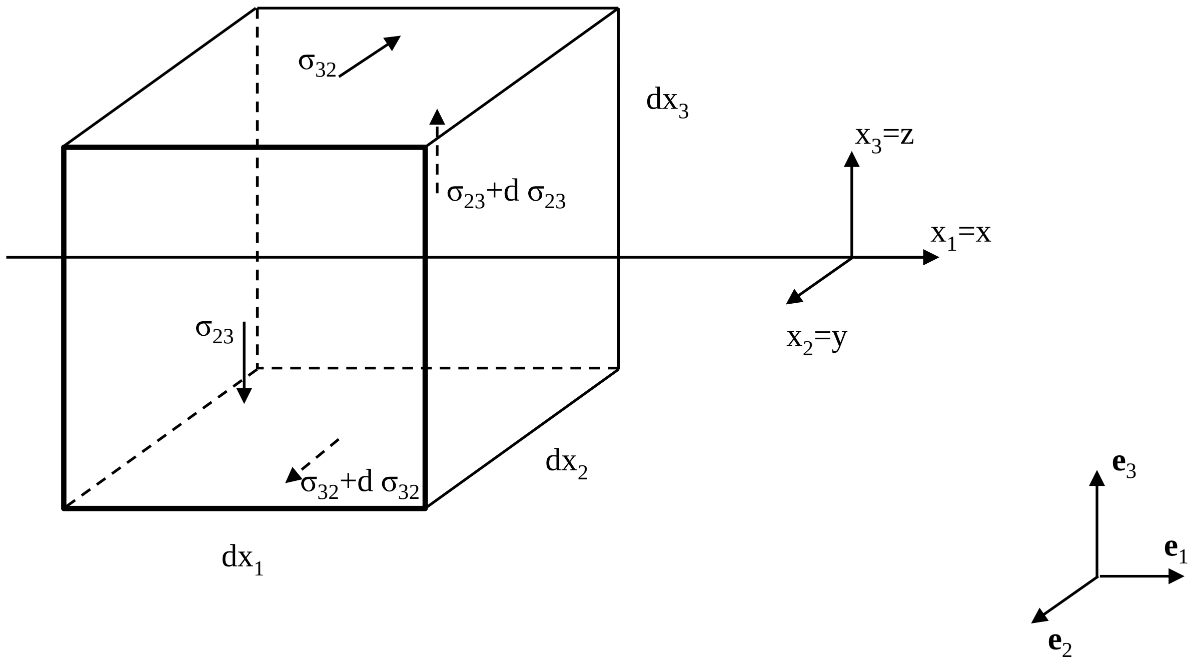
# Momentum balance

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \sigma_{ij} \quad (i, j = 1,2,3)$$



# Momentum balance

Components 23 and 32 must be identical (the component related to the increase of the stress $d\sigma_{ij}$, when multiplied by the dimension $dx_k$ is neglected as it is a product of two infinitesimal quantities

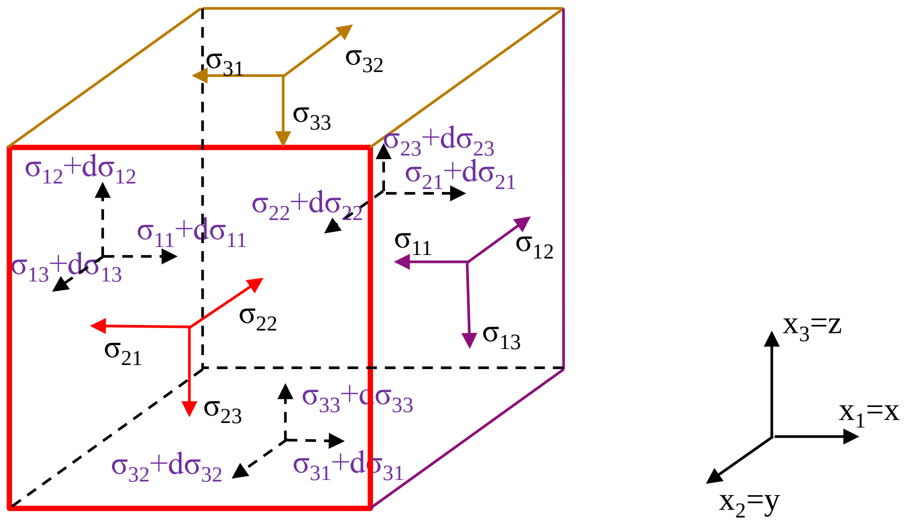


# Momentum balance

Similarly all other ij , i ≠ j components:

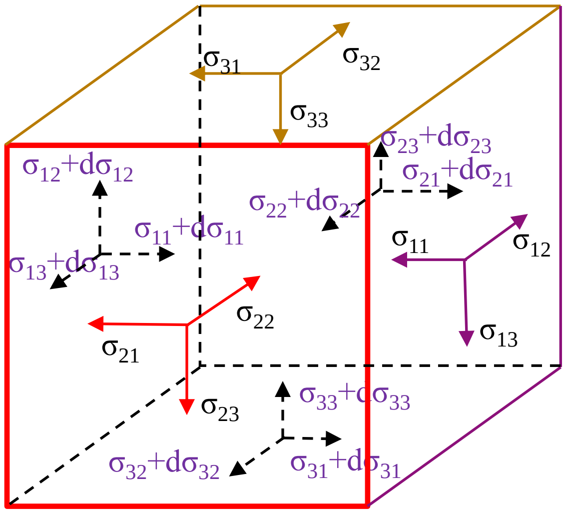
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ & \sigma_{yy} & \sigma_{yz} \\ . & & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ . & & \sigma_{33} \end{bmatrix} = \sigma_{ij} \quad (i, j = 1,2,3), \; \sigma_{ij} = \sigma_{ji}$$

# Equilibrium equations



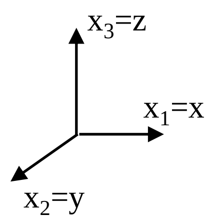

For the stress cube to be in balance, the total of forces acting in each direction must be zero

# Equilibrium equations




$$\sigma_{11}dx_2dx_3 - (\sigma_{11} + d\sigma_{11})dx_2dx_3 + \sigma_{31}dx_1dx_2 - (\sigma_{31} + d\sigma_{31})dx_1dx_2 + \sigma_{21}dx_1dx_3 - (\sigma_{21} + d\sigma_{21})dx_1dx_3 = 0$$

$$d\sigma_{11}dx_2dx_3 + d\sigma_{31}dx_1dx_2 + d\sigma_{21}dx_1dx_3 = 0$$




For the stress cube to be in balance, the total of forces acting in each direction must be zero

# Equilibrium equations

$$\sigma_{11} dx_2 dx_3 - (\sigma_{11} + d\sigma_{11}) dx_2 dx_3 + \sigma_{31} dx_1 dx_2 - (\sigma_{31} + d\sigma_{31}) dx_1 dx_2 + \sigma_{21} dx_1 dx_3 - (\sigma_{21} + d\sigma_{21}) dx_1 dx_3 = 0$$

$$d\sigma_{11} dx_2 dx_3 + d\sigma_{31} dx_1 dx_2 + d\sigma_{21} dx_1 dx_3 = 0$$

We need to get rid of the infinitesimal dimensions…

Stress would vary over their direction only, hence:

$$d\sigma_{11} = \frac{\partial \sigma_{11}}{\partial x_1} dx_1 \qquad d\sigma_{12} = \frac{\partial \sigma_{12}}{\partial x_2} dx_2 \qquad d\sigma_{13} = \frac{\partial \sigma_{13}}{\partial x_3} dx_3$$

And we get…

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0$$

# Equilibrium equations

Taking into account some external force acting on the cube, for example unit weight, we end up with

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \gamma_x = 0$$

Which also works in 2 other directions:

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \gamma_y = 0 \qquad \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \gamma_z = 0$$

# Equilibrium equations

Using short notation…

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \gamma_x = 0 \qquad \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \gamma_y = 0$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \gamma_z = 0$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \gamma_i = 0 \ \ , \ where\ i, j = 1,2,3$$

# Strain

Many definitions of strain are possible…

- for definition derived from displacement gradient, see Zienkiewicz book, appendix C

- for even better definitions, see & Brannon book

- for **decent** grade, you do not have to bother with those too much, especially if you did not hear about them beforehand

- **simplest definition given here is required!**

# Strain: simplest definition

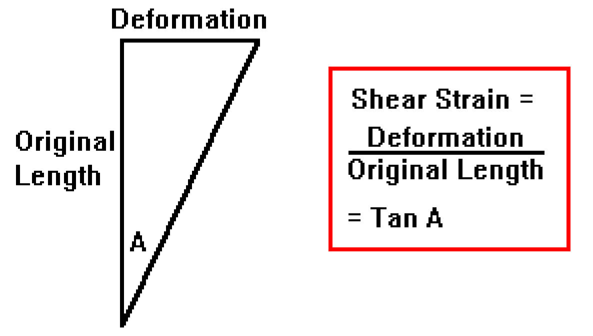
Strain = amount of displacement in given length of material

$$\varepsilon_{11} = \varepsilon_x = -\frac{\partial u}{\partial x} = -\frac{\partial u_1}{\partial x_1} \qquad \varepsilon_{22} = \varepsilon_y = -\frac{\partial v}{\partial y} = -\frac{\partial u_2}{\partial x_2} \qquad \varepsilon_{33} = \varepsilon_z = -\frac{\partial w}{\partial z} = -\frac{\partial u_3}{\partial x_3}$$

But… we have the shear stress… and the cube may not change the volume but will change the shape… what do do?

# Shear Strain: simplest definition

…. Hey, it seem to be related to displacement difference in orthogonal direction… so:

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# Shear Strain: simplest definition

And in 3D

$$\varepsilon_{12} = \varepsilon_{xy} = 0.5\gamma_{xy} = 0.5\left( -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = 0.5\left( -\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$$

$$\varepsilon_{13} = \varepsilon_{xz} = 0.5\gamma_{xz} = 0.5\left( -\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) = 0.5\left( -\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3} \right)$$

$$\varepsilon_{23} = \varepsilon_{yz} = 0.5\gamma_{yz} = 0.5\left( -\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = 0.5\left( -\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right)$$

## All components definition:

$$\varepsilon_{ij} = 0.5\left( -\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right), \quad i, j = 1,2,3$$

# Finally, we have

**15 unknowns**: 6 stresses, 6 strains and three displacements
**9 equations**: 3 equilibrium and 6 compatibility (strain definitions)

**6 equations needed: linking stress and strain**

**CONSTITUTIVE RELATIONSHIP**

$$d\boldsymbol{\sigma} = \mathbf{D}d\boldsymbol{\varepsilon}$$

**D** is a matrix defining the stress-strain relationship

# Elasticity

$$d\boldsymbol{\sigma} = \mathbf{D}d\boldsymbol{\varepsilon}$$

**Simplest definition of matrix D:**
- **constant**

**if material is isotropic (behaves the same in every direction), we can have it defined with only 2 constants!**

# Elasticity

$$d\boldsymbol{\sigma} = \mathbf{D}d\boldsymbol{\varepsilon}$$

**Pick any two… bulk modulus K, shear modulus G, Young's modulus E, Poisson ratio E, Lame's first parameter λ**

| | | K | G | E | v | λ |
|---|---|---|---|---|---|---|
| K & G | | K | G | $\frac{9KG}{3K+G}$ | $\frac{3K-2G}{2(3K+G)}$ | $K-\frac{2}{3}G$ |
| E & v | | $\frac{E}{3(1-2v)}$ | $\frac{E}{2(1+v)}$ | E | $v$ | $\frac{Ev}{(1+v)(1-2v)}$ |
| K & v | | K | $\frac{3K(1-2v)}{2(1+v)}$ | $3K(1-2v)$ | $v$ | $(1-2v)$ |
| λ & v | | $\frac{\lambda(1+v)}{3v}$ | $\frac{\lambda(1-2v)}{2v}$ | $\frac{\lambda(1+v)(1-2v)}{v}$ | $v$ | λ |
| λ & G | | $\lambda+\frac{2}{3}G$ | G | $\frac{G(3\lambda+2G)}{\lambda+G}$ | $\frac{\lambda}{2(\lambda+G)}$ | λ |

# Elasticity

$$d\boldsymbol{\sigma} = \mathbf{D}d\boldsymbol{\varepsilon}$$

$$\begin{cases} \Delta\sigma_{11} = \dfrac{E}{(1+v)(1-2v)}\left([1-v]\Delta\varepsilon_{11} + v[\Delta\varepsilon_{22} + \Delta\varepsilon_{33}]\right) \\ \Delta\sigma_{22} = \dfrac{E}{(1+v)(1-2v)}\left([1-v]\Delta\varepsilon_{22} + v[\Delta\varepsilon_{11} + \Delta\varepsilon_{33}]\right) \\ \Delta\sigma_{33} = \dfrac{E}{(1+v)(1-2v)}\left([1-v]\Delta\varepsilon_{33} + v[\Delta\varepsilon_{11} + \Delta\varepsilon_{22}]\right) \\ \Delta\sigma_{12} = \dfrac{E(1-2v)}{(1+v)(1-2v)}\Delta\varepsilon_{12} \\ \Delta\sigma_{13} = \dfrac{E(1-2v)}{(1+v)(1-2v)}\Delta\varepsilon_{13} \\ \Delta\sigma_{23} = \dfrac{E(1-2v)}{(1+v)(1-2v)}\Delta\varepsilon_{23} \end{cases}$$

# Elasticity

$$d\boldsymbol{\sigma} = \mathbf{D}\,d\boldsymbol{\varepsilon}$$

$$\left\{ \overbrace{\begin{bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \\ \Delta\sigma_{13} \\ \Delta\sigma_{23} \end{bmatrix}}^{\Delta\boldsymbol{\sigma}} = \overbrace{\frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ & & & 1-2v & 0 & 0 \\ & & & & 1-2v & 0 \\ . & & & & & 1-2v \end{bmatrix}}^{\mathbf{D}} \overbrace{\begin{bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{33} \\ \Delta\varepsilon_{12} \\ \Delta\varepsilon_{13} \\ \Delta\varepsilon_{23} \end{bmatrix}}^{\Delta\boldsymbol{\varepsilon}} \right.$$

# Elasticity in triaxial space

$$d\boldsymbol{\sigma} = \mathbf{D}d\boldsymbol{\varepsilon}$$

$$\left\{ \underbrace{\begin{bmatrix} p \\ q \end{bmatrix}}_{\Delta\boldsymbol{\sigma}} = \underbrace{\begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} \Delta\varepsilon_v \\ \Delta\varepsilon_q \end{bmatrix}}_{\Delta\boldsymbol{\varepsilon}} \right.$$

$$p = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$$

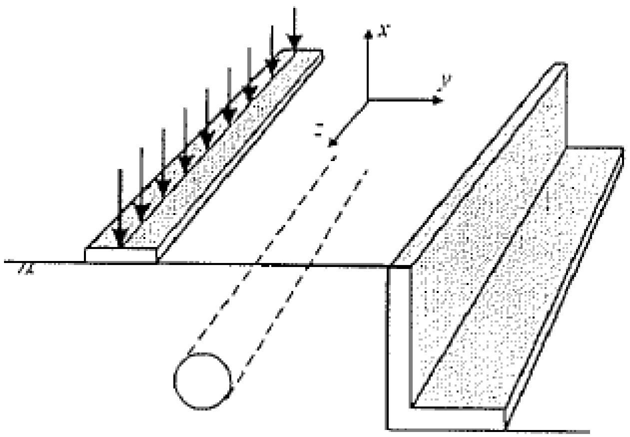
$$q = \sqrt{\frac{1}{2}\left(\left(\sigma_{11} - \sigma_{22}\right)^2 + \left(\sigma_{11} - \sigma_{33}\right)^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 + 3\left(\sigma_{12}^2 + \sigma_{21}^2 + \sigma_{13}^2 + \sigma_{31}^2 + \sigma_{23}^2 + \sigma_{32}^2\right)\right)}$$

$$\varepsilon_v = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$\varepsilon_q = \frac{\sqrt{2}}{3}\sqrt{\left[\left(\varepsilon_{11} - \varepsilon_{22}\right)^2 + \left(\varepsilon_{11} - \varepsilon_{33}\right)^2 + \left(\varepsilon_{22} - \varepsilon_{33}\right)^2\right] + 3\left(\varepsilon_{12}^2 + \varepsilon_{21}^2 + \varepsilon_{13}^2 + \varepsilon_{31}^2 + \varepsilon_{23}^2 + \varepsilon_{32}^2\right)}$$

# Plane strain

$$\left\{ \overbrace{\begin{bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \\ \Delta\sigma_{13} \\ \Delta\sigma_{23} \end{bmatrix}}^{\Delta\boldsymbol{\sigma}} = \overbrace{\begin{bmatrix} \text{3 columns} \\ \text{6 rows} \end{bmatrix}}^{\mathbf{D}} \overbrace{\begin{bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{12} \end{bmatrix}}^{\Delta\boldsymbol{\varepsilon}} \right.$$



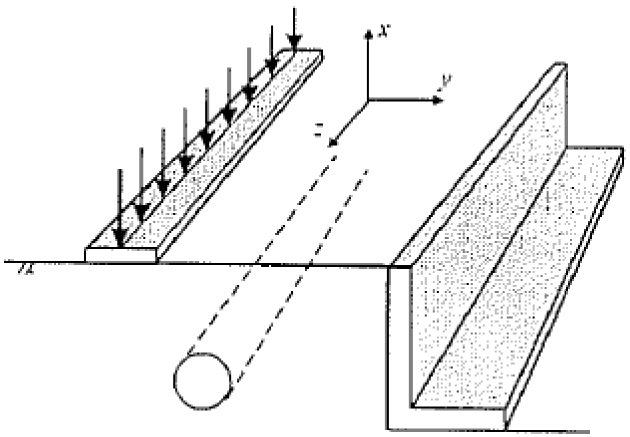

$$\Delta\sigma_{13} = \frac{E(1-2v)}{(1+v)(1-2v)}\Delta\varepsilon_{13} \qquad \Delta\sigma_{23} = \frac{E(1-2v)}{(1+v)(1-2v)}\Delta\varepsilon_{23}$$

$$\left\{ \overbrace{\begin{bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \end{bmatrix}}^{\Delta\boldsymbol{\sigma}} = \overbrace{\begin{bmatrix} \text{3 columns} \\ \text{4 rows} \end{bmatrix}}^{\mathbf{D}} \overbrace{\begin{bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{12} \end{bmatrix}}^{\Delta\boldsymbol{\varepsilon}} \right.$$

no deformations perpendicular to the plane

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$$

# Plane strain



no deformations perpendicular to the plane

When used in modelling – changes a 3D problem into 2D

- very useful, saves resources & time
- 2D approximation is usually fine from engineering perspective
- 3D analytical solutions are rare

- use when the construction is long – embankments, strip foundations, pipelines, roads, slopes etc…

# Thank you