Aalto University
School of Engineering

GEO – E1050
Finite Element Method in Geoengineering

Finite difference method
Wojciech Sołowski

Finite Difference (FD) method

Bases on Taylor's theorem…

… solves differential equations directly

… solves STRONG form of the differential equations

… perhaps easiest numerically and to code

… always gives results

… errors are difficult to control …

…as we are just discretising the equations directly, we do not have the comfort of physical interpretations, global energy balance and so on…

To learn

Why use Finite Difference method?

What are the approximations done?

When it is useful?

What is the algorithm / idea behind the Finite Difference method?

Finite Difference (FD) method

Taylor's theorem:

Let U(x) have n continuous derivatives over the interval (a, b). Then for $a < x_o$, $x_o+h < b$,

$$U(x_o+h)=U(x_o)+hU_x(x_o)+h^2\frac{U_{xx}(x_o)}{2!}+\ldots+h^{n-1}\frac{U_{(n-1)}(x_o)}{(n-1)!}+O(h^n)\,,$$

where,

- $U_x=\frac{dU}{dx}, U_{xx}=\frac{d^2U}{dx^2}, \ldots, U_{(n-1)}=\frac{d^{n-1}U}{dx^{n-1}}$.
- $U_x(x_o)$ is the derivative of U with respect to x *evaluated* at $x = x_o$.

Finite Difference (FD) method

Taylor's theorem:

Let $U(x)$ have n continuous derivatives over the interval (a, b). Then for $a < x_o$, $x_o+h < b$,

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$$U(x_o+h)=U(x_o)+hU_x(x_o)+O(h^2)$$

$$U_x(x_o)=\frac{U(x_o+h)-U(x_o)}{h}+\frac{O(h^2)}{h}$$

$$=\frac{U(x_o+h)-U(x_o)}{h}+O(h)$$

$$U_x(x_o) \approx \frac{U(x_o+h)-U(x_o)}{h}$$

Finite difference method

$$U_x(x_o) \approx \frac{U(x_o+h)-U(x_o)}{h}$$

first order FD approximation to $U_x(x_o)$ since the approximation error = O(h)

forward FD approximation since we start

at x_o and step forwards to the point x_o+h. h is called the *step size* (h > 0).

Finite difference method: example

$$U_x(x_o) \approx \frac{U(x_o+h)-U(x_o)}{h}$$

Function: $U(x) = x^2.$

$$U_x(x_o) \approx \frac{(x_o+h)^2 - x_o{}^2}{h}$$

What is the FD derivative, at point x=3, h=0.1?

$$U_x(3) \approx \frac{(3+0.1)^2 - 3^2}{0.1} = 6.1$$

Finite difference method: with time

$$U(t, x_o + \Delta x) = U(t, x_o) + \Delta x U_x(t, x_o) + \frac{\Delta x^2}{2!} U_{xx}(t, x_o) + \ldots + \frac{\Delta x^{n-1}}{(n-1)!} U_{(n-1)}(t, x_o) + O(\Delta x^n)$$

$$U(t, x_o + \Delta x) = U(t, x_o) + \Delta x U_x(t, x_o) + \frac{\Delta x^2}{2!} U_{xx}(t, x_o) + \frac{\Delta x^3}{3!} U_{xxx}(t, x_o) + O(\Delta x^4)$$

$$U(t, x_o - \Delta x) = U(t, x_o) - \Delta x U_x(t, x_o) + \frac{\Delta x^2}{2!} U_{xx}(t, x_o) - \frac{\Delta x^3}{3!} U_{xxx}(t, x_o) + O(\Delta x^4)$$

$$U(t, x_o + \Delta x) + U(t, x_o - \Delta x) = 2U(t, x_o) + \Delta x^2 U_{xx}(t, x_o) + O(\Delta x^4)$$

At given time t, the second derivative can be computed based on +h and –h values of function:

$$U_{i+1}^n + U_{i-1}^n = 2U_i^n + \Delta x^2 U_{xx}(t_n, x_i) + O(\Delta x^4)$$

Finite difference method: with time

$$U_{i+1}^{n}+U_{i-1}^{n}=2U_{i}^{n}+\Delta x^{2}U_{xx}(t_{n},x_{i})+O(\Delta x^{4})$$

At given time t, the second derivative can be computed based on +h and –h values of function:

Pure diffusion equation 1D: $U_t = K_x U_{xx}$

$$\frac{u_i^{n+1}-u_i^{n}}{\Delta t}=K_x\frac{u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n}}{\Delta x^{2}}$$

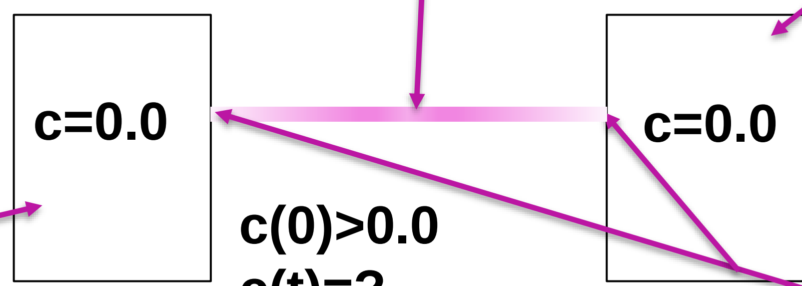
$$u_i^{n+1}=u_i^{n}+\frac{\Delta t K_x}{\Delta x^{2}}\left(u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n}\right)$$

We almost solved it – now we just discretise in space for Δx

Finite difference method: with time

initial non-zero concentration

solute concentration 0.0



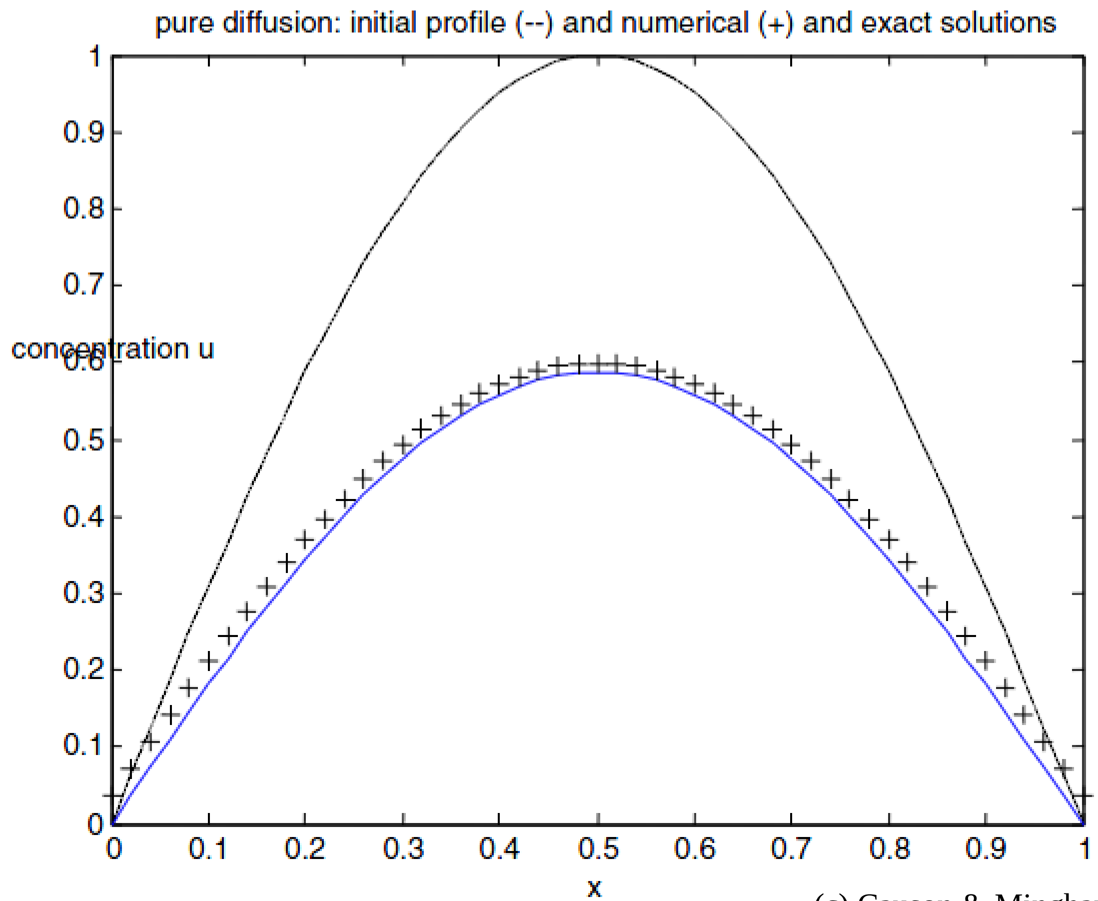
solute concentration 0.0

c(0)>0.0
c(t)=?
c(t-->infinity)=0

membranes through which diffusion can occur

Compute the concentration in the connecting pipe at given time t

Finite difference method: with time



(c) Causon & Mingham

Analytical solution: $U(t,x)=e^{-K_x\pi^2 t}\sin(\pi x)$

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Boundary element method

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Boundary element method

2 components:

a) some analytical solution we will use in our approximation (FUNDAMENTAL SOLUTION)

b) discretisation, so we use our fundamental solution over and over again, over some finite domain

of course… **fundamental solution must exists**

… and must be known…

… and generally should not be too complex…

Boundary element method

Method reduce the dimension of the problem by 1 so:

2D problem becomes 1D problem;

requires 2D fundamental solution

3D problem becomes 2D problem;

requires 3D fundamental solution

We discretise the boundary of the problem only – hence the name: ‘boundary element method’

Boundary element method

Reducing dimensions of the problem is **HUGE**

however…

Fundamental solutions only can be analytically computed for simple cases…

- linear elasticity
- Poisson equation problems
- and similar…

Cannot be used for elasto-plasticity (at least not easily)

Boundary element method

For example, for Poisson equation,

$$\mathbf{q} = -\mathbf{D}\nabla u$$

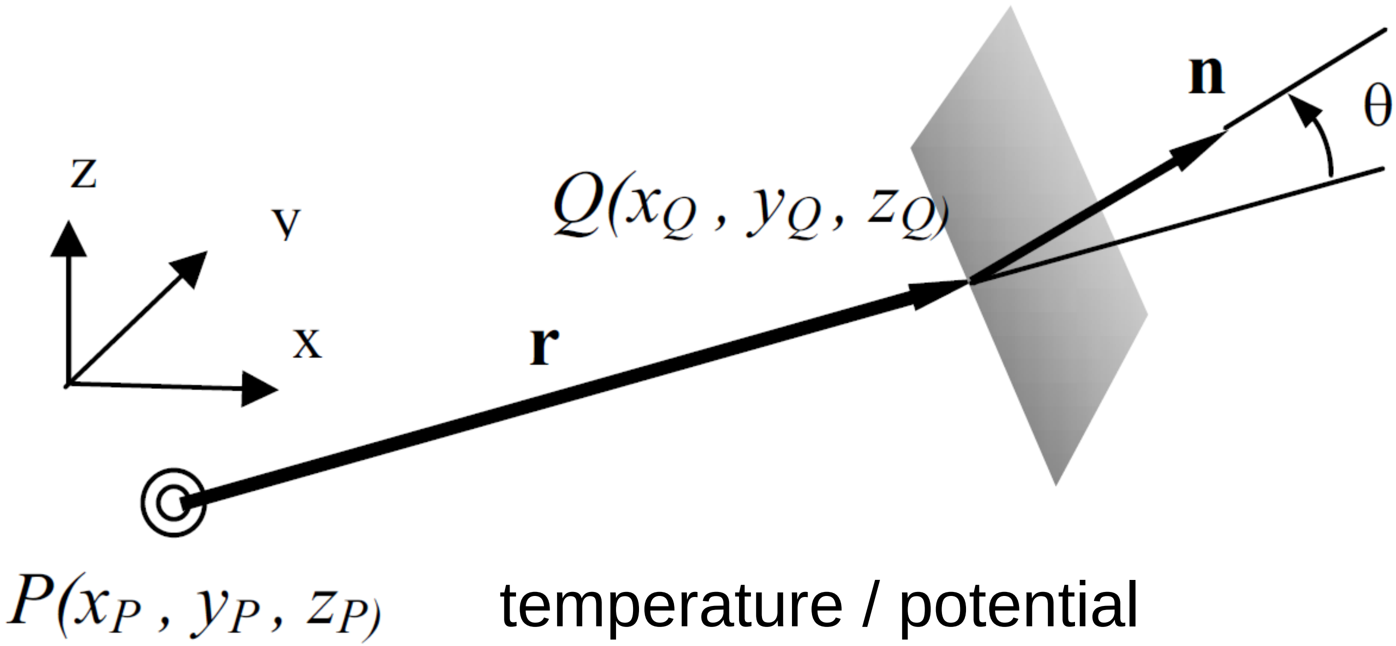
We assume some source at point P(x_p,y_p,z_p) in infinite space and at some point Q the temperature / potential is:

$$U(P,Q) = \frac{1}{4\pi rk}$$

And if we assume flow in x direction, the flow at point P is:

$$T(P,Q) = \frac{\cos\theta}{4\pi r^2}$$

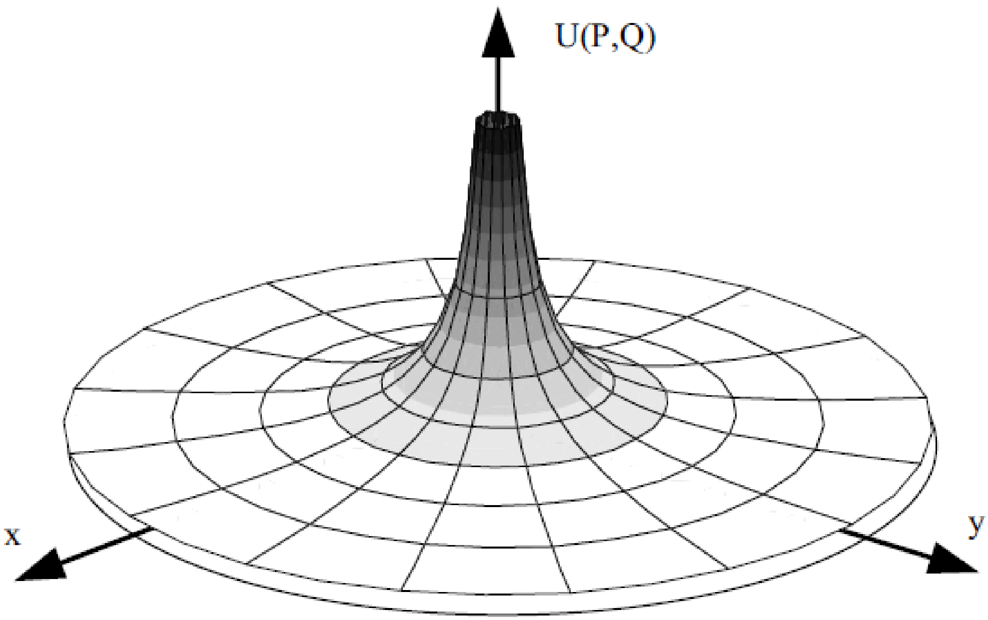
Boundary element method



temperature / potential

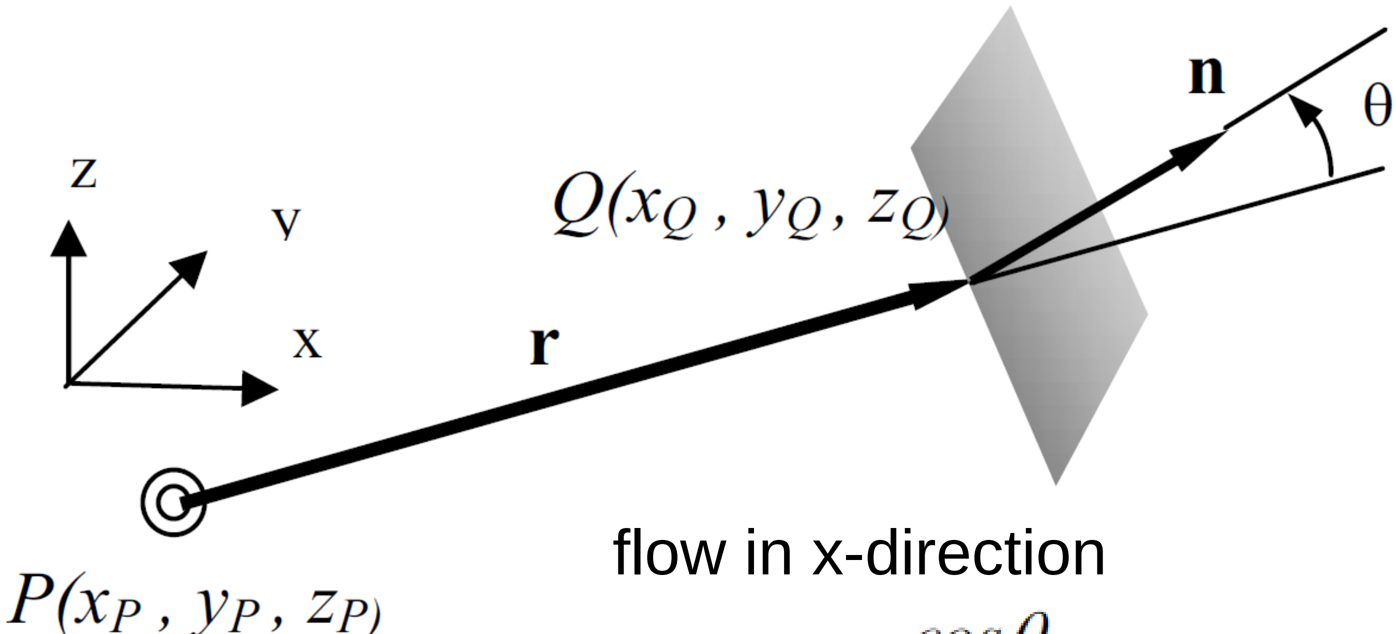
$$U(P,Q) = \frac{1}{4\pi rk}$$

Boundary element method



Variation of fundamental solution *U* (potential/temperature) in the x-y plane for 3-D potential problems (source at origin of coordinate system)

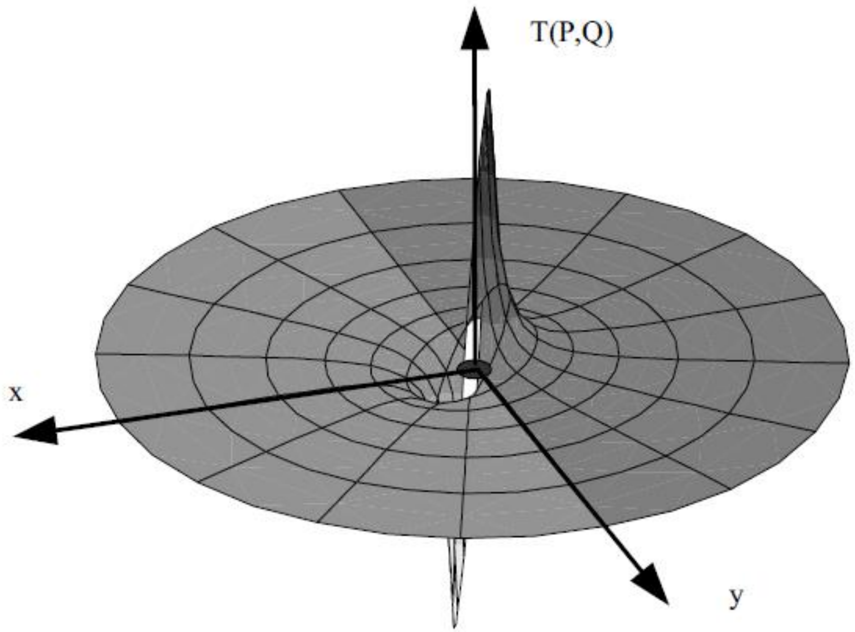
Boundary element method



flow in x-direction

$$T(P,Q) = \frac{\cos\theta}{4\pi r^2}$$

Boundary element method



Variation of fundamental solution for **n** = {1,0,0} (flow in x-direction) in x-y plane for 3-D potential problems (e.g. temperature changes if flow is happening)

Boundary element method

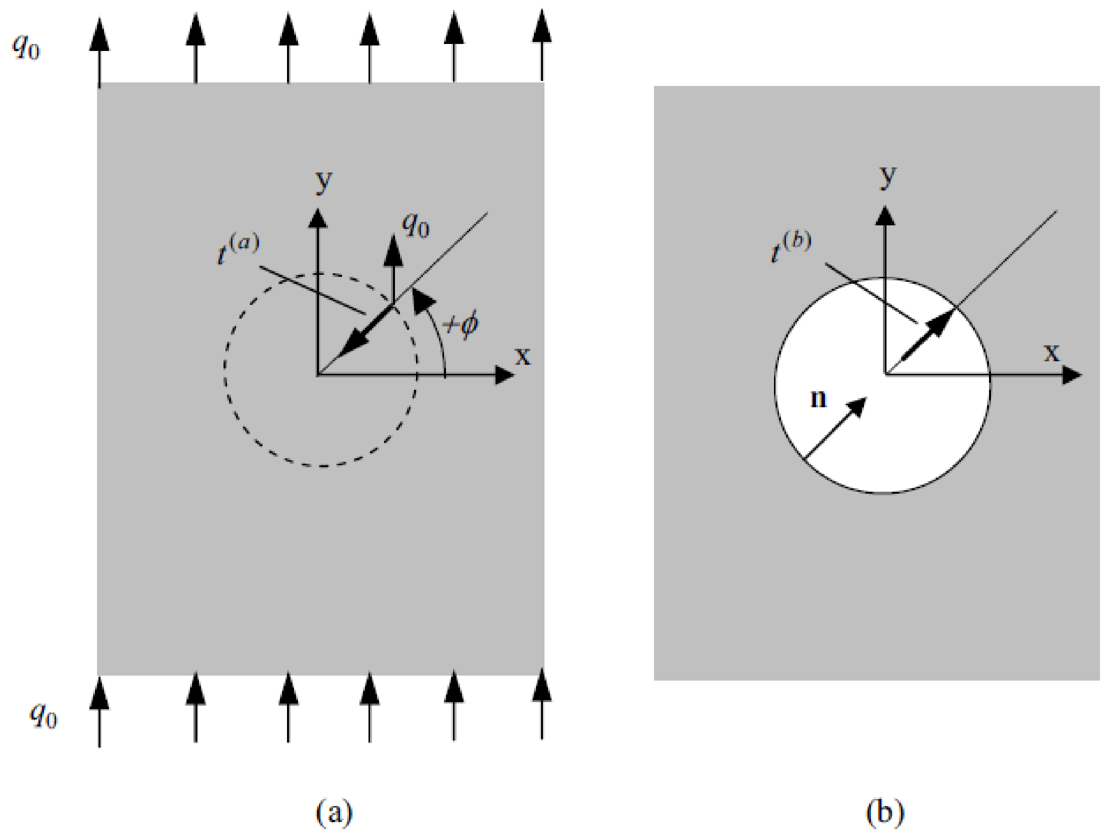


Figure 5.1 Heat flow in an infinite domain, case (a) and (b)

Boundary element method

We know the heat flow solution

We want to solve this one, with a perfect insulator inside, same boundary conditions…

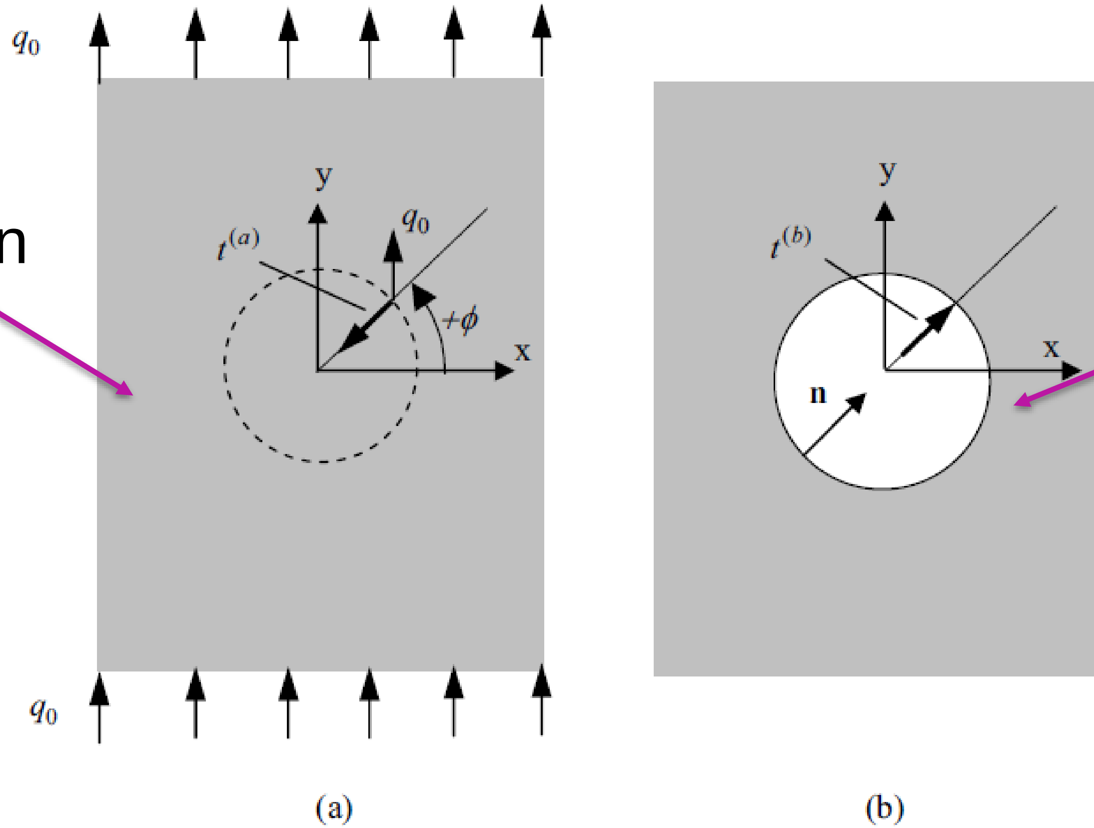
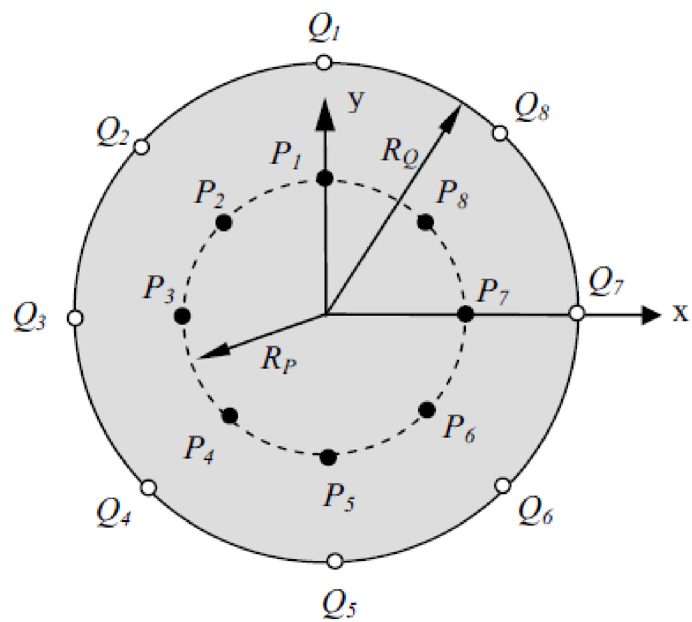


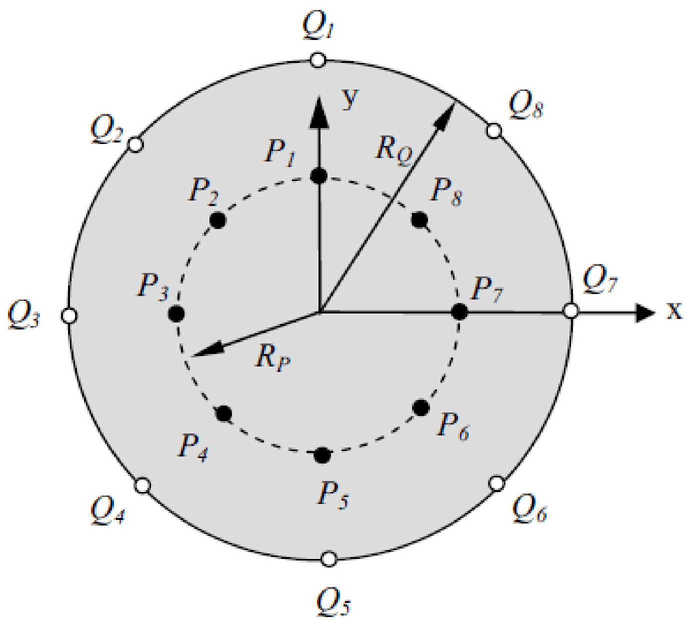
Figure 5.1 Heat flow in an infinite domain, case (a) and (b)

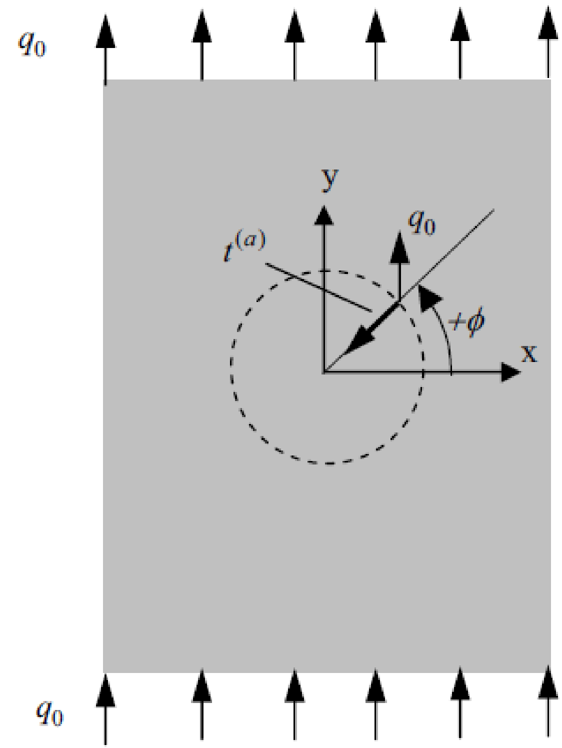
Boundary element method



We want to approximate the known solution on the outside by the sources (we have the fundamental solution for those)

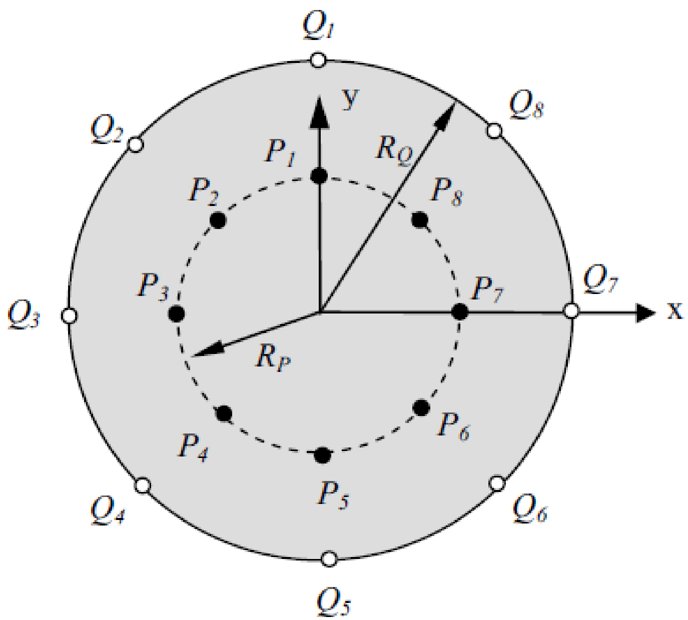
Boundary element method

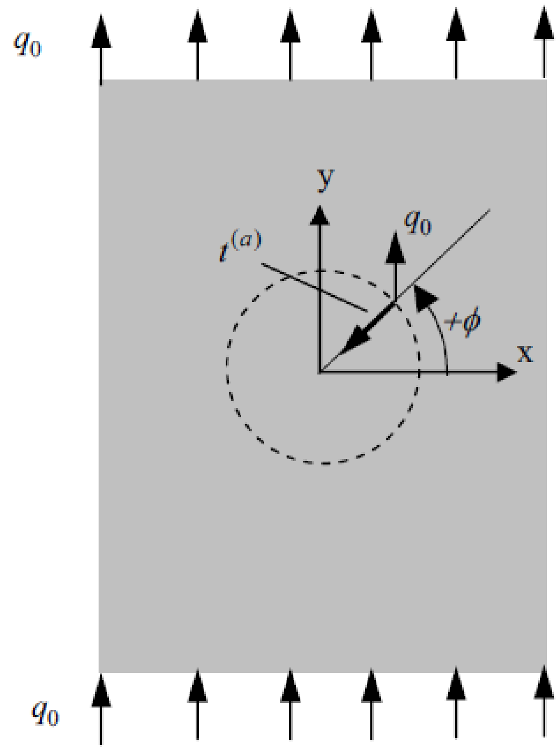




The sources should be such, that on the boundary we have exactly same solution as the one without the insulator…

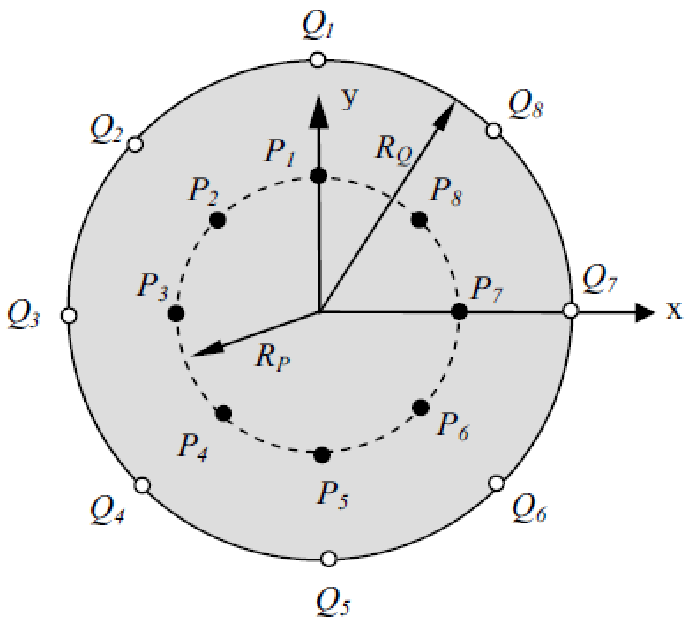
Boundary element method

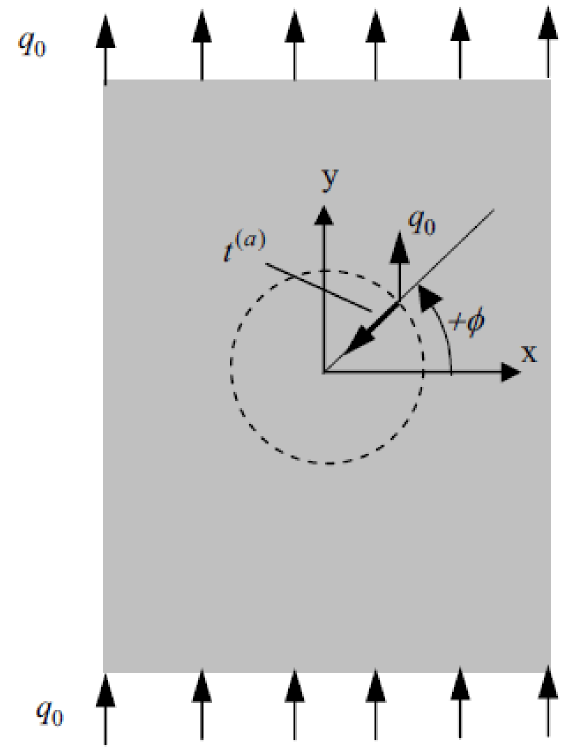




Now, to get the solution with the insulator, we use the superposition – from the known solution we subtract the one obtained with the sources... **And we are done…**

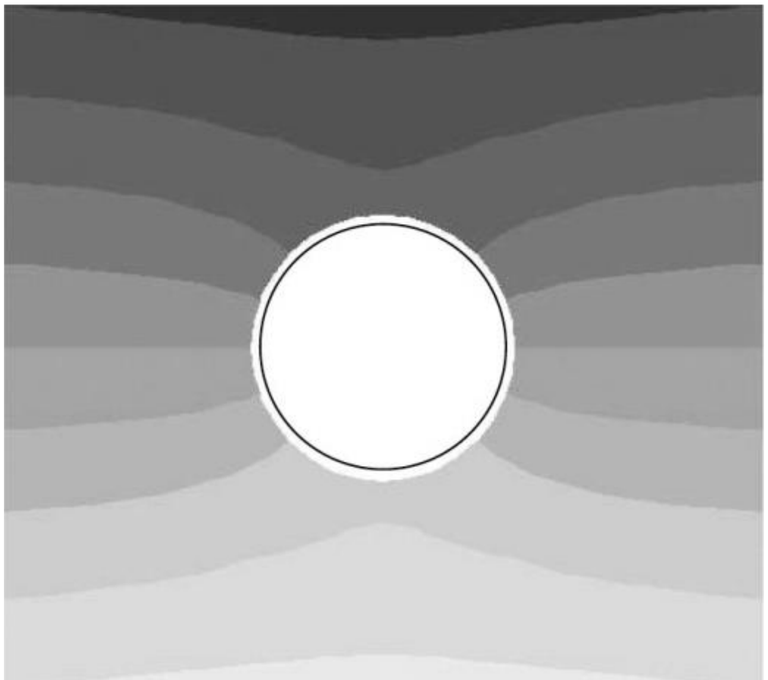
Boundary element method

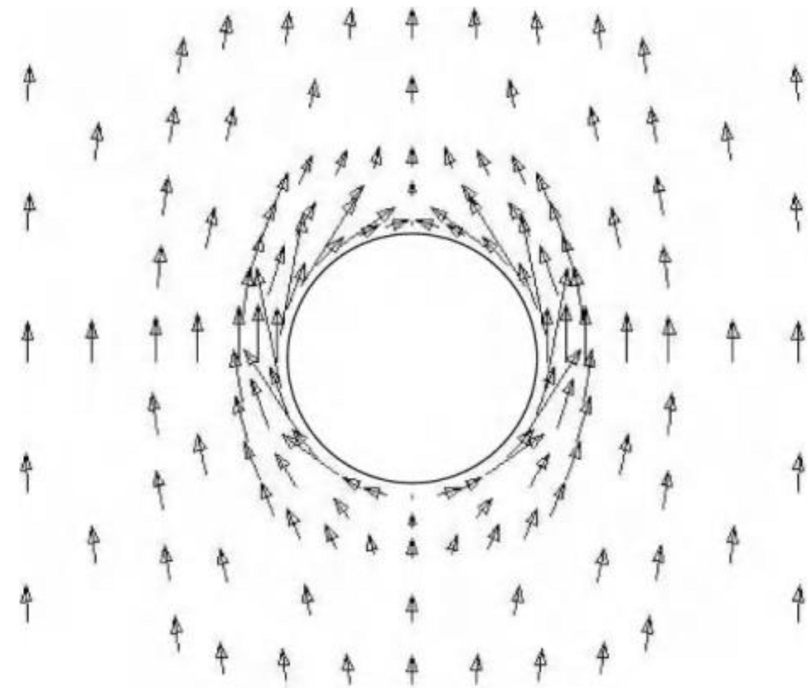




The better we discretize the boundary of the insulator, the more sources we can add, the better the solution…

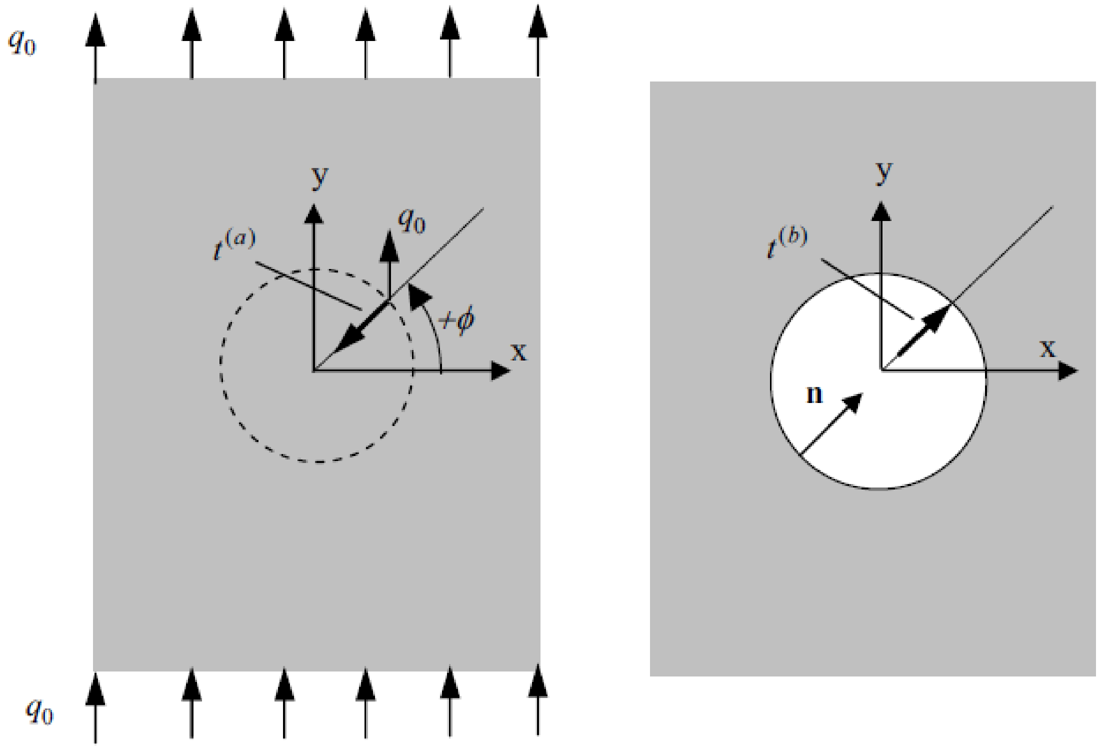
Boundary element method





Temperature and flow vectors for the solved problem

Boundary element method



How to solve the problem on the right, where we have void and the boundary condition on the void is T=const.

Thank you