Aalto University
School of Engineering

GEO – E1050
Finite Element Method in Geoengineering

Lecture 2. Basics of FEM: ‘easy derivation’

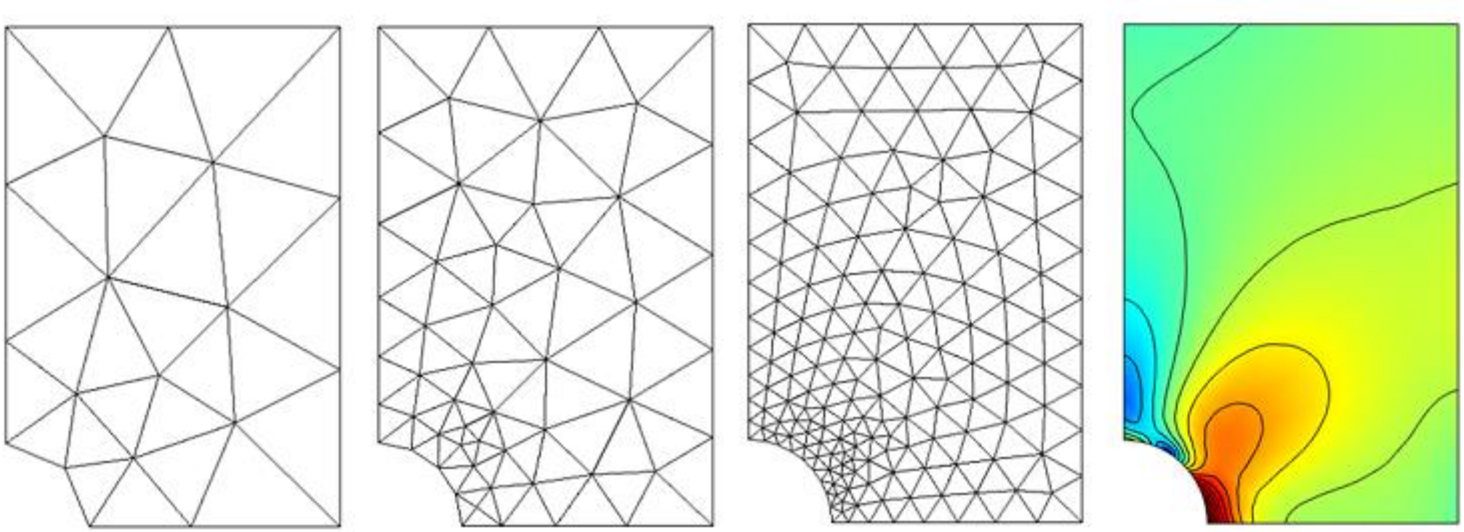
To learn...

- Understand in detail the finite element algorithm for linear elasticity
- Understand how the shape choice of the finite element affect the finite element solution
- Understand what is the role of element shape functions
- Understand what is the role of element stiffness matrix and how it is computed
- Understand how the global stiffness matrix is assembled
- Understand how the boundary conditions are applied.

Literature

- Zienkiewicz et al., The Finite Element Method: Its Basis and Fundamentals, Chapter 1 & 2
- Potts & Zdravkovic, Finite element analysis in geotechnical engineering. Theory. Chapter 1 & 2

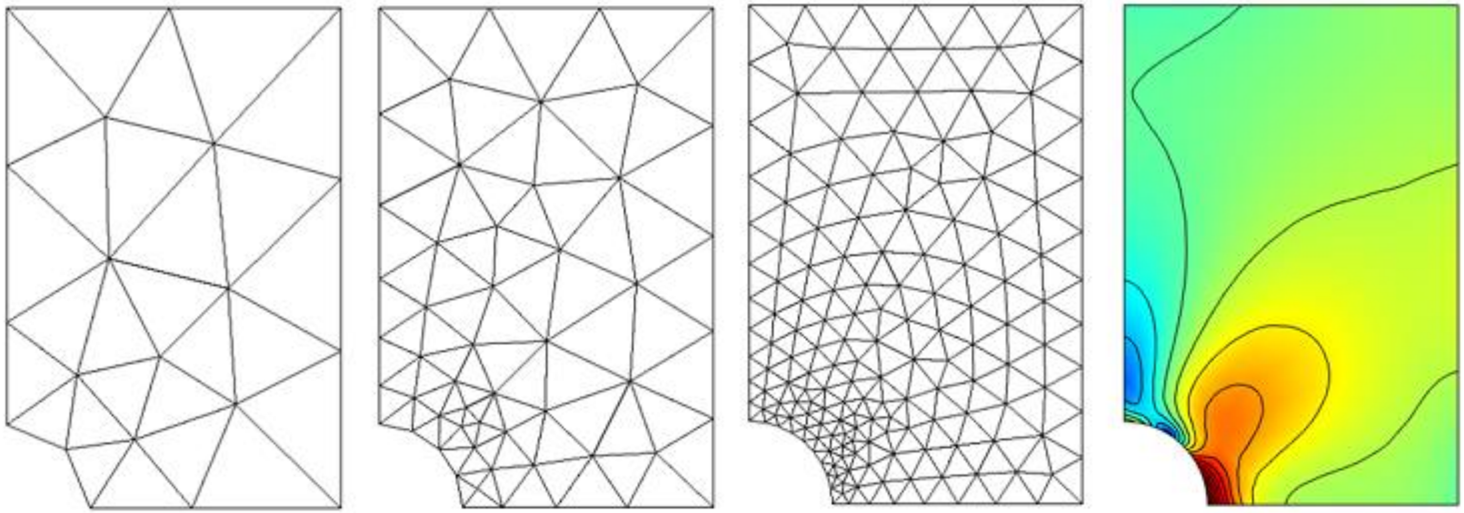
FEM algorithm: elasticity



(c) COMSOL

We discretise the problem with finite elements. For each element, we have the link between the work done by forces acting on nodes and internal work (giving set of linear equations to solve). We add all the equations together to solve for all the elements we discretise the problem with…

FEM algorithm: elasticity



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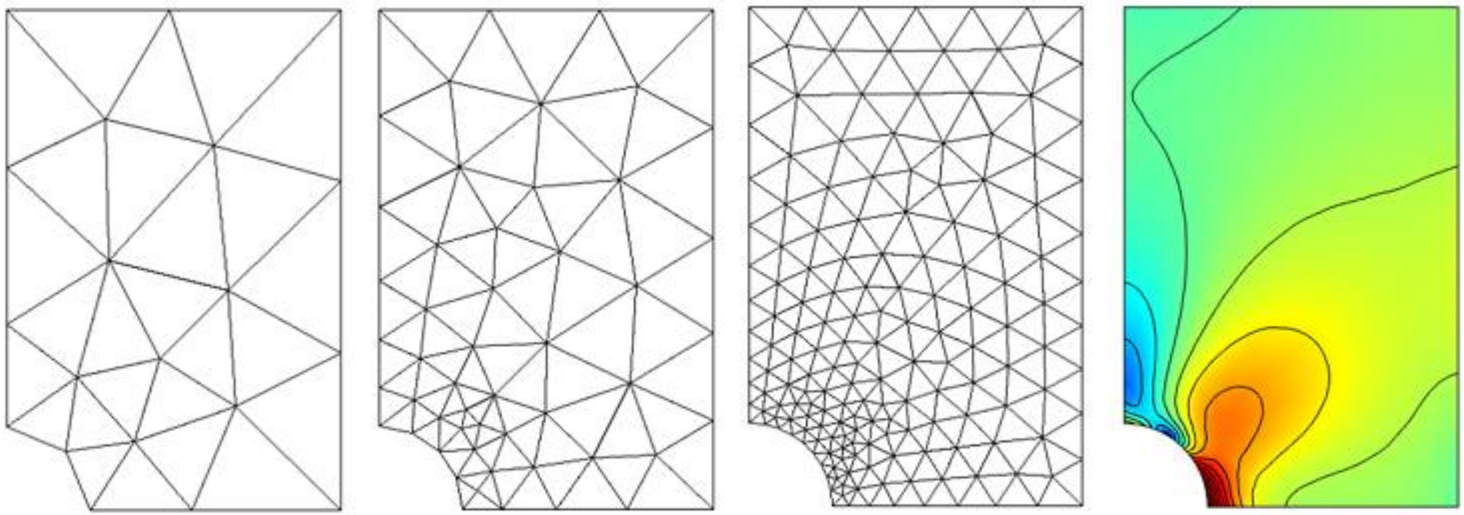
The work done by forces displacing the element nodes and the work done by internal forces in the elements should be the same (we solve large amount of equations). We take the internal forces in the elements as the solution of the problem.

Solved ☺

FEM algorithm: elasticity



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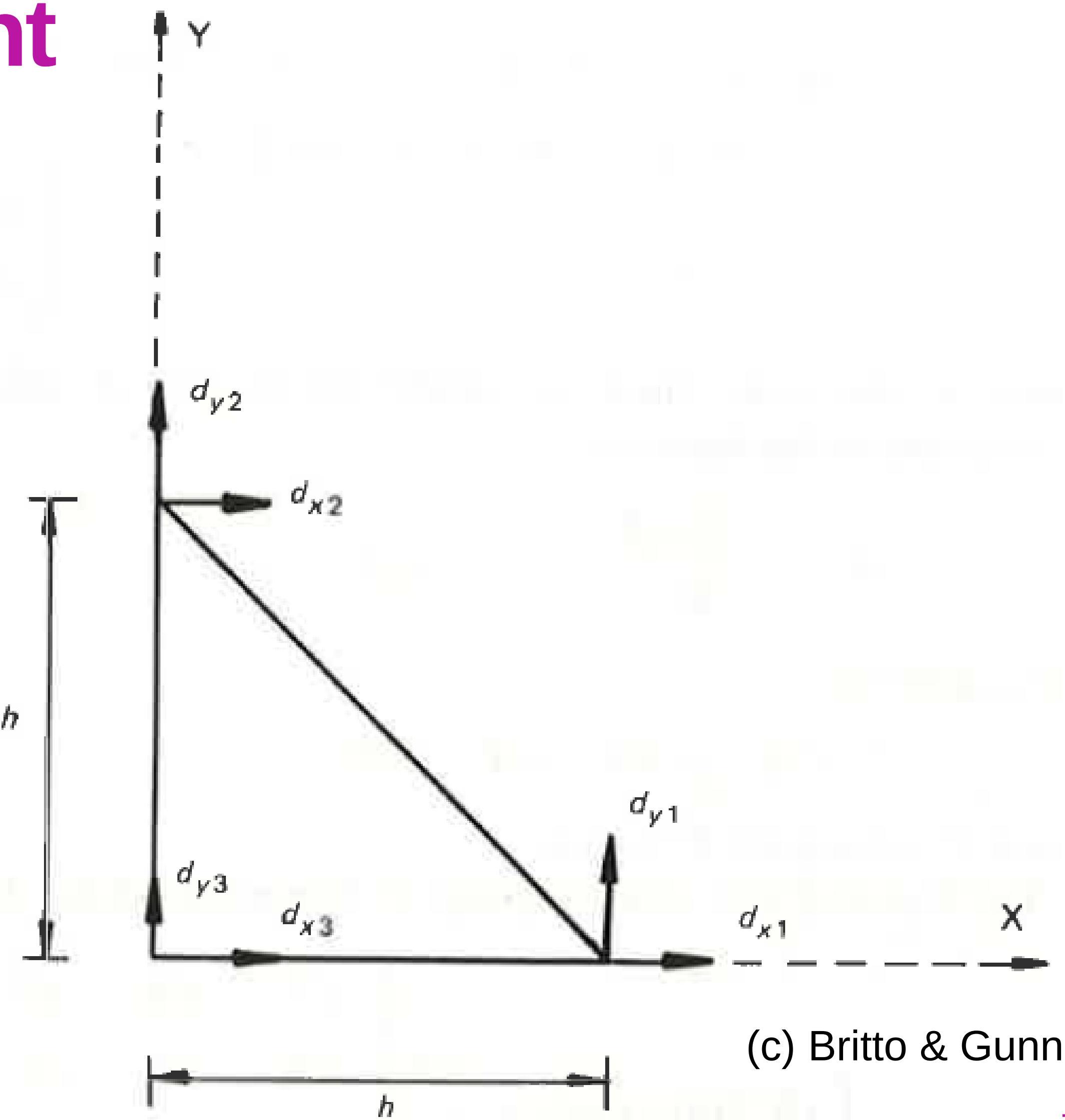
Plan for today ☺

First we need element equations (linking displacement of the nodes and internal forces), and global equations (linking all the elements). So, first we need to derive the element, and than introduce algorithm linking all the elements together…

Finite element



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FEM algorithm

1. **Mesh generation**

$$\underbrace{\mathbf{K}_E}_{\substack{\text{Element stiffnes}\\ \text{marix}}}\ \underbrace{\Delta\mathbf{d}_E}_{\substack{\text{Vector containing}\\ \text{displacements increments}\\ \text{of element nodes}}} = \underbrace{\Delta\mathbf{R}_E}_{\substack{\text{Vector containing}\\ \text{forces at element nodes}\\ \text{(and increments of them)}}}$$

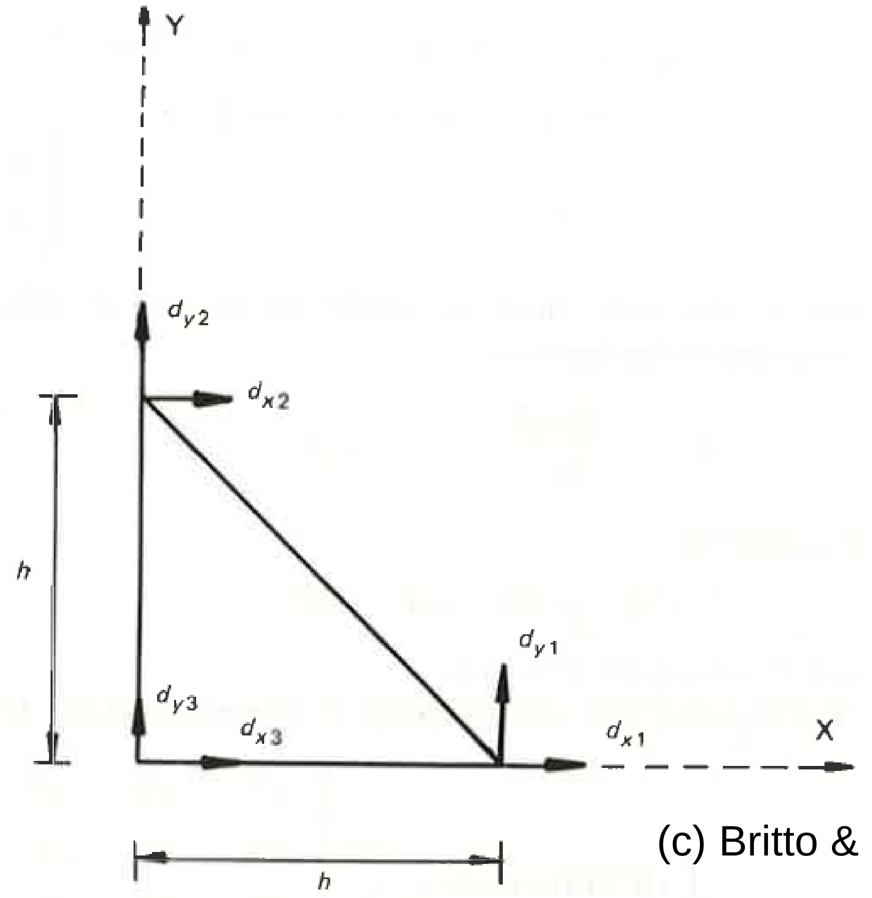
$$\underbrace{\Delta\mathbf{u}}_{\substack{\text{Vector containing}\\ \text{displacement increments}\\ \text{at some point within the}\\ \text{finite element}}} = \underbrace{\mathbf{N}}_{\substack{\text{Matrix containing}\\ \text{shape functions values}\\ \text{at some point within}\\ \text{the finite element}}}\ \underbrace{\Delta\mathbf{d}_E}_{\substack{\text{Vector containing increments}\\ \text{of displacements of element nodes}}}$$

2. **Global Equations**

$$\underbrace{\mathbf{K}_G}_{\substack{\text{Global stiffnes}\\ \text{marix}}}\ \underbrace{\Delta\mathbf{d}_G}_{\substack{\text{Vector containing}\\ \text{displacement increments}\\ \text{of all element nodes}}} = \underbrace{\Delta\mathbf{R}_G}_{\substack{\text{Vector containing forces}\\ \text{and their increments}\\ \text{at all element nodes}}}$$

3. **Boundary conditions**
4. **Solution of the global equations**

Constant strain triangle



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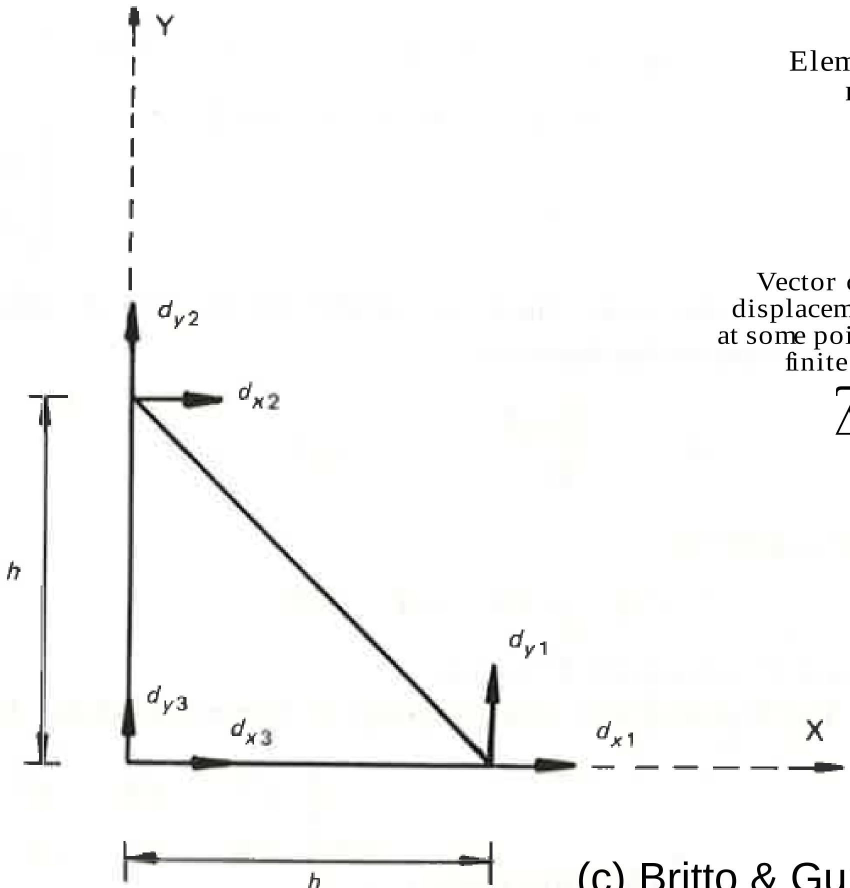
Constant strain triangle

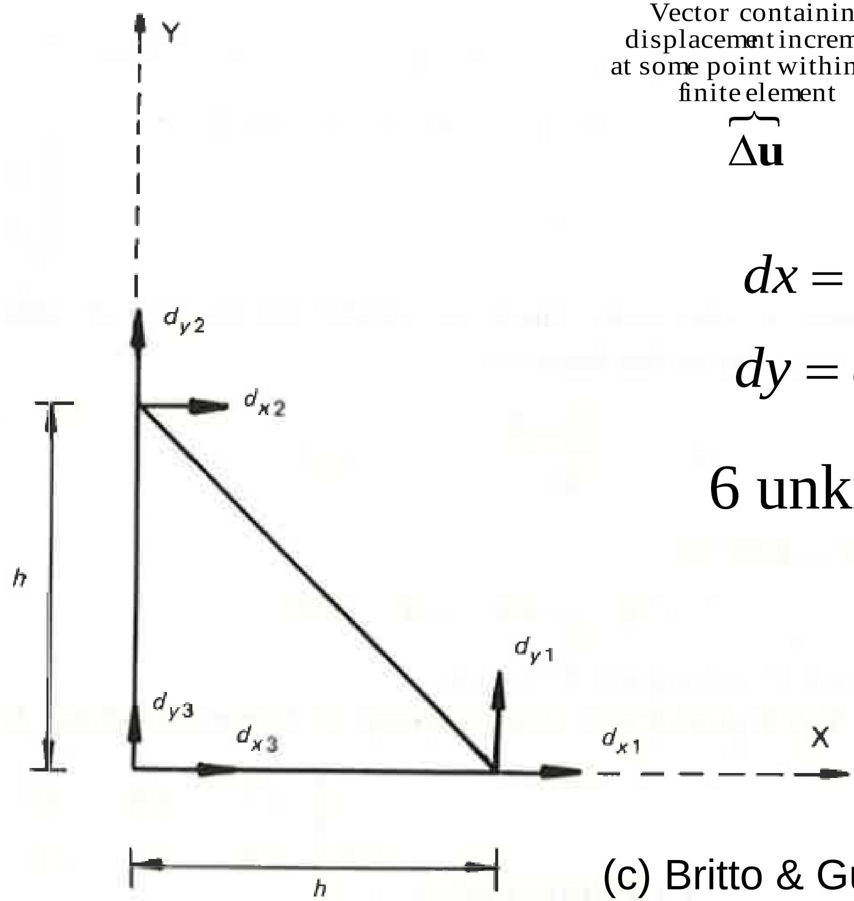


(c) Britto & Gunn

$$\overbrace{\mathbf{K}_E}^{\text{Element stiffnes marix}} \overbrace{\Delta\mathbf{d}_E}^{\text{Vector containing displacements increments of element nodes}} = \overbrace{\Delta\mathbf{R}_E}^{\text{Vector containing forces at element nodes (and increments of them)}}$$

$$\overbrace{\Delta\mathbf{u}}^{\text{Vector containing displacement increments at some point within the finite element}} = \overbrace{\mathbf{N}}^{\text{Matrix containing shape functions values at some point within the finite element}} \overbrace{\Delta\mathbf{d}_E}^{\text{Vector containing increments of displacements of element nodes}}$$

Shape functions



$$\underbrace{\Delta \mathbf{u}}_{\substack{\text{Vector containing}\\ \text{displacement increments}\\ \text{at some point within the}\\ \text{finite element}}} = \underbrace{\mathbf{N}}_{\substack{\text{Matrix containing}\\ \text{shape functions values}\\ \text{at some point within}\\ \text{the finite element}}} \underbrace{\Delta \mathbf{d}_E}_{\substack{\text{Vector containing increments}\\ \text{of displacements of element nodes}}}$$

$$dx = c_0 + c_1 x + c_2 y$$

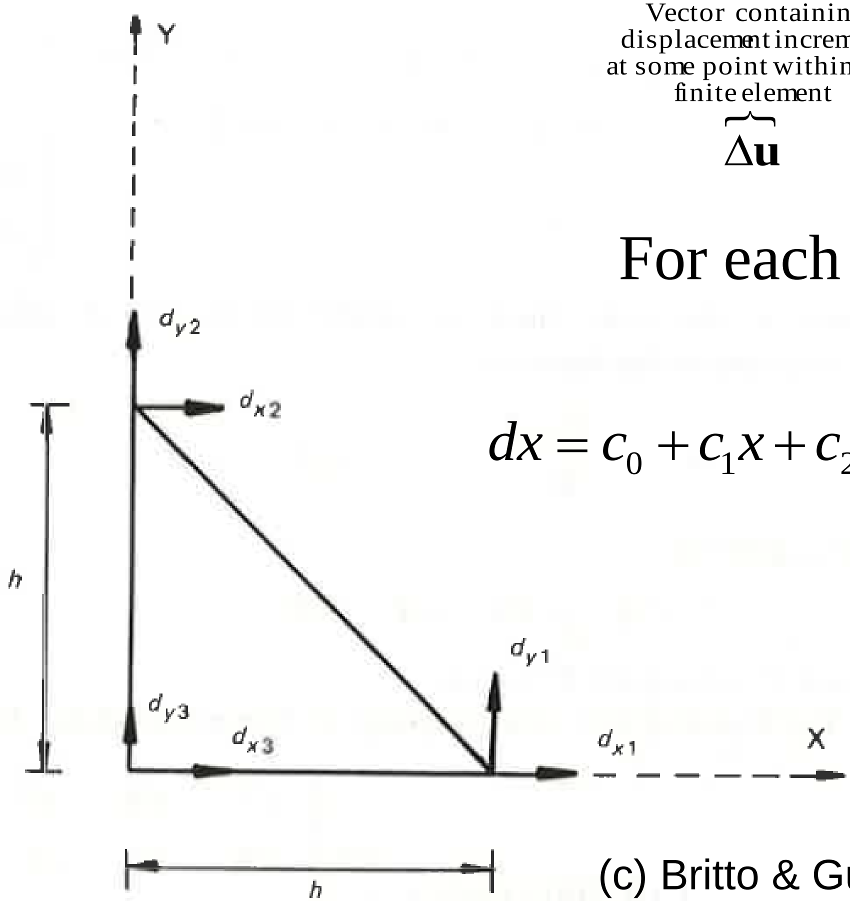
$$dy = c_3 + c_4 x + c_5 y$$

6 unknowns - (c_0-c_6)

(c) Britto & Gunn

Shape functions

$$\underbrace{\Delta\mathbf{u}}_{\text{Vector containing displacement increments at some point within the finite element}} = \underbrace{\mathbf{N}}_{\text{Matrix containing shape functions values at some point within the finite element}} \underbrace{\Delta\mathbf{d}_E}_{\text{Vector containing increments of displacements of element nodes}}$$



For each corner, we know the displacements:

$$dx = c_0 + c_1 x + c_2 y \quad \begin{cases} d_{x1} = c_0 + c_1 h \\ d_{x2} = c_0 + c_2 h \\ d_{x3} = c_0 \end{cases}$$

$$dy = c_3 + c_4 x + c_5 y \quad \begin{cases} d_{y1} = c_3 + c_4 h \\ d_{y2} = c_3 + c_5 h \\ d_{y3} = c_3 \end{cases}$$

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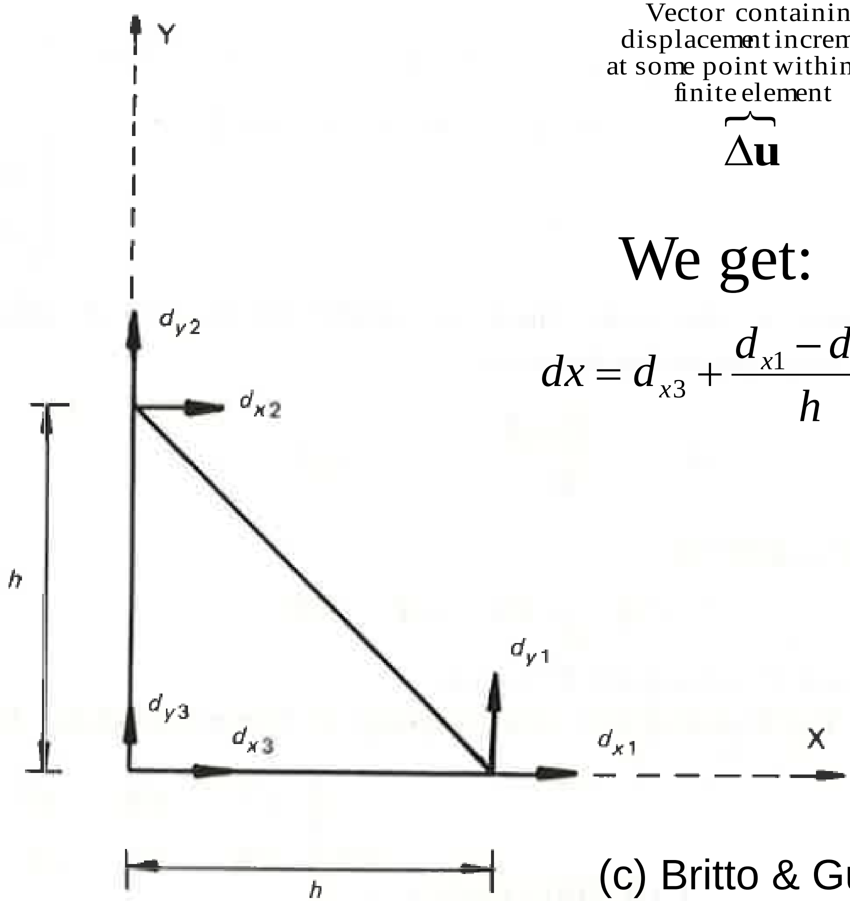
Shape functions

$$\overbrace{\Delta\mathbf{u}}^{\substack{\text{Vector containing}\\ \text{displacement increments}\\ \text{at some point within the}\\ \text{finite element}}} = \overbrace{\mathbf{N}}^{\substack{\text{Matrix containing}\\ \text{shape functions values}\\ \text{at some point within}\\ \text{the finite element}}} \overbrace{\Delta\mathbf{d}_E}^{\substack{\text{Vector containing increments}\\ \text{of displacements of element nodes}}}$$



We get:

$$dx = c_0 + c_1 x + c_2 y$$

$$dx = d_{x3} + \frac{d_{x1} - d_{x3}}{h} x + \frac{d_{x2} - d_{x3}}{h} y = \frac{x}{h} d_{x1} + \frac{y}{h} d_{x2} + \left(1 - \frac{x}{h} - \frac{y}{h}\right) d_{x3}$$

$$dy = c_3 + c_4 x + c_5 y$$

$$dy = \frac{x}{h} d_{y1} + \frac{y}{h} d_{y2} + \left(1 - \frac{x}{h} - \frac{y}{h}\right) d_{y3}$$

(c) Britto & Gunn

Shape functions

$$\underbrace{\Delta \mathbf{u}}_{\substack{\text{Vector containing} \\ \text{displacement increments} \\ \text{at some point within the} \\ \text{finite element}}} = \underbrace{\mathbf{N}}_{\substack{\text{Matrix containing} \\ \text{shape functions values} \\ \text{at some point within} \\ \text{the finite element}}} \underbrace{\Delta \mathbf{d}_E}_{\substack{\text{Vector containing increments} \\ \text{of displacements of element nodes}}}$$

We get:

$$dx = \frac{x}{h} d_{x1} + \frac{y}{h} d_{x2} + \left(1 - \frac{x}{h} - \frac{y}{h}\right) d_{x3}$$

$$dy = \frac{x}{h} d_{y1} + \frac{y}{h} d_{y2} + \left(1 - \frac{x}{h} - \frac{y}{h}\right) d_{y3}$$

$$\Delta \mathbf{u} = \begin{bmatrix} du_1 \\ du_2 \end{bmatrix} = \begin{bmatrix} dx \\ dy \end{bmatrix} = \mathbf{N} \Delta \mathbf{d}_E =$$

$$\overbrace{\begin{bmatrix} x/h & 0 & y/h & 0 & 1 - x/h - y/h & 0 \\ 0 & x/h & 0 & y/h & 0 & 1 - x/h - y/h \end{bmatrix}}^{\mathbf{N}} \overbrace{\begin{bmatrix} d_{x1} \\ d_{y1} \\ d_{x2} \\ d_{y2} \\ d_{x3} \\ d_{y3} \end{bmatrix}}^{\Delta \mathbf{d}_E}$$

Strains

$$\varepsilon_{ij} = 0.5\left(-\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right), \quad i, j = 1,2,3$$

We get:

$$\varepsilon_{11} = \varepsilon_{xx} = 0.5\left(-\frac{\partial dx}{\partial x} - \frac{\partial dx}{\partial x}\right) = -\frac{\partial\left(\frac{x}{h}d_{x1} + \frac{y}{h}d_{x2} + \left(1 - \frac{x}{h} - \frac{y}{h}\right)d_{x3}\right)}{\partial x} = -\frac{1}{h}d_{x1} + \frac{1}{h}d_{x3}$$

$$\varepsilon_{22} = \varepsilon_{yy} = 0.5\left(-\frac{\partial dy}{\partial y} - \frac{\partial dy}{\partial y}\right) = -\frac{\partial\left(\frac{x}{h}d_{y1} + \frac{y}{h}d_{y2} + \left(1 - \frac{x}{h} - \frac{y}{h}\right)d_{y3}\right)}{\partial y} = -\frac{1}{h}d_{y2} + \frac{1}{h}d_{y3}$$

$$\varepsilon_{12} = \varepsilon_{xy} = 0.5\left(-\frac{\partial dx}{\partial y} - \frac{\partial dy}{\partial x}\right) = 0.5\left(-\frac{1}{h}d_{x2} + \frac{1}{h}d_{x3}\right) + 0.5\left(-\frac{1}{h}d_{y1} + \frac{1}{h}d_{y3}\right)$$

Strains

$$\varepsilon_{11}=\varepsilon_{xx}=0.5\left(-\frac{\partial dx}{\partial x}-\frac{\partial dx}{\partial x}\right)=-\frac{\partial\left(\frac{x}{h}d_{x1}+\frac{y}{h}d_{x2}+\left(1-\frac{x}{h}-\frac{y}{h}\right)d_{x3}\right)}{\partial x}=-\frac{1}{h}d_{x1}+\frac{1}{h}d_{x3}$$

$$\varepsilon_{22}=\varepsilon_{yy}=0.5\left(-\frac{\partial dy}{\partial y}-\frac{\partial dy}{\partial y}\right)=-\frac{\partial\left(\frac{x}{h}d_{y1}+\frac{y}{h}d_{y2}+\left(1-\frac{x}{h}-\frac{y}{h}\right)d_{y3}\right)}{\partial y}=-\frac{1}{h}d_{y2}+\frac{1}{h}d_{y3}$$

$$\varepsilon_{12}=\varepsilon_{xy}=0.5\left(-\frac{\partial dx}{\partial y}-\frac{\partial dy}{\partial x}\right)=0.5\left(-\frac{1}{h}d_{x2}+\frac{1}{h}d_{x3}\right)+0.5\left(-\frac{1}{h}d_{y1}+\frac{1}{h}d_{y3}\right)$$

$$\Delta\boldsymbol{\varepsilon}=\begin{bmatrix}d\varepsilon_{11}\\ d\varepsilon_{22}\\ d\varepsilon_{12}\end{bmatrix}=\mathbf{B}\,\Delta\mathbf{d}_E=\overbrace{\begin{bmatrix}-1/h & 0 & 0 & 0 & 1/h & 0\\ 0 & 0 & 0 & -1/h & 0 & 1/h\\ 0 & -0.5/h & -0.5/h & 0 & 0.5/h & 0.5/h\end{bmatrix}}^{\mathbf{B}}\overbrace{\begin{bmatrix}d_{x1}\\ d_{y1}\\ d_{x2}\\ d_{y2}\\ d_{x3}\\ d_{y3}\end{bmatrix}}^{\Delta\mathbf{d}_E}$$

Strains

$$\Delta\boldsymbol{\varepsilon} = \begin{bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{12} \end{bmatrix} = \mathbf{B}\,\Delta\mathbf{d}_E = \overbrace{\begin{bmatrix} -1/h & 0 & 0 & 0 & 1/h & 0 \\ 0 & 0 & 0 & -1/h & 0 & 1/h \\ 0 & -0.5/h & -0.5/h & 0 & 0.5/h & 0.5/h \end{bmatrix}}^{\mathbf{B}} \overbrace{\begin{bmatrix} d_{x1} \\ d_{y1} \\ d_{x2} \\ d_{y2} \\ d_{x3} \\ d_{y3} \end{bmatrix}}^{\Delta\mathbf{d}_E}$$

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \dots & \dots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \dots & \dots & 0 & \frac{\partial N_n}{\partial y} \\ 0.5\frac{\partial N_1}{\partial y} & 0.5\frac{\partial N_1}{\partial x} & 0.5\frac{\partial N_2}{\partial y} & 0.5\frac{\partial N_2}{\partial x} & \dots & \dots & 0.5\frac{\partial N_n}{\partial y} & 0.5\frac{\partial N_n}{\partial x} \end{bmatrix}$$

Remarks

1. Constant strain
2. Continuus displacement field

   2a. Even though the stress and strain fields are not continuous, many computer programs will display them as continuous, interpolating based on the values in elements. Usually this option can be turned off (but it is usually on by default).

3. Strain approximation as constant is quite inaccurate.

Element Stifness matrix

Change of elastic energy inside the element and on the nodes the same:

$$dE = 0.5\int_V d\boldsymbol{\varepsilon}^T d\boldsymbol{\sigma} dV = 0.5\int_V d\boldsymbol{\varepsilon}^T \mathbf{D} d\boldsymbol{\varepsilon} dV =$$

$$= 0.5\int_V \left(\mathbf{B} d\mathbf{d}_E\right)^T \mathbf{D}\mathbf{B} d\mathbf{d}_E dV = 0.5\int_V d\mathbf{d}_E^T \mathbf{B}^T \mathbf{D}\mathbf{B} d\mathbf{d}_E dV$$

work at nodes:

$$dL = \int_S d\mathbf{d}_E^T \mathbf{T} dS = d\mathbf{d}_E^T \Delta\mathbf{R}$$

Element Stifness matrix

Change of elastic energy inside the element:

$$dE = 0.5\int_V d\mathbf{d}_E^T \mathbf{B}^T \mathbf{D}\mathbf{B} d\mathbf{d}_E dV$$

$$E = 0.5 A d\mathbf{d}_E^T \mathbf{B}^T \mathbf{D}\mathbf{B} d\mathbf{d}_E$$

A is the area of the element:

A=0.5 h * h

$$E = 0.25 h^2 d\mathbf{d}_E^T \mathbf{B}^T \mathbf{D}\mathbf{B} d\mathbf{d}_E$$

Element Stifness matrix

Change of elastic energy inside the element:

$$E = 0.25 h^2 d\mathbf{d}_E^T \mathbf{B}^T \mathbf{D}\mathbf{B} d\mathbf{d}_E$$

Equals to the nodal work:

$$0.25 h^2 d\mathbf{d}_E^T \mathbf{B}^T \mathbf{D}\mathbf{B} d\mathbf{d}_E = d\mathbf{d}_E^T \Delta\mathbf{R}$$

$$0.25 h^2 \mathbf{B}^T \mathbf{D}\mathbf{B} d\mathbf{d}_E = \Delta\mathbf{R}$$

Element Stifness matrix

$$0.25h^2\mathbf{B}^T\mathbf{D}\mathbf{B}d\mathbf{d}_E = \Delta\mathbf{R}$$

$$\left\{ \overbrace{\begin{bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \end{bmatrix}}^{\Delta\boldsymbol{\sigma}} = \overbrace{\frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ v & v & 0 \\ 0 & 0 & 1-2v \end{bmatrix}}^{\mathbf{D}} \overbrace{\begin{bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{12} \end{bmatrix}}^{\Delta\boldsymbol{\varepsilon}} \right.$$

$$\mathbf{K}_E = 0.25h^2 \overbrace{\begin{bmatrix} -1/h & 0 & 0 & 0 & 1/h & 0 \\ 0 & 0 & 0 & -1/h & 0 & 1/h \\ 0 & -0.5/h & -0.5/h & 0 & 0.5/h & 0.5/h \end{bmatrix}}^{\mathbf{B}}{}^{T} \cdot \overbrace{\frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ v & v & 0 \\ 0 & 0 & 1-2v \end{bmatrix}}^{\mathbf{D}}$$

$$\cdot \overbrace{\begin{bmatrix} -1/h & 0 & 0 & 0 & 1/h & 0 \\ 0 & 0 & 0 & -1/h & 0 & 1/h \\ 0 & -0.5/h & -0.5/h & 0 & 0.5/h & 0.5/h \end{bmatrix}}^{\mathbf{B}}$$

Element Stifness matrix

$$\mathbf{K}_E = 0.25h^2 \overbrace{\begin{bmatrix} -1/h & 0 & 0 & 0 & 1/h & 0 \\ 0 & 0 & 0 & -1/h & 0 & 1/h \\ 0 & -0.5/h & -0.5/h & 0 & 0.5/h & 0.5/h \end{bmatrix}}^{\mathbf{B}}{}^{T} \cdot \overbrace{\frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ v & v & 0 \\ 0 & 0 & 1-2v \end{bmatrix}}^{\mathbf{D}}$$

$$\cdot \overbrace{\begin{bmatrix} -1/h & 0 & 0 & 0 & 1/h & 0 \\ 0 & 0 & 0 & -1/h & 0 & 1/h \\ 0 & -0.5/h & -0.5/h & 0 & 0.5/h & 0.5/h \end{bmatrix}}^{\mathbf{B}}$$

K independent on the element size!

$$\mathbf{K}_E = 0.25 \frac{E}{(1+v)(1-2v)} \overbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -0.5 & -0.5 & 0 & 0.5 & 0.5 \end{bmatrix}}^{\mathbf{B}}{}^{T} \cdot \overbrace{\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ v & v & 0 \\ 0 & 0 & 1-2v \end{bmatrix}}^{\mathbf{D}}$$

$$\cdot \overbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -0.5 & -0.5 & 0 & 0.5 & 0.5 \end{bmatrix}}^{\mathbf{B}}$$

Element Stifness matrix

$$\mathbf{K}_E = 0.25\frac{E}{(1+v)(1-2v)}\overbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -0.5 & -0.5 & 0 & 0.5 & 0.5 \end{bmatrix}}^{\mathbf{B}}{}^{T} \cdot \overbrace{\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ v & v & 0 \\ 0 & 0 & 1-2v \end{bmatrix}}^{\mathbf{D}}$$

$$\cdot \overbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -0.5 & -0.5 & 0 & 0.5 & 0.5 \end{bmatrix}}^{\mathbf{B}}$$

Element stiffness matrix is constant only if we assume linear elasticity.

Numerical integration for element stiffness matrix is common (instead of multiplying number of matrices)

Global Stifness matrix

$$\underbrace{\mathbf{K}_E}_{\substack{\text{Element stiffnes}\\ \text{marix}}} \underbrace{\Delta\mathbf{d}_E}_{\substack{\text{Vector containing}\\ \text{displacements increments}\\ \text{of element nodes}}} = \underbrace{\Delta\mathbf{R}_E}_{\substack{\text{Vector containing}\\ \text{forces at element nodes}\\ \text{(and increments of them)}}}$$

$$\underbrace{\mathbf{K}_G}_{\substack{\text{Global stiffnes}\\ \text{marix}}} \underbrace{\Delta\mathbf{d}_G}_{\substack{\text{Vector containing}\\ \text{displacement increments}\\ \text{of all element nodes}}} = \underbrace{\Delta\mathbf{R}_G}_{\substack{\text{Vector containing forces}\\ \text{and their increments}\\ \text{at all element nodes}}}$$

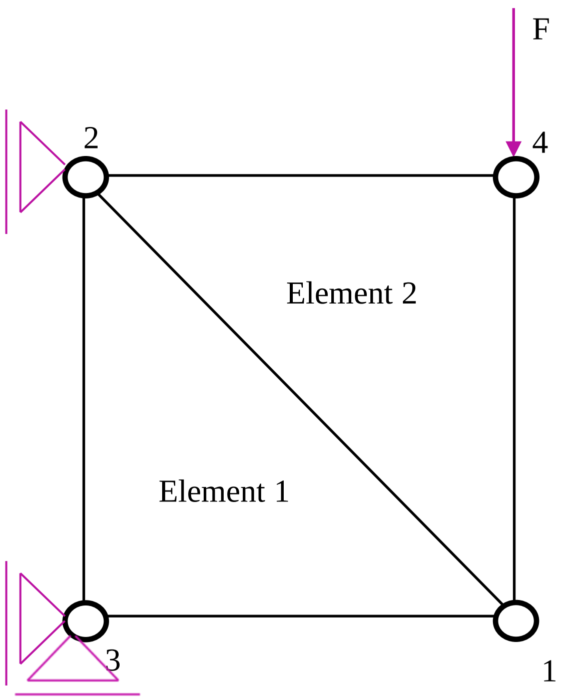
$$\mathbf{K}_E = 0.25\frac{E}{(1+v)(1-2v)} \overbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -0.5 & -0.5 & 0 & 0.5 & 0.5 \end{bmatrix}}^{\mathbf{B}}{}^{T} \cdot \overbrace{\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ v & v & 0 \\ 0 & 0 & 1-2v \end{bmatrix}}^{\mathbf{D}}$$

$$\cdot \overbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -0.5 & -0.5 & 0 & 0.5 & 0.5 \end{bmatrix}}^{\mathbf{B}}$$

Global Stifness matrix

$$\underbrace{\mathbf{K}_E}_{\text{Element stiffnes marix}} \underbrace{\Delta\mathbf{d}_E}_{\text{Vector containing displacements increments of element nodes}} = \underbrace{\Delta\mathbf{R}_E}_{\text{Vector containing forces at element nodes (and increments of them)}}$$

$$\underbrace{\mathbf{K}_G}_{\text{Global stiffnes marix}} \underbrace{\Delta\mathbf{d}_G}_{\text{Vector containing displacement increments of all element nodes}} = \underbrace{\Delta\mathbf{R}_G}_{\text{Vector containing forces and their increments at all element nodes}}$$



Element 1:

$$\mathbf{K}_E^1 \Delta\mathbf{d}_E^1 = \overbrace{\begin{bmatrix} K^1_{11xx} & K^1_{11xy} & K^1_{12xx} & K^1_{12xy} & K^1_{13xx} & K^1_{13xy} \\ & K^1_{11yy} & K^1_{12yx} & K^1_{12yy} & K^1_{13yx} & K^1_{13yy} \\ & & K^1_{22xx} & K^1_{22xy} & K^1_{23xx} & K^1_{23xy} \\ & & & K^1_{22yy} & K^1_{23yx} & K^1_{23yy} \\ & & & & K^1_{33xx} & K^1_{33xy} \\ . & & & & & K^1_{yy} \end{bmatrix}}^{\mathbf{K}_E^1} \overbrace{\begin{bmatrix} d_{x1} \\ d_{y1} \\ d_{x2} \\ d_{y2} \\ d_{x3} \\ d_{y3} \end{bmatrix}}^{\Delta\mathbf{d}_E^1} = \overbrace{\begin{bmatrix} \Delta R_{x1} \\ \Delta R_{y1} \\ \Delta R_{x2} \\ \Delta R_{y2} \\ \Delta R_{x3} \\ \Delta R_{y3} \end{bmatrix}}^{\Delta\mathbf{R}_E^1} = \Delta\mathbf{R}_E^1$$

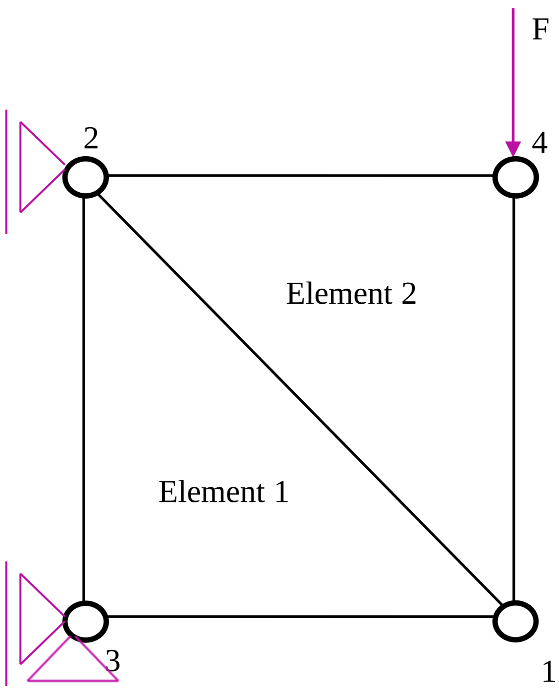
Meaning of indices:

$$K^1_{12yx} \; - \; K^{1-\text{element number}}_{1-\text{node number }\; 2-\text{due to displacement of node number }\; y-\text{resulting reaction direction }\; x-\text{due to displacement direction}}$$

Global Stifness matrix

$$\overbrace{\mathbf{K}_E}^{\text{Element stiffnes marix}} \; \overbrace{\Delta\mathbf{d}_E}^{\text{Vector containing displacements increments of element nodes}} = \overbrace{\Delta\mathbf{R}_E}^{\text{Vector containing forces at element nodes (and increments of them)}}$$

$$\overbrace{\mathbf{K}_G}^{\text{Global stiffnes marix}} \; \overbrace{\Delta\mathbf{d}_G}^{\text{Vector containing displacement increments of all element nodes}} = \overbrace{\Delta\mathbf{R}_G}^{\text{Vector containing forces and their increments at all element nodes}}$$



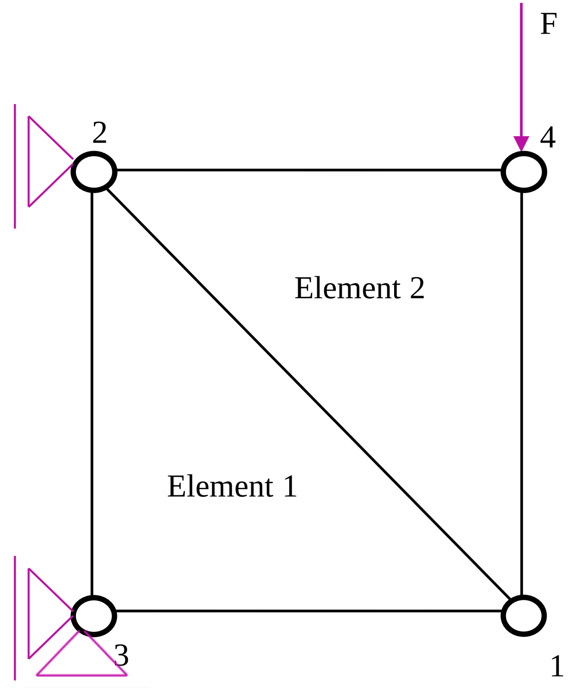
Element 2:

$$\mathbf{K}_E^2 \Delta\mathbf{d}_E^2 = \overbrace{\begin{bmatrix} K^2_{11xx} & K^2_{11xy} & K^2_{12xx} & K^2_{12xy} & K^2_{14xx} & K^2_{14xy} \\ & K^2_{11yy} & K^2_{12yx} & K^2_{12yy} & K^2_{14yx} & K^2_{14yy} \\ & & K^2_{22xx} & K^2_{22xy} & K^2_{24xx} & K^2_{24xy} \\ & & & K^2_{22yy} & K^2_{24yx} & K^2_{24yy} \\ & & & & K^2_{44xx} & K^2_{44xy} \\ . & & & & & K^2_{44yy} \end{bmatrix}}^{\mathbf{K}_E^2} \overbrace{\begin{bmatrix} d_{x1} \\ d_{y1} \\ d_{x2} \\ d_{y2} \\ d_{x4} \\ d_{y4} \end{bmatrix}}^{\Delta\mathbf{d}_E^2} = \overbrace{\begin{bmatrix} \Delta R_{x1} \\ \Delta R_{y1} \\ \Delta R_{x2} \\ \Delta R_{y2} \\ \Delta R_{x4} \\ \Delta R_{y4} \end{bmatrix}}^{\Delta\mathbf{R}_E^2} = \Delta\mathbf{R}_E^2$$

Meaning of indices:

$K^1_{12\,yx}$ - $K^{1-element\ number}_{1-node\ number\ 2-due\ to\ displacement\ of\ node\ number\ \ y-resulting\ reaction\ direction\ \ x-due\ to\ displacement\ direction}$

Global Stifness matrix

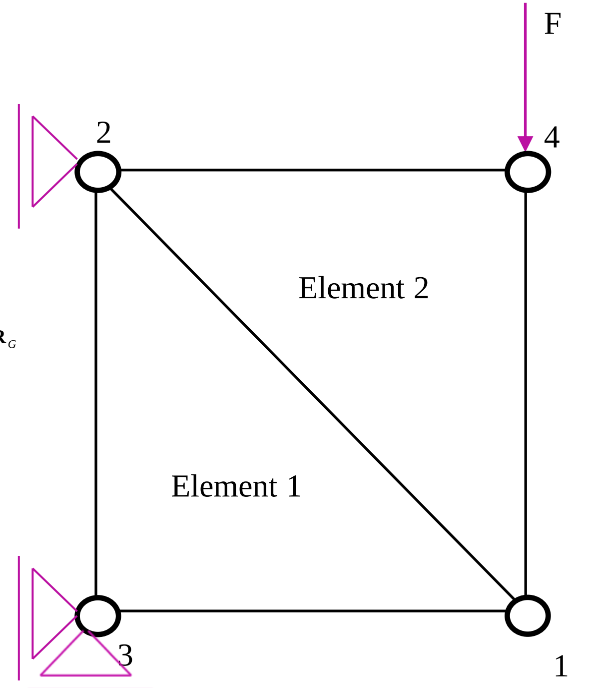


Global:

$$
\mathbf{K}_G \Delta \mathbf{d}_G = \overbrace{\begin{bmatrix}
K^1_{11xx}+K^2_{11xx} & K^1_{11xy}+K^2_{11xy} & K^1_{12xx}+K^2_{12xx} & K^1_{12xy}+K^2_{12xy} & K^1_{13xx}+0 & K^1_{13xy}+0 & K^2_{14xx}+0 & K^2_{14xy}+0 \\
 & K^1_{11yy}+K^2_{11yy} & K^1_{12yx}+K^2_{12yx} & K^1_{12yy}+K^2_{12yy} & K^1_{13yx}+0 & K^1_{13yy}+0 & K^2_{14yx}+0 & K^2_{14yy}+0 \\
 & & K^1_{22xx}+K^2_{22xx} & K^1_{22xy}+K^2_{22xy} & K^1_{23xx}+0 & K^1_{23xy}+0 & K^2_{24xx}+0 & K^2_{24xy}+0 \\
 & & & K^1_{22yy}+K^2_{22yy} & K^1_{23yx}+0 & K^1_{23yy}+0 & K^2_{24yx}+0 & K^2_{24yy}+0 \\
 & & & & K^1_{33xx}+0 & K^1_{33xy}+0 & 0 & 0 \\
 & & & & & K^1_{33yy}+0 & 0 & 0 \\
 & & & & & & K^2_{44xx}+0 & K^2_{44xy}+0 \\
 . & & & & & & & K^2_{44yy}+0
\end{bmatrix}}^{\mathbf{K}_G}
\overbrace{\begin{bmatrix} d_{x1} \\ d_{y1} \\ d_{x2} \\ d_{y2} \\ d_{x3} \\ d_{y3} \\ d_{x4} \\ d_{y4} \end{bmatrix}}^{\Delta \mathbf{d}_G}
=
\overbrace{\begin{bmatrix} \Delta R_{x1} \\ \Delta R_{y1} \\ \Delta R_{x2} \\ \Delta R_{y2} \\ \Delta R_{x3} \\ \Delta R_{y3} \\ \Delta R_{x4} \\ \Delta R_{y4} \end{bmatrix}}^{\Delta \mathbf{R}_G}
= \Delta \mathbf{R}_G
$$

Boundary conditions

$$\mathbf{K}_G \Delta \mathbf{d}_G = \overbrace{\begin{bmatrix} K^1_{11xx}+K^2_{11xx} & K^1_{11xy}+K^2_{11xy} & K^1_{12xx}+K^2_{12xx} & K^1_{12xy}+K^2_{12xy} & K^1_{13xx}+0 & K^1_{13xy}+0 & K^2_{14xx}+0 & K^2_{14xy}+0 \\ & K^1_{11yy}+K^2_{11yy} & K^1_{12yx}+K^2_{12yx} & K^1_{12yy}+K^2_{12yy} & K^1_{13yx}+0 & K^1_{13yy}+0 & K^2_{14yx}+0 & K^2_{14yy}+0 \\ & & K^1_{22xx}+K^2_{22xx} & K^1_{22xy}+K^2_{22xy} & K^1_{23xx}+0 & K^1_{23xy}+0 & K^2_{24xx}+0 & K^2_{24xy}+0 \\ & & & K^1_{22yy}+K^2_{22yy} & K^1_{23yx}+0 & K^1_{23yy}+0 & K^2_{24yx}+0 & K^2_{24yy}+0 \\ & & & & K^1_{33xx}+0 & K^1_{33xy}+0 & 0 & 0 \\ & & & & & K^1_{33yy}+0 & 0 & 0 \\ & & & & & & K^2_{44xx}+0 & K^2_{44xy}+0 \\ . & & & & & & & K^2_{44yy}+0 \end{bmatrix}}^{\mathbf{K}_G} \overbrace{\begin{bmatrix} d_{x1} \\ d_{y1} \\ d_{x2} \\ d_{y2} \\ d_{x3} \\ d_{y3} \\ d_{x4} \\ d_{y4} \end{bmatrix}}^{\Delta\mathbf{d}_G} = \overbrace{\begin{bmatrix} \Delta R_{x1} \\ \Delta R_{y1} \\ \Delta R_{x2} \\ \Delta R_{y2} \\ \Delta R_{x3} \\ \Delta R_{y3} \\ \Delta R_{x4} \\ \Delta R_{y4} \end{bmatrix}}^{\Delta\mathbf{R}_G} = \Delta \mathbf{R}_G$$

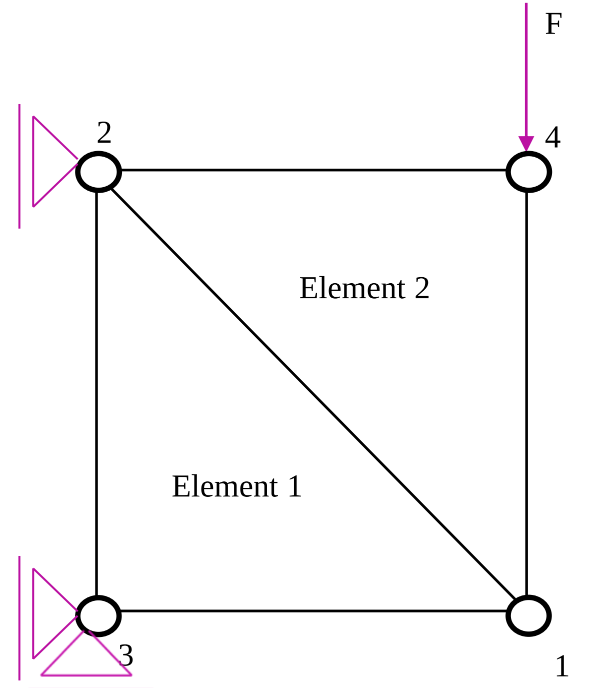


Global + boundary conditions:

$$\mathbf{K}_G \Delta \mathbf{d}_G = \overbrace{\begin{bmatrix} K^1_{11xx}+K^2_{11xx} & K^1_{11xy}+K^2_{11xy} & K^1_{12xx}+K^2_{12xx} & K^1_{12xy}+K^2_{12xy} & K^1_{13xx}+0 & K^1_{13xy}+0 & K^2_{14xx}+0 & K^2_{14xy}+0 \\ & K^1_{11yy}+K^2_{11yy} & K^1_{12yx}+K^2_{12yx} & K^1_{12yy}+K^2_{12yy} & K^1_{13yx}+0 & K^1_{13yy}+0 & K^2_{14yx}+0 & K^2_{14yy}+0 \\ & & K^1_{22xx}+K^2_{22xx} & K^1_{22xy}+K^2_{22xy} & K^1_{23xx}+0 & K^1_{23xy}+0 & K^2_{24xx}+0 & K^2_{24xy}+0 \\ & & & K^1_{22yy}+K^2_{22yy} & K^1_{23yx}+0 & K^1_{23yy}+0 & K^2_{24yx}+0 & K^2_{24yy}+0 \\ & & & & K^1_{33xx}+0 & K^1_{33xy}+0 & 0 & 0 \\ & & & & & K^1_{33yy}+0 & 0 & 0 \\ & & & & & & K^2_{44xx}+0 & K^2_{44xy}+0 \\ . & & & & & & & K^2_{44yy}+0 \end{bmatrix}}^{\mathbf{K}_G} \overbrace{\begin{bmatrix} d_{x1} \\ d_{y1} \\ 0 \\ d_{y2} \\ 0 \\ 0 \\ d_{x4} \\ d_{y4} \end{bmatrix}}^{\Delta\mathbf{d}_G} = \overbrace{\begin{bmatrix} 0 \\ 0 \\ \Delta R_{x2} \\ 0 \\ \Delta R_{x3} \\ \Delta R_{y3} \\ 0 \\ F \end{bmatrix}}^{\Delta\mathbf{R}_G} = \Delta \mathbf{R}_G$$

Boundary conditions

$$\mathbf{K}_G \Delta\mathbf{d}_G = \overbrace{\begin{bmatrix} K^1_{11xx}+K^2_{11xx} & K^1_{11xy}+K^2_{11xy} & K^1_{12xx}+K^2_{12xx} & K^1_{12xy}+K^2_{12xy} & K^1_{13xx}+0 & K^1_{13xy}+0 & K^2_{14xx}+0 & K^2_{14xy}+0 \\ & K^1_{11yy}+K^2_{11yy} & K^1_{12yx}+K^2_{12yx} & K^1_{12yy}+K^2_{12yy} & K^1_{13yx}+0 & K^1_{13yy}+0 & K^2_{14yx}+0 & K^2_{14yy}+0 \\ & & K^1_{22xx}+K^2_{22xx} & K^1_{22xy}+K^2_{22xy} & K^1_{23xx}+0 & K^1_{23xy}+0 & K^2_{24xx}+0 & K^2_{24xy}+0 \\ & & & K^1_{22yy}+K^2_{22yy} & K^1_{23yx}+0 & K^1_{23yy}+0 & K^2_{24yx}+0 & K^2_{24yy}+0 \\ & & & & K^1_{33xx}+0 & K^1_{33xy}+0 & 0 & 0 \\ & & & & & K^1_{33yy}+0 & 0 & 0 \\ & & & & & & K^2_{44xx}+0 & K^2_{44xy}+0 \\ . & & & & & & & K^2_{44yy}+0 \end{bmatrix}}^{\mathbf{K}_G} \overbrace{\begin{bmatrix} d_{x1} \\ d_{y1} \\ 0 \\ d_{y2} \\ 0 \\ 0 \\ d_{x4} \\ d_{y4} \end{bmatrix}}^{\Delta\mathbf{d}_G} = \overbrace{\begin{bmatrix} 0 \\ 0 \\ \Delta R_{x2} \\ 0 \\ \Delta R_{x3} \\ \Delta R_{y3} \\ 0 \\ F \end{bmatrix}}^{\Delta\mathbf{R}_G} = \Delta\mathbf{R}_G$$



Global + boundary conditions:

$$\mathbf{K}_G \Delta\mathbf{d}_G = \overbrace{\begin{bmatrix} K^1_{11xx}+K^2_{11xx} & K^1_{11xy}+K^2_{11xy} & 0 & K^1_{12xy}+K^2_{12xy} & 0 & 0 & K^2_{14xx}+0 & K^2_{14xy}+0 \\ & K^1_{11yy}+K^2_{11yy} & 0 & K^1_{12yy}+K^2_{12yy} & 0 & 0 & K^2_{14yx}+0 & K^2_{14yy}+0 \\ & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & K^1_{22yy}+K^2_{22yy} & 0 & 0 & K^2_{24yx}+0 & K^2_{24yy}+0 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & K^2_{44xx}+0 & K^2_{44xy}+0 \\ . & & & & & & & K^2_{44yy}+0 \end{bmatrix}}^{\mathbf{K}_G} \overbrace{\begin{bmatrix} d_{x1} \\ d_{y1} \\ 0 \\ d_{y2} \\ 0 \\ 0 \\ d_{x4} \\ d_{y4} \end{bmatrix}}^{\Delta\mathbf{d}_G} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F \end{bmatrix}$$

Finally…

Having calculated all nodal displacements, how do we get strains and stresses?

$$\Delta\boldsymbol{\varepsilon} = \begin{bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{12} \end{bmatrix} = \mathbf{B}\,\Delta\mathbf{d}_E = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \ldots & \ldots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \ldots & \ldots & 0 & \frac{\partial N_n}{\partial y} \\ 0.5\frac{\partial N_1}{\partial y} & 0.5\frac{\partial N_1}{\partial x} & 0.5\frac{\partial N_2}{\partial y} & 0.5\frac{\partial N_2}{\partial x} & \ldots & \ldots & 0.5\frac{\partial N_n}{\partial y} & 0.5\frac{\partial N_n}{\partial x} \end{bmatrix} \overbrace{\begin{bmatrix} d_{x1} \\ d_{y1} \\ d_{x2} \\ d_{y2} \\ d_{x3} \\ d_{y3} \end{bmatrix}}^{\Delta\mathbf{d}_E}$$

$$\left\{ \overbrace{\begin{bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \end{bmatrix}}^{\Delta\boldsymbol{\sigma}} = \overbrace{\frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ v & v & 0 \\ 0 & 0 & 1-2v \end{bmatrix}}^{\mathbf{D}} \overbrace{\begin{bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{12} \end{bmatrix}}^{\Delta\boldsymbol{\varepsilon}} \right.$$

Summary?

Questions?

Can you repeat what has been done here?

Thank you