Aalto University
School of Engineering

GEO – E1050
Finite Element Method in Geoengineering

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School of Engineering

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Lecture 3. Selected advancements in FEM

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Refresh of lecture 2…

Refresh

1. What element shape functions are for?

Refresh

1. What element shape functions are for?

$$\underbrace{\Delta\mathbf{u}}_{\text{Vector containing displacement at some point within the finite element}} = \underbrace{\mathbf{N}}_{\text{Matrix containing shape functions values at some point within the finite element}} \underbrace{\Delta\mathbf{d}_E}_{\text{Vector containing displacements of element nodes}}$$

Refresh

2. How to compute strains within finite element, knowing nodal displacements?

Refresh

2. How to compute strains within finite element, knowing nodal displacements?

$$\varepsilon_{ij} = 0.5\left(-\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right), \quad i, j = 1,2,3$$

$$\Delta\boldsymbol{\varepsilon} = \begin{bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{12} \end{bmatrix} = \mathbf{B}\,\Delta\mathbf{d}_E = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \dots & \dots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \dots & \dots & 0 & \frac{\partial N_n}{\partial y} \\ 0.5\frac{\partial N_1}{\partial y} & 0.5\frac{\partial N_1}{\partial x} & 0.5\frac{\partial N_2}{\partial y} & 0.5\frac{\partial N_2}{\partial x} & \dots & \dots & 0.5\frac{\partial N_n}{\partial y} & 0.5\frac{\partial N_n}{\partial x} \end{bmatrix} \overbrace{\begin{bmatrix} d_{x1} \\ d_{y1} \\ d_{x2} \\ d_{y2} \\ d_{x3} \\ d_{y3} \end{bmatrix}}^{\Delta\mathbf{d}_E}$$

Refresh

3. What is element stiffness matrix for?

Refresh

3. What is element stiffness matrix for?

$$\underbrace{\mathbf{K}_E}_{\text{Element stiffnes marix}} \underbrace{\Delta\mathbf{d}_E}_{\text{Vector containing displacements increments of element nodes}} = \underbrace{\Delta\mathbf{R}_E}_{\text{Vector containing forces at element nodes (and increments of them)}}$$

Refresh

4. What is global stiffness matrix?

Refresh

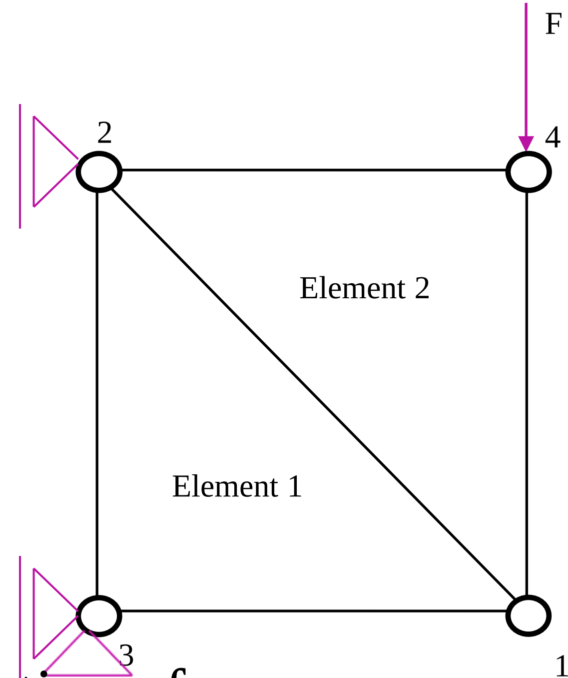
4. What is global stiffness matrix?

$$\underbrace{\mathbf{K}_G}_{\text{Global stiffnes marix}} \underbrace{\Delta\mathbf{d}_G}_{\text{Vector containing displacement increments of all element nodes}} = \underbrace{\Delta\mathbf{R}_G}_{\text{Vector containing forces and their increments at all element nodes}}$$

Refresh

5. How we apply boundary conditions?

Refresh



5. How we apply boundary conditions?

element number

stiffness of node number 1 in direction of x

K_{12xy}^{1}

due to movement of node number 2 in direction of y

$$
\mathbf{K}_G \Delta \mathbf{d}_G = \overbrace{\begin{bmatrix}
K_{11xx}^1 + K_{11xx}^2 & K_{11xy}^1 + K_{11xy}^2 & K_{12xx}^1 + K_{12xx}^2 & K_{12xy}^1 + K_{12xy}^2 & K_{13xx}^1 + 0 & K_{13xy}^1 + 0 & K_{14xx}^2 + 0 & K_{14xy}^2 + 0 \\
 & K_{11yy}^1 + K_{11yy}^2 & K_{12yx}^1 + K_{12yx}^2 & K_{12yy}^1 + K_{12yy}^2 & K_{13yx}^1 + 0 & K_{13yy}^1 + 0 & K_{14yx}^2 + 0 & K_{14yy}^2 + 0 \\
 & & K_{22xx}^1 + K_{22xx}^2 & K_{22xy}^1 + K_{22xy}^2 & K_{23xx}^1 + 0 & K_{23xy}^1 + 0 & K_{24xx}^2 + 0 & K_{24xy}^2 + 0 \\
 & & & K_{22yy}^1 + K_{22yy}^2 & K_{23yx}^1 + 0 & K_{23yy}^1 + 0 & K_{24yx}^2 + 0 & K_{24yy}^2 + 0 \\
 & & & & K_{33xx}^1 + 0 & K_{33xy}^1 + 0 & 0 & 0 \\
 & & & & & K_{33yy}^1 + 0 & 0 & 0 \\
 & & & & & & K_{44xx}^2 + 0 & K_{44xy}^2 + 0 \\
 & . & & & & & & K_{44yy}^2 + 0
\end{bmatrix}}^{\mathbf{K}_G}
\overbrace{\begin{bmatrix} d_{x1} \\ d_{y1} \\ 0 \\ d_{y2} \\ 0 \\ 0 \\ d_{x4} \\ d_{y4} \end{bmatrix}}^{\Delta \mathbf{d}_G}
=
\overbrace{\begin{bmatrix} 0 \\ 0 \\ \Delta R_{x2} \\ 0 \\ \Delta R_{x3} \\ \Delta R_{y3} \\ 0 \\ F \end{bmatrix}}^{\Delta \mathbf{R}_G}
= \Delta \mathbf{R}_G
$$

Refresh

6. Having calculated all nodal displacements, how do we get strains and stresses?

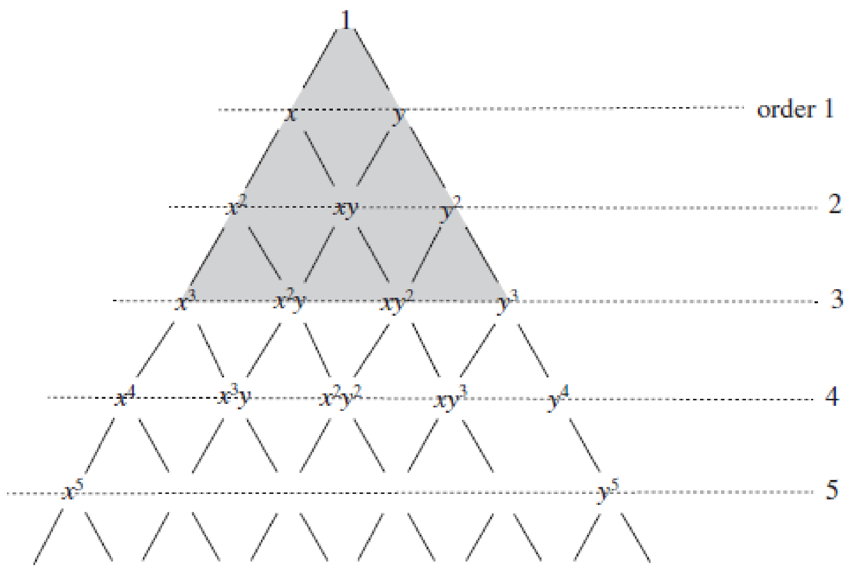
$$\Delta\boldsymbol{\varepsilon} = \begin{bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{12} \end{bmatrix} = \mathbf{B}\,\Delta\mathbf{d}_E = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \dots & \dots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \dots & \dots & 0 & \frac{\partial N_n}{\partial y} \\ 0.5\frac{\partial N_1}{\partial y} & 0.5\frac{\partial N_1}{\partial x} & 0.5\frac{\partial N_2}{\partial y} & 0.5\frac{\partial N_2}{\partial x} & \dots & \dots & 0.5\frac{\partial N_n}{\partial y} & 0.5\frac{\partial N_n}{\partial x} \end{bmatrix} \overbrace{\begin{bmatrix} d_{x1} \\ d_{y1} \\ d_{x2} \\ d_{y2} \\ d_{x3} \\ d_{y3} \end{bmatrix}}^{\Delta\mathbf{d}_E}$$

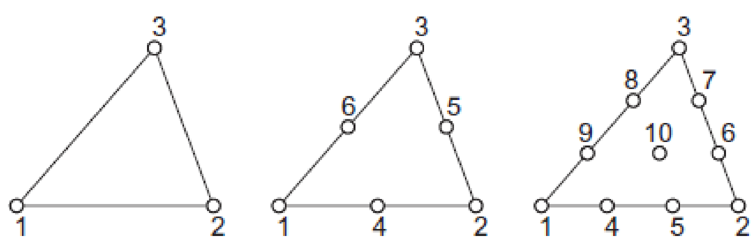
$$\left\{ \overbrace{\begin{bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \end{bmatrix}}^{\Delta\boldsymbol{\sigma}} = \overbrace{\frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ v & v & 0 \\ 0 & 0 & 1-2v \end{bmatrix}}^{\mathbf{D}} \overbrace{\begin{bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{12} \end{bmatrix}}^{\Delta\boldsymbol{\varepsilon}} \right.$$

To learn today…

1. Higher order elements – how they are created?
2. Understand the idea behind isoparametric elements
3. Understand the advantages (and disadvantages) of use of higher order elements
4. Understand the concept of numerical integration of a function and its extension to two and three dimensional problems
5. Understand the concepts behind error control and automatic mesh refinement, as well as suggest cases when such automatic error control may not working as intended.

Higher order triangular elements





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To get a full polynomial we need:

3 nodes for linear one

6 nodes for quadratic

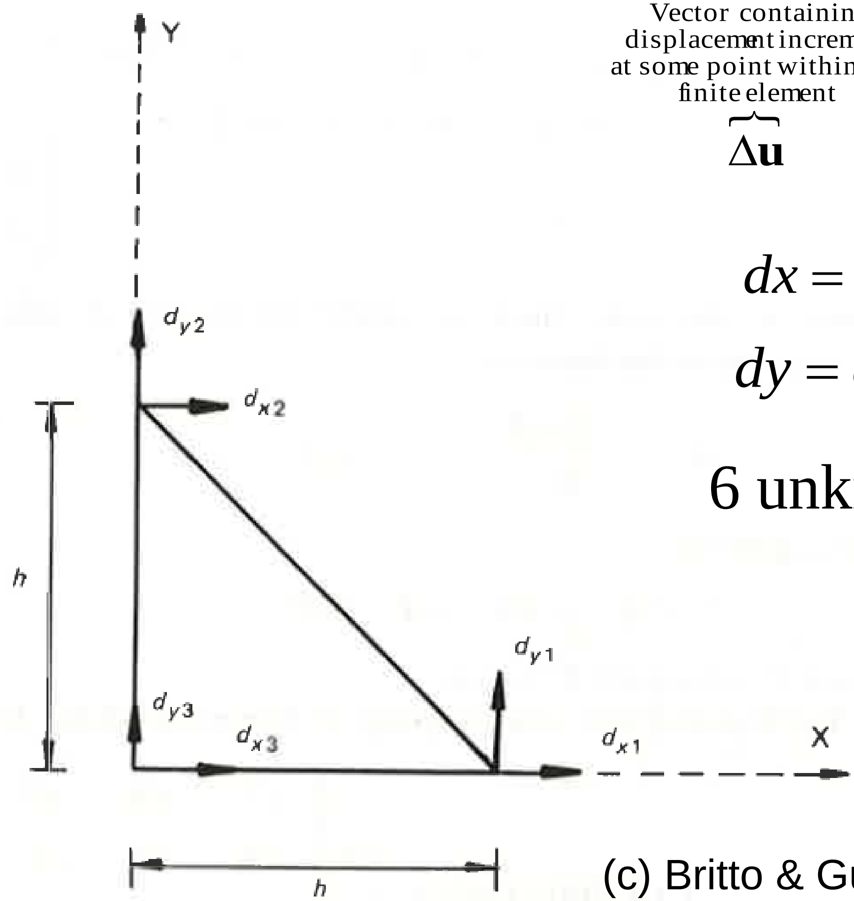
10 nodes for third order

15 nodes for fourth order

21 nodes for fifth order

…

Shape functions



$$\underbrace{\Delta\mathbf{u}}_{\substack{\text{Vector containing}\\\text{displacement increments}\\\text{at some point within the}\\\text{finite element}}} = \underbrace{\mathbf{N}}_{\substack{\text{Matrix containing}\\\text{shape functions values}\\\text{at some point within}\\\text{the finite element}}} \underbrace{\Delta\mathbf{d}_E}_{\substack{\text{Vector containing increments}\\\text{of displacements of element nodes}}}$$

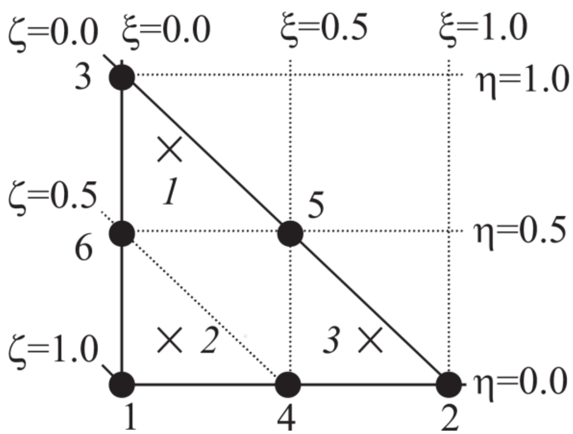
$$dx = c_0 + c_1 x + c_2 y$$

$$dy = c_3 + c_4 x + c_5 y$$

6 unknowns - (c_0-c_6)

(c) Britto & Gunn

Higher order elements



$N_1 = \zeta(2\zeta - 1)$

$N_2 = \xi(2\xi - 1)$

$N_3 = \eta(2\eta - 1)$

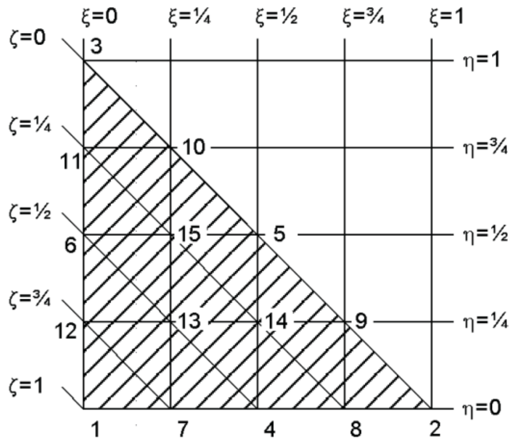
$N_4 = 4\zeta\xi$

$N_5 = 4\xi\eta$

$N_6 = 4\eta\zeta$

$$dx = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2$$

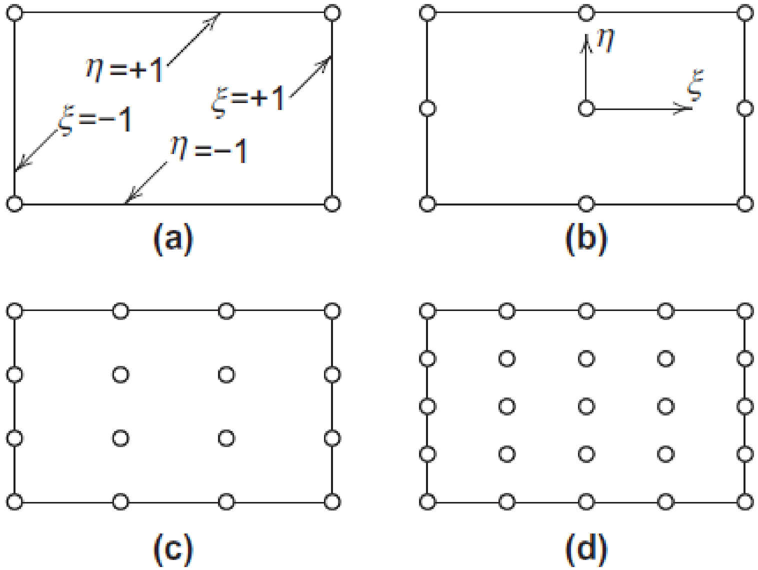
Higher order elements

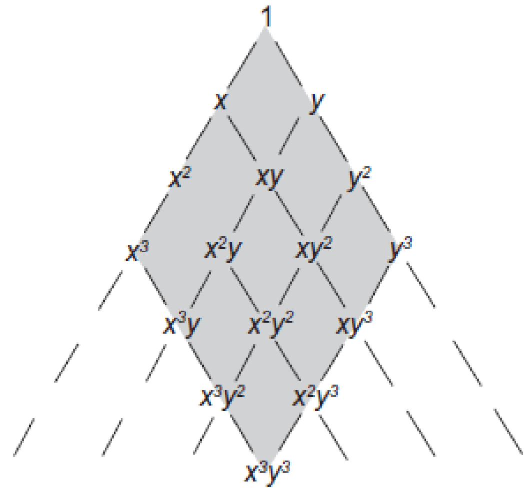


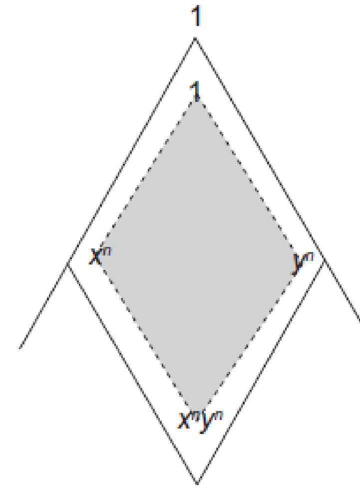
$$N_1 = \zeta(4\zeta - 1)(4\zeta - 2)(4\zeta - 3)/6$$
$$N_2 = \xi(4\xi - 1)(4\xi - 2)(4\xi - 3)/6$$
$$N_3 = \eta(4\eta - 1)(4\eta - 2)(4\eta - 3)/6$$
$$N_4 = 4\zeta\xi(4\zeta - 1)(4\xi - 1)$$
$$N_5 = 4\xi\eta(4\xi - 1)(4\eta - 1)$$
$$N_6 = 4\eta\zeta(4\eta - 1)(4\zeta - 1)$$
$$N_7 = \xi\zeta(4\zeta - 1)(4\zeta - 2) * 8/3$$
$$N_8 = \zeta\xi(4\xi - 1)(4\xi - 2) * 8/3$$
$$N_9 = \eta\xi(4\xi - 1)(4\xi - 2) * 8/3$$
$$N_{10} = \xi\eta(4\eta - 1)(4\eta - 2) * 8/3$$
$$N_{11} = \zeta\eta(4\eta - 1)(4\eta - 2) * 8/3$$
$$N_{12} = \eta\zeta(4\zeta - 1)(4\zeta - 2) * 8/3$$
$$N_{13} = 32\eta\xi\zeta(4\zeta - 1)$$
$$N_{14} = 32\eta\xi\zeta(4\xi - 1)$$
$$N_{15} = 32\eta\xi\zeta(4\eta - 1)$$

For high order elements numerical integration is necessary

Higher order Lagrangian quadrilaterals







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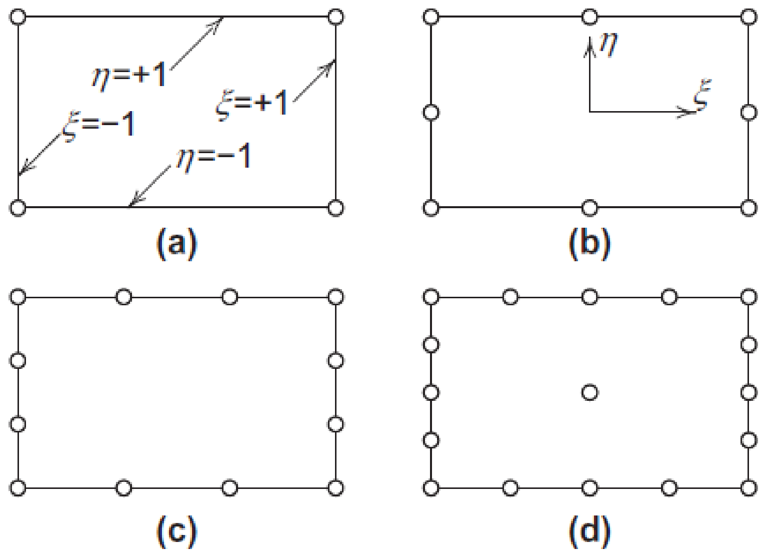
To get a ‘most efficient’ element we need:
4 nodes for a ‘linear’
9 nodes for a ‘quadratic’
16 nodes for a ‘third order’
25 nodes for ‘fourth order’ …

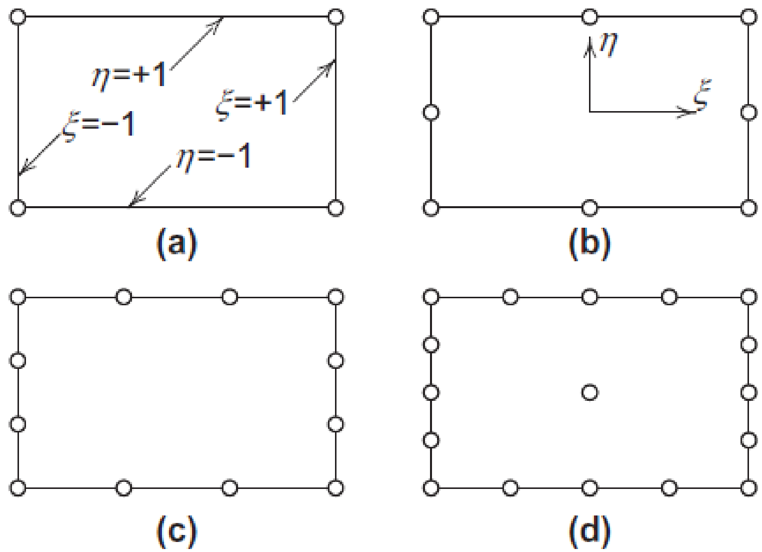
Higher order Serendipity quadrilaterals



Functions were originally derived by inspection…
Name of this family “serendipity” after the famous princes of Serendip.

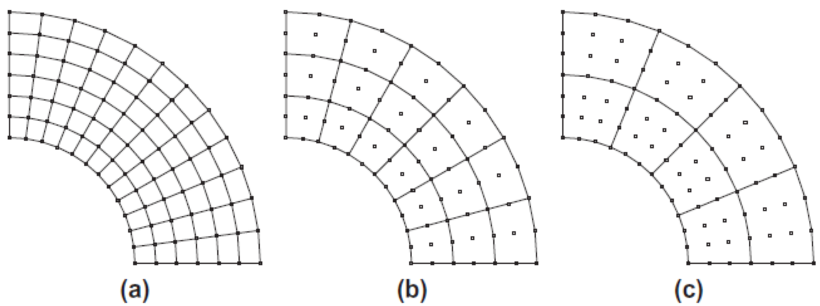
https://en.wikipedia.org/wiki/The_Three_Princes_of_Serendip

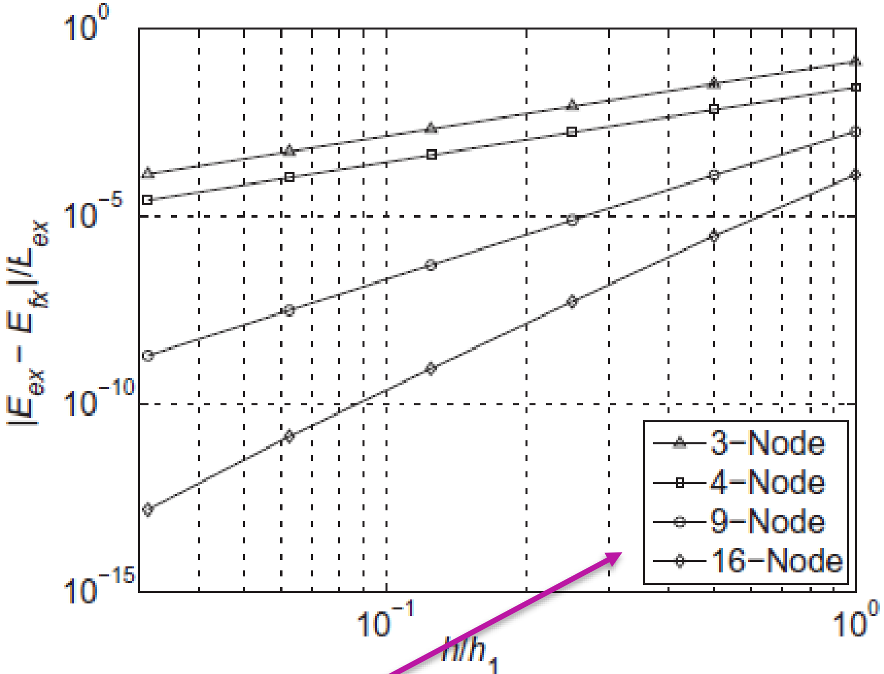
Higher order Serendipity quadrilaterals



Now we can derive higher order quadrilaterals, but generally they fell out of fashion, as almost everyone is using high order triangles…

Higher order elements





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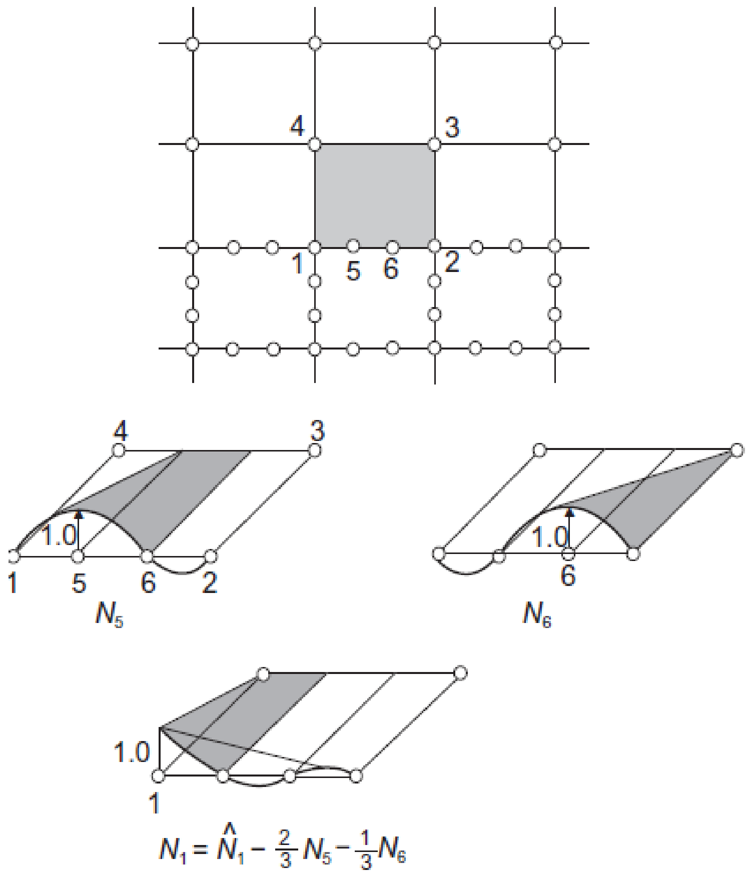
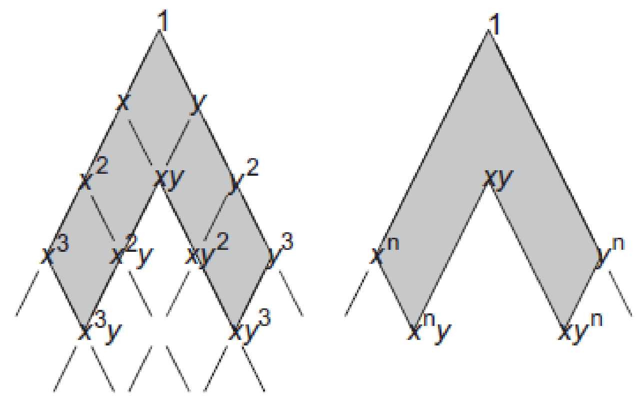
4, 9, 16 node Lagrangian quadrilaterals

Transition elements





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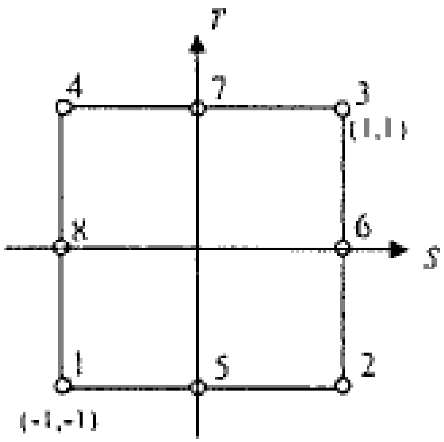
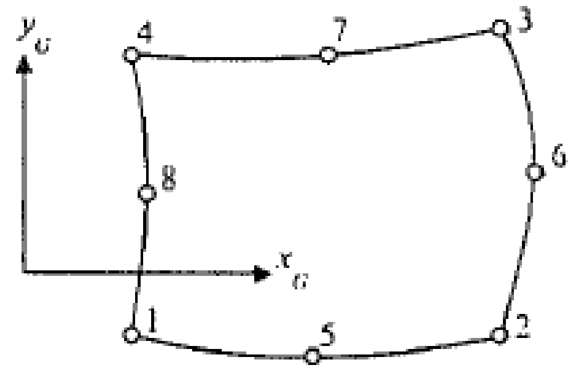
Isoparametric elements

Isoparametric elements / negative Jacobian errors





Mid - side nodes:

$$N_5 = \tfrac{1}{2}(1 - S^2)(1 - T)$$
$$N_6 = \tfrac{1}{2}(1 + S)(1 - T^2)$$
$$N_7 = \tfrac{1}{2}(1 - S^2)(1 + T)$$
$$N_8 = \tfrac{1}{2}(1 - S)(1 - T^2)$$

Corner nodes:

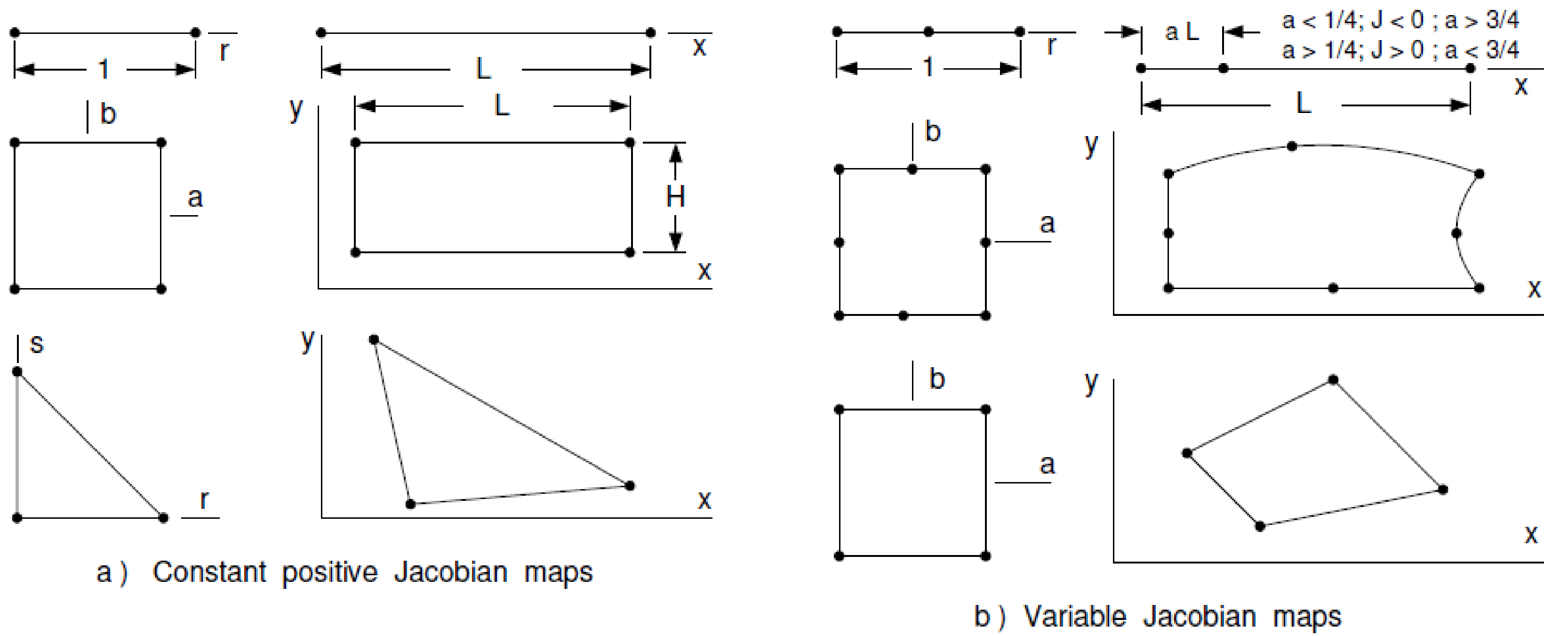
$$N_1 = \tfrac{1}{4}(1 - S)(1 - T) - \tfrac{1}{2}N_5 - \tfrac{1}{2}N_8$$
$$N_2 = \tfrac{1}{4}(1 + S)(1 - T) - \tfrac{1}{2}N_5 - \tfrac{1}{2}N_6$$
$$N_3 = \tfrac{1}{4}(1 + S)(1 + T) - \tfrac{1}{2}N_6 - \tfrac{1}{2}N_7$$
$$N_4 = \tfrac{1}{4}(1 - S)(1 + T) - \tfrac{1}{2}N_7 - \tfrac{1}{2}N_8$$

Global coordinate system vs element coordinate system

$$\begin{bmatrix} \frac{\partial N}{\partial S} & \frac{\partial N}{\partial T} \end{bmatrix}^T = \begin{bmatrix} \frac{\partial x}{\partial S} & \frac{\partial y}{\partial S} \\ \frac{\partial x}{\partial T} & \frac{\partial y}{\partial T} \end{bmatrix} \begin{bmatrix} \frac{\partial N}{\partial x} & \frac{\partial N}{\partial y} \end{bmatrix}^T = \mathbf{J} \begin{bmatrix} \frac{\partial N}{\partial x} & \frac{\partial N}{\partial y} \end{bmatrix}^T \qquad \mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial S} & \frac{\partial y}{\partial S} \\ \frac{\partial x}{\partial T} & \frac{\partial y}{\partial T} \end{bmatrix}$$

Once the element is too distorted, the Jacobian J may not be invertible anymore (and beforehand, large errors occur)

Isoparametric elements / negative Jacobian errors



Once the element is too distorted, the Jacobian J may not be invertible anymore (and beforehand, large errors occur)

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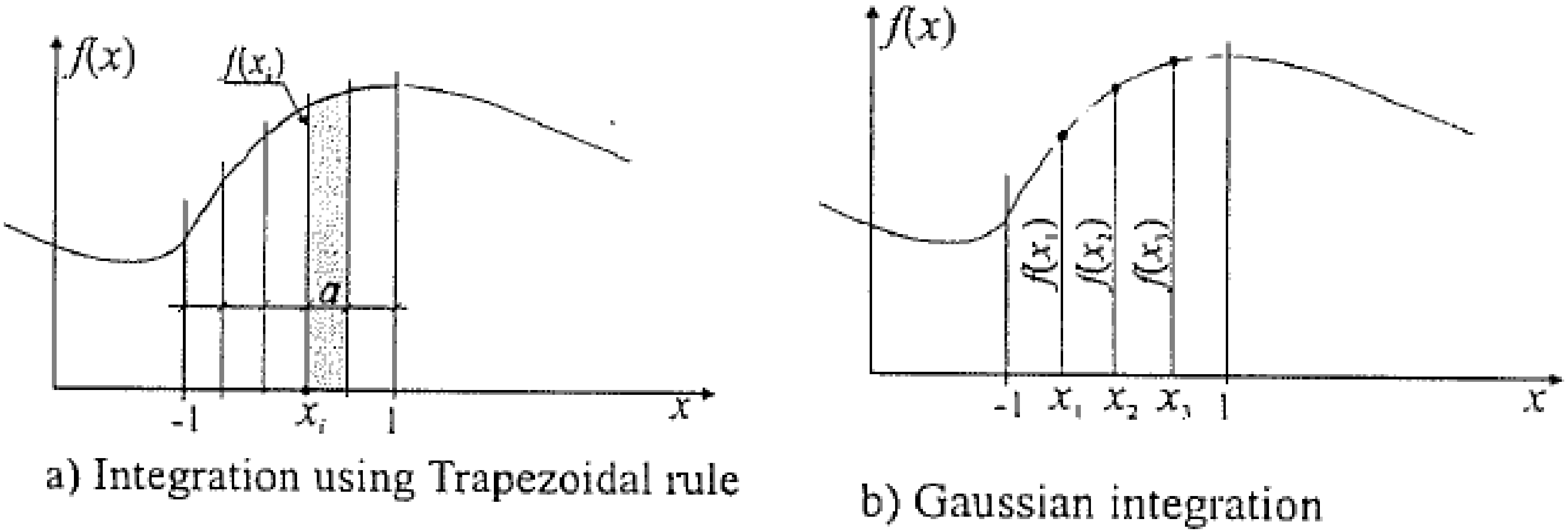
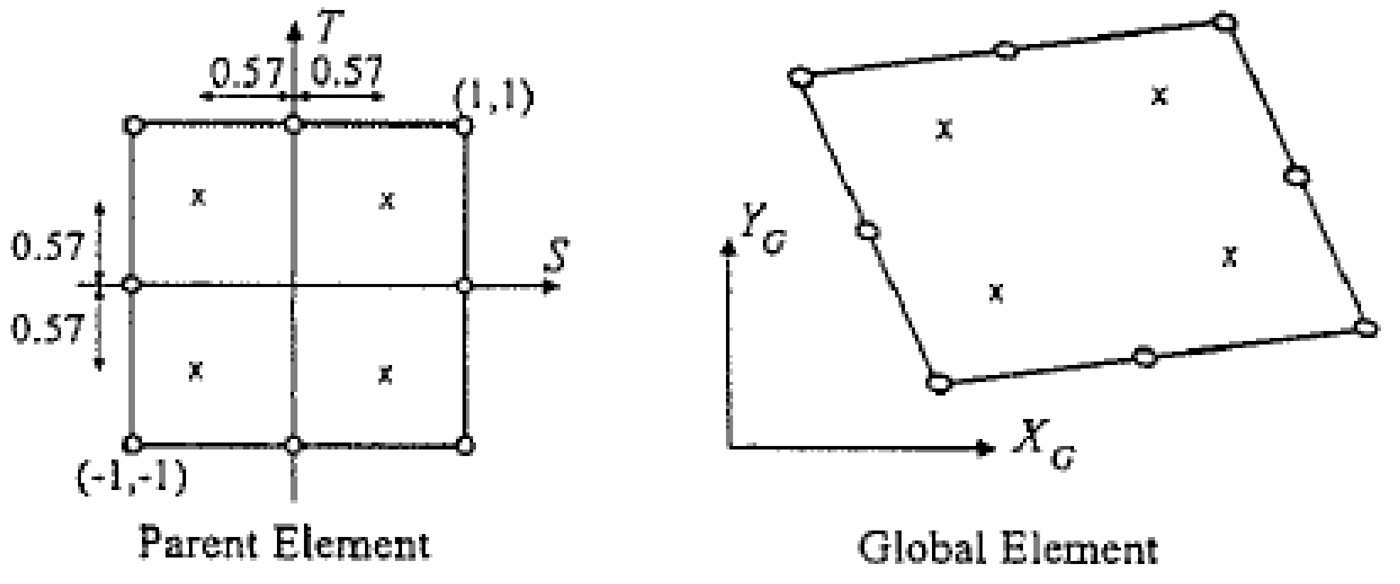
Gauss integration

Gauss integration



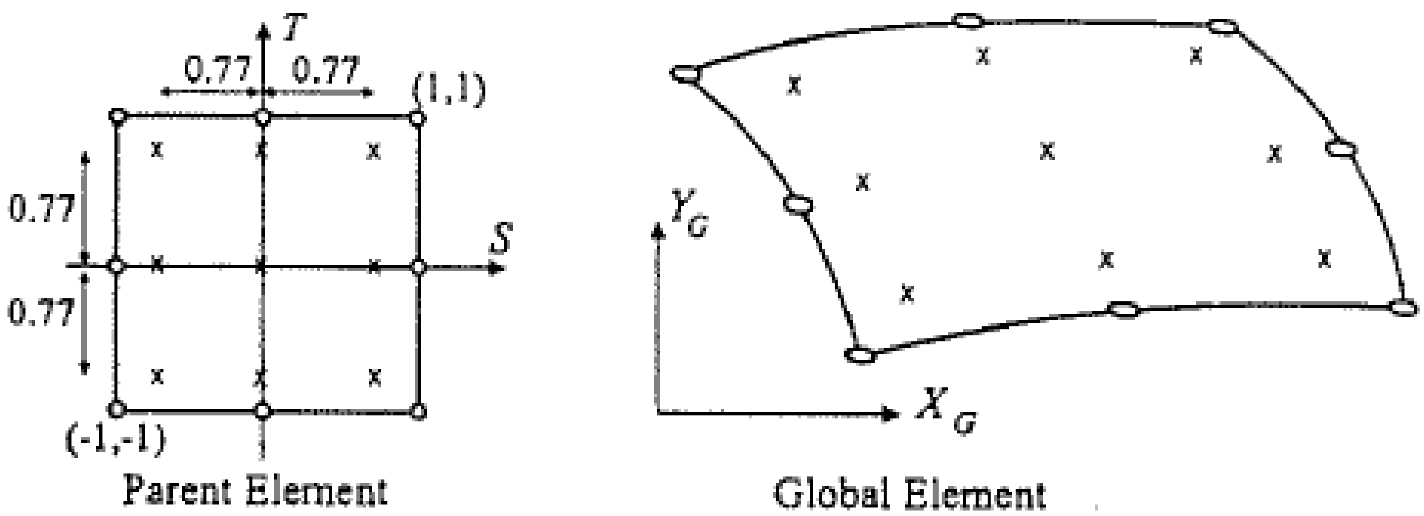
$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{3} W_i f(x_i) = W_1 f(x_1) + W_2 f(x_2) + W_3 f(x_3)$$

Gauss integration



a) 2x2 order

Gauss integration



b) 3x3 order

Gauss integration

n	ξ i = ηi	Wi
1	ξ 1 = 0	W1 = 2
2	ξ 1= ξ 2= ±0.577350269189626	W1 = W2 = 1
3	ξ1=ξ3= ±0.774596669241483 ξ 2 = 0	W1 = W3 = 0.555555555555556 W2 = 0.888888888888889
4	ξ1=ξ4= ±0.861134311594053 ξ2=ξ3= ±0.339981043584856	W1 = W4 = 0.347854845137454 W2 = W3 = 0.652145154862546
5	ξ1=ξ5=± 0.906179845938664 ξ2=ξ4=± 0.538469310105683 ξ3=0	W1 = W5 = 0.236926885056189 W2 = W4 = 0.478628670499366 W3 = 0.568888888888889
6	ξ1=ξ6=± 0.932469514203152 ξ2=ξ5=± 0.661209386466265 ξ3=ξ4= ±0.238619186003152	W1 = W6 = 0.171324492379170 W2 = W5 = 0.360761573048139 W3 = W4 = 0.467913934572691
7	ξ1=ξ7=±0.949107912342759 ξ2=ξ6=±0.741531185599394 ξ3=ξ5=±0.405845151377397 ξ4= 0	W1 = W7 = 0.129484966168870 W2 = W6 = 0.279705391489277 W3 = W5 = 0.381830050505119 W4 = 0.417959183673469

Table 3.1:- Gauss point coordinates and weights in rectangular elements

Engineers:

Engineers hate risk. They try to eliminate it whenever they can. This is understandable, given that when a scientist makes a mistake, the media will treat it like it's a big deal or something. Examples of Bad Press for Engineers: Hindenburg, Challenger, Apollo 13, Titanic, Ford Pinto

The risk/reward calculation for scientists looks something like this:
RISK: Public humiliation and the death of thousands of innocent people.
REWARD: A certificate of appreciation in a handsome plastic frame.

Being practical people, engineers evaluate this balance of risks and rewards and decide that risk is not a good thing. The best way to avoid risk is by advising that any activity is technically impossible for reasons that are far too complicated to explain.

If that approach is not sufficient to halt a project, then the engineer will fall back to a second line of defence: "It's technically possible but it will cost too much."

Error control and mesh refinement

Where is the approximation of Finite Element Algorithm?

How to make a method which would be accurate?

Error control and mesh refinement

Where is the approximation of Finite Element Algorithm?

$$dE = 0.5\int_V d\boldsymbol{\varepsilon}^T d\boldsymbol{\sigma} dV = 0.5\int_V d\boldsymbol{\varepsilon}^T \mathbf{D} d\boldsymbol{\varepsilon} dV = \ldots$$

Approximation: substituting strain field by the nodal displacements and some functions when creating the element stiffness matrices!

$$= 0.5\int_V \left(\mathbf{B} d\mathbf{d}_E\right)^T \mathbf{D}\mathbf{B} d\mathbf{d}_E dV = 0.5\int_V d\mathbf{d}_E^T \mathbf{B}^T \mathbf{D}\mathbf{B} d\mathbf{d}_E dV$$

Another minor approximation: numerical integration in cases where **D** matrix is highly nonlinear

Error control and mesh refinement

Typically we define error in strain energy, that is:

$$E = \sqrt{\int_V (\boldsymbol{\varepsilon} - \hat{\boldsymbol{\varepsilon}})^T \mathbf{D} (\boldsymbol{\varepsilon} - \hat{\boldsymbol{\varepsilon}}) dV}$$

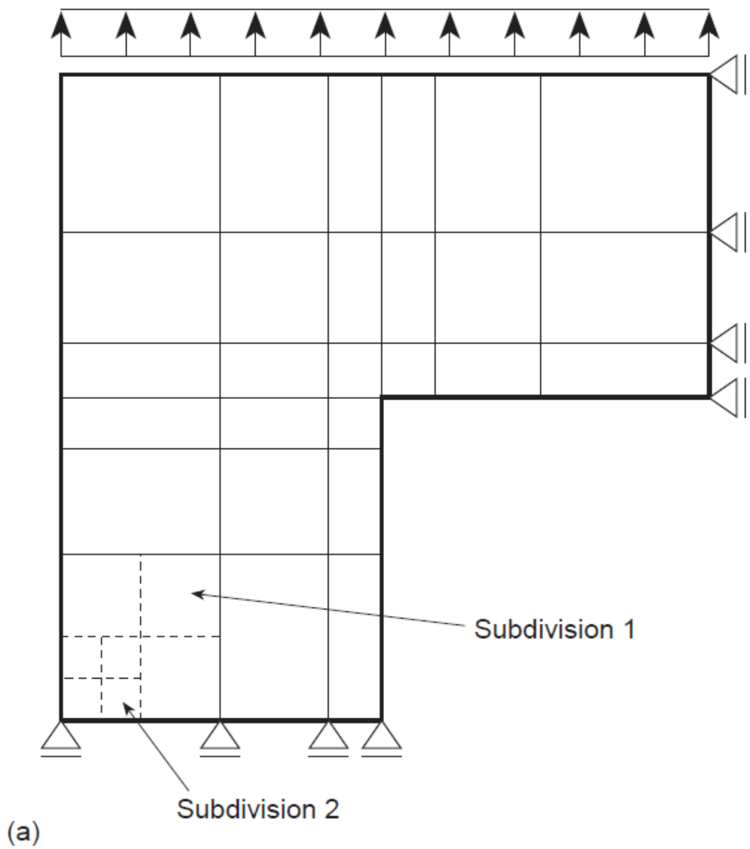
And the strain energy as:

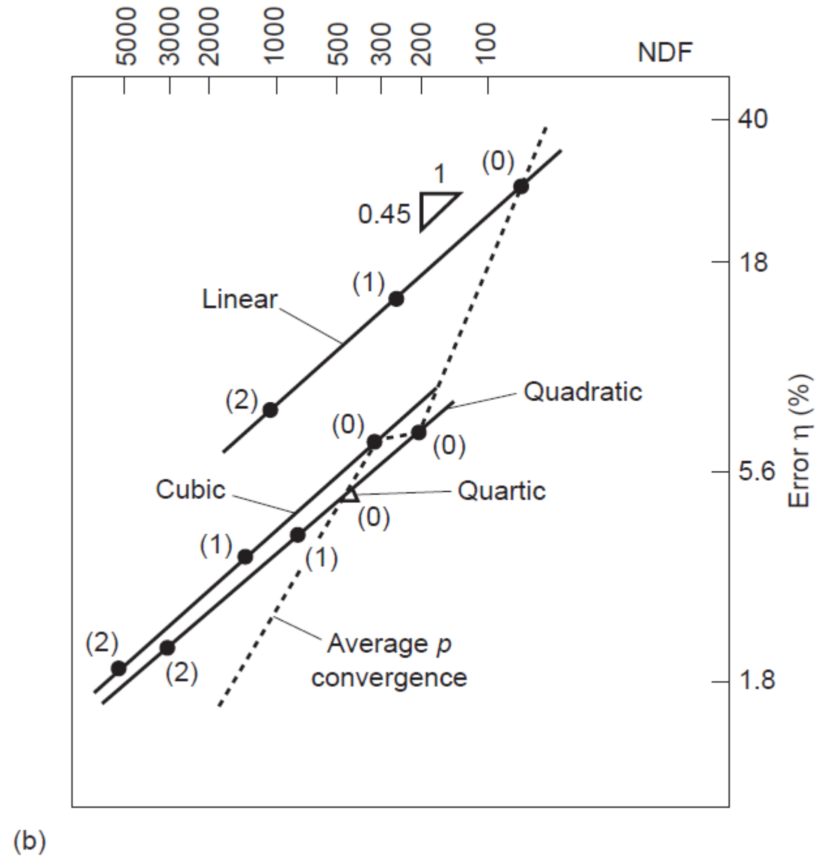
$$E_{Tot} = \sqrt{\int_V \boldsymbol{\varepsilon}^T \mathbf{D} \boldsymbol{\varepsilon} dV}$$

The relative error will be:

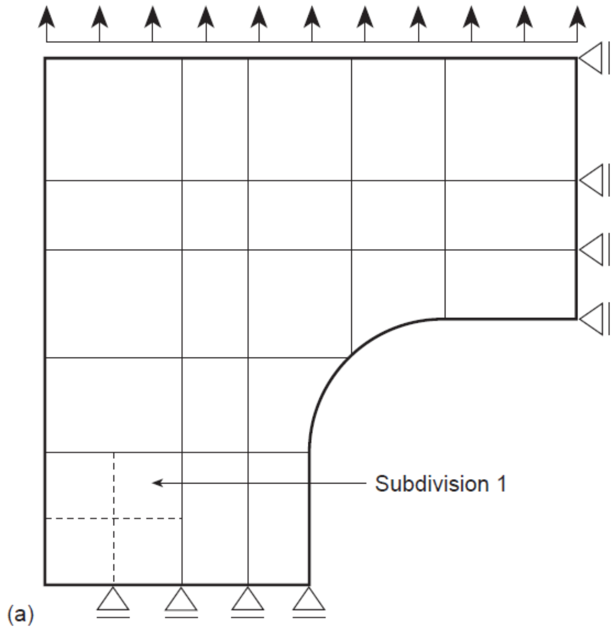
$$e = \eta = \frac{E}{E_{Tot}}$$

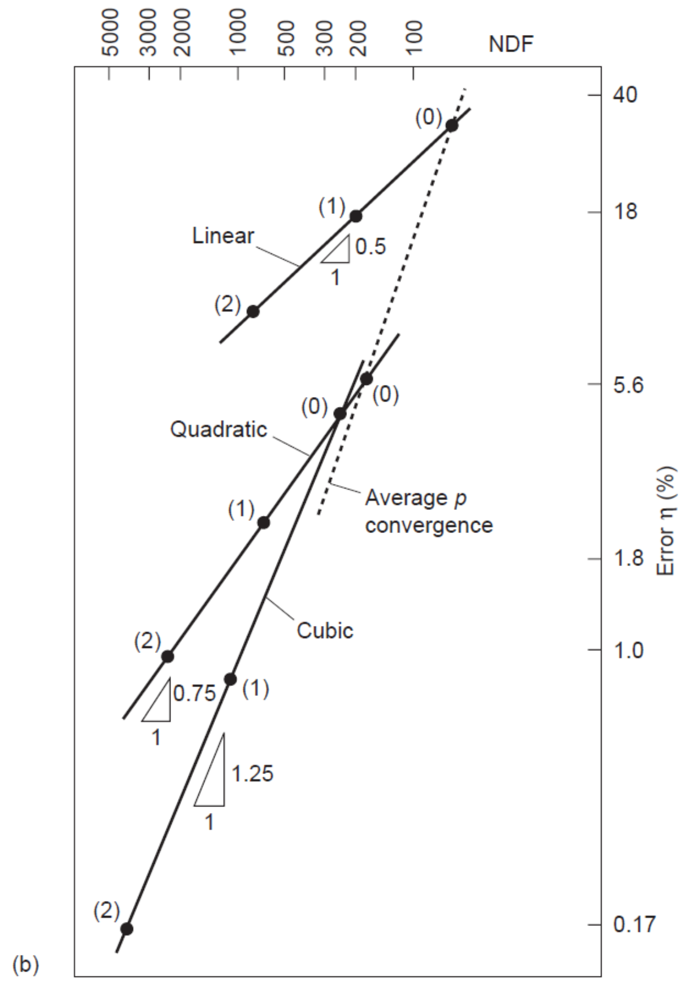
Error control and mesh refinement



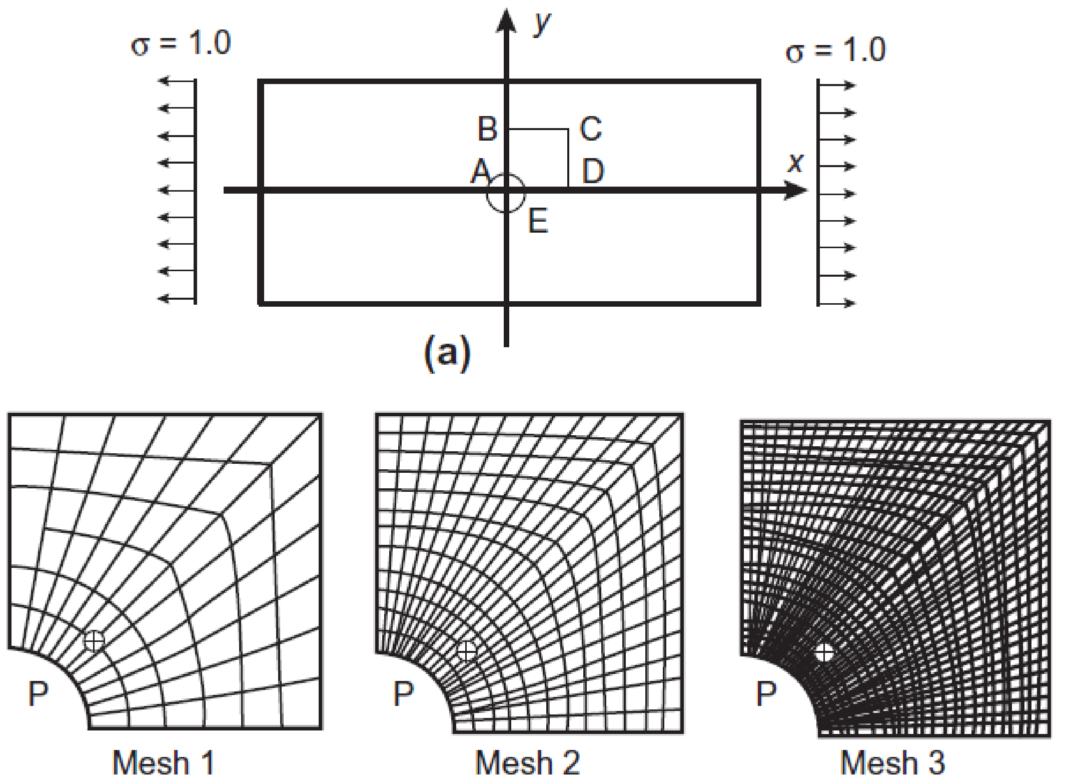


Error control and mesh refinement





Errors of stresses around circular hole



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Errors of stresses around circular hole

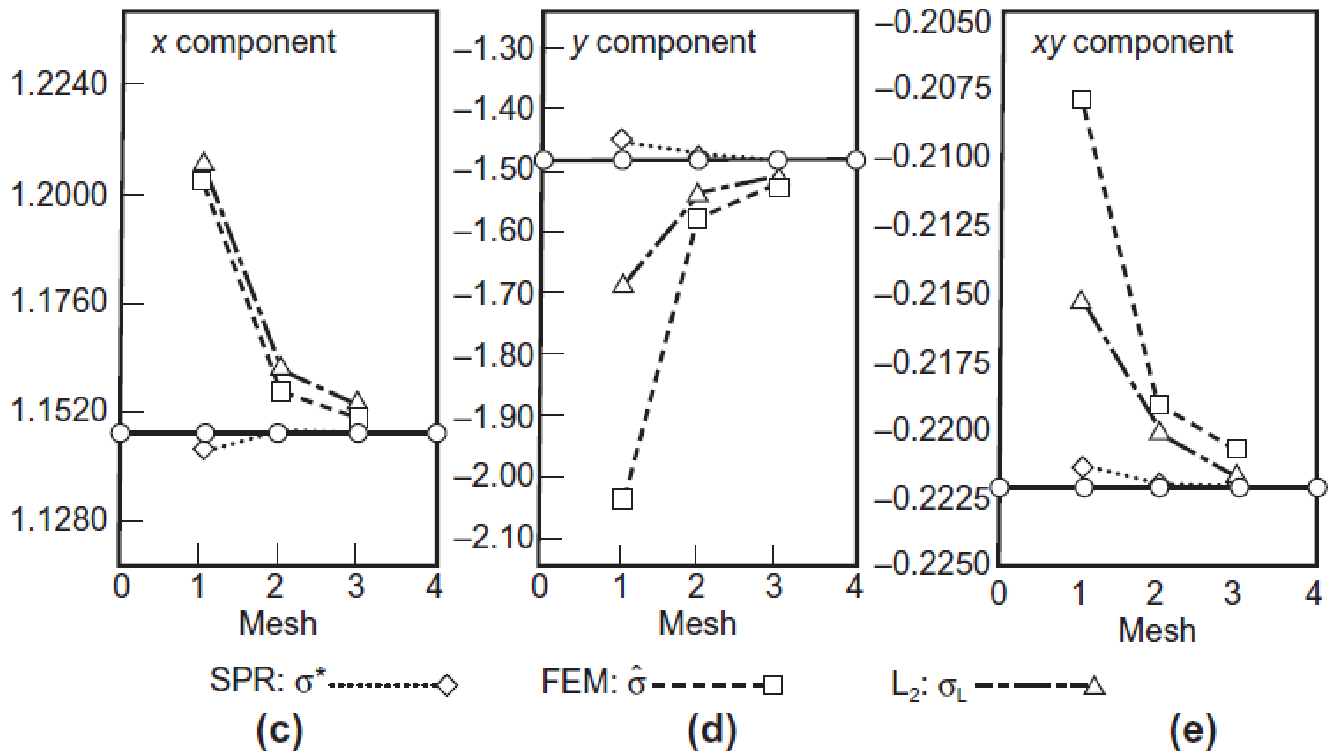

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FIGURE 15.13

Plane stress analysis of stresses around a circular hole in a uniaxial field.

Zero energy modes

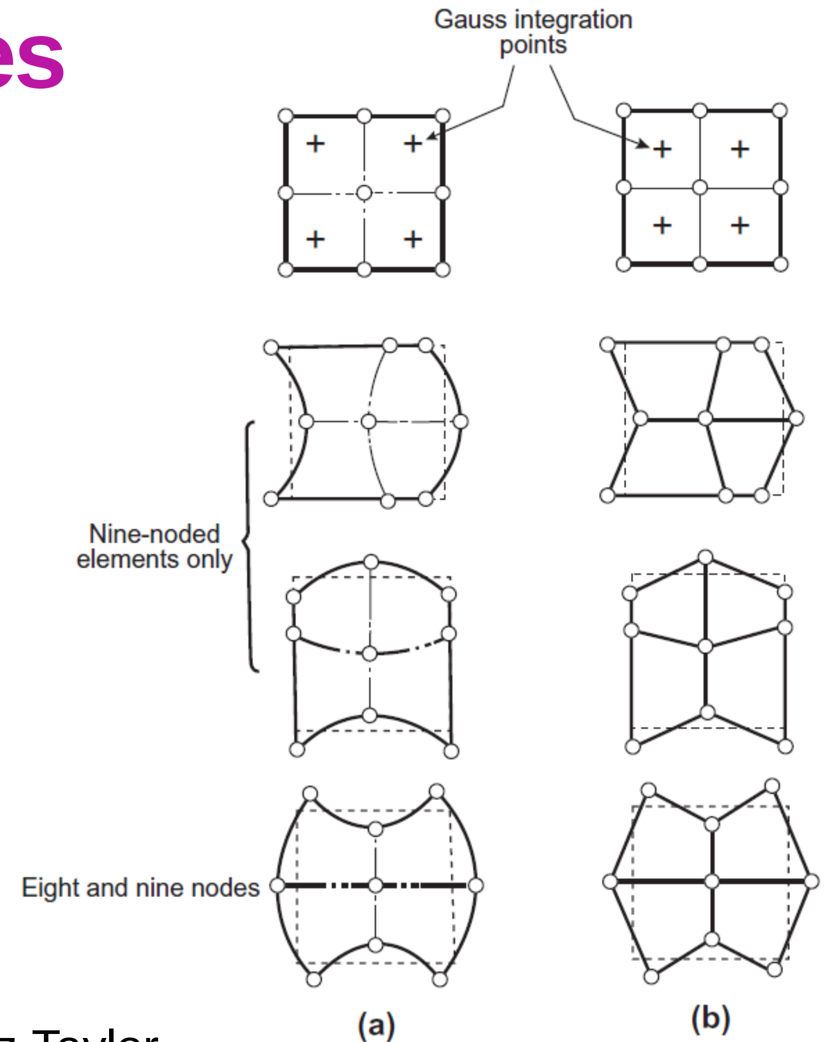

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FIGURE 8.5

(a) Zero-energy (singular) modes for eight- and nine-node quadratic elements and (b) for a patch of bilinear elements with single integration points.

Thank you