Aalto University
School of Engineering

Partial exam 1

Wojciech Sołowski

Partial exam 1

1. **In MyCourses (quiz, opens 10.15, closes 11.00)**

2. Four questions
3. Total of 10 points
4. In the essay questions you can submit the answer in the box provided or by uploading a file (e.g. photo of your written answer). You can also write in the box and submit parts of the answer as a file…
5. Lecture notes and other materials are allowed. You are graded as individual and **team work is not allowed.** If I spot answers answers which are too similar, that will be a problem.

Aalto University
School of Engineering

GEO – E1050
Finite Element Method in Geoengineering
Introduction to constitutive modelling

Wojciech Sołowski

To learn today…

1. Reminder – elasticity

2. Elasto- plastic theory of material behaviour: concept

Philosophy

Geomaterials behaviour – COMPLEX

Constitutive models – **huge simplifications**

usually the simplification designer is making is mainly about choice of model and its parameters

Model and choice of its parameters must be clearly selected for given problem. No constitutive model for geomaterials is good for solving all problems.

(these which are trying to be good for everything, like the MIT constitutive model for geomaterials are extremely complex and difficult to use, validate and calibrate)

Need to balance simplicity, accuracy and problem needs

Philosophy

Challenge:

- understand the simplifications

- use them to your benefit

- get accurate results from crude tools

- no results is ‘right’ - each result is because of the assumptions made

- if you do good job with modelling, you will get result which is close to reality…

also because you have chosen the right model…

yet primarily because **you have chosen the right set of parameters** for the model and the problem

Element Stifness matrix

$$0.25h^2\mathbf{B}^T\mathbf{D}\mathbf{B}d\mathbf{d}_E = \Delta\mathbf{R}$$

$$\left\{ \overbrace{\begin{bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \end{bmatrix}}^{\Delta\boldsymbol{\sigma}} = \overbrace{\frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ v & v & 0 \\ 0 & 0 & 1-2v \end{bmatrix}}^{\mathbf{D}} \overbrace{\begin{bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{12} \end{bmatrix}}^{\Delta\boldsymbol{\varepsilon}} \right.$$

$$\mathbf{K}_E = 0.25h^2 \overbrace{\begin{bmatrix} -1/h & 0 & 0 & 0 & 1/h & 0 \\ 0 & 0 & 0 & -1/h & 0 & 1/h \\ 0 & -0.5/h & -0.5/h & 0 & 0.5/h & 0.5/h \end{bmatrix}}^{\mathbf{B}}{}^T \cdot \overbrace{\frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ v & v & 0 \\ 0 & 0 & 1-2v \end{bmatrix}}^{\mathbf{D}}$$

$$\cdot \overbrace{\begin{bmatrix} -1/h & 0 & 0 & 0 & 1/h & 0 \\ 0 & 0 & 0 & -1/h & 0 & 1/h \\ 0 & -0.5/h & -0.5/h & 0 & 0.5/h & 0.5/h \end{bmatrix}}^{\mathbf{B}}$$

Elasticity

This lecture covers the very basics of constitutive modelling…

You know elasticity, we will build on that…

$$\left\{ \overbrace{\begin{bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \end{bmatrix}}^{\Delta\boldsymbol{\sigma}} = \overbrace{\frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ v & v & 0 \\ 0 & 0 & 1-2v \end{bmatrix}}^{\mathbf{D}} \overbrace{\begin{bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{12} \end{bmatrix}}^{\Delta\boldsymbol{\varepsilon}} \right.$$

More on modelling in Advanced Soil Mechanics…
Even more in the Numerical Methods in Geotechnics…

Outline

1. Constitutive modelling: Elasticity
   a. Linear elasticity
   b. Non-linear elasticity
   c. Hypoelasticity

2. Constitutive modelling: Elasto-plasticity
   a. What is elasto-plasticity, basic concepts and theory

Later lectures:
   a. Mohr-Coulomb
   b. Other elastic-perfectly plastic constitutive models

More on modelling in Advanced Soil Mechanics…
Even more in the Numerical Methods in Geotechnics…

Elasticity

Idea:

- one of the oldest approaches to describe material behaviour

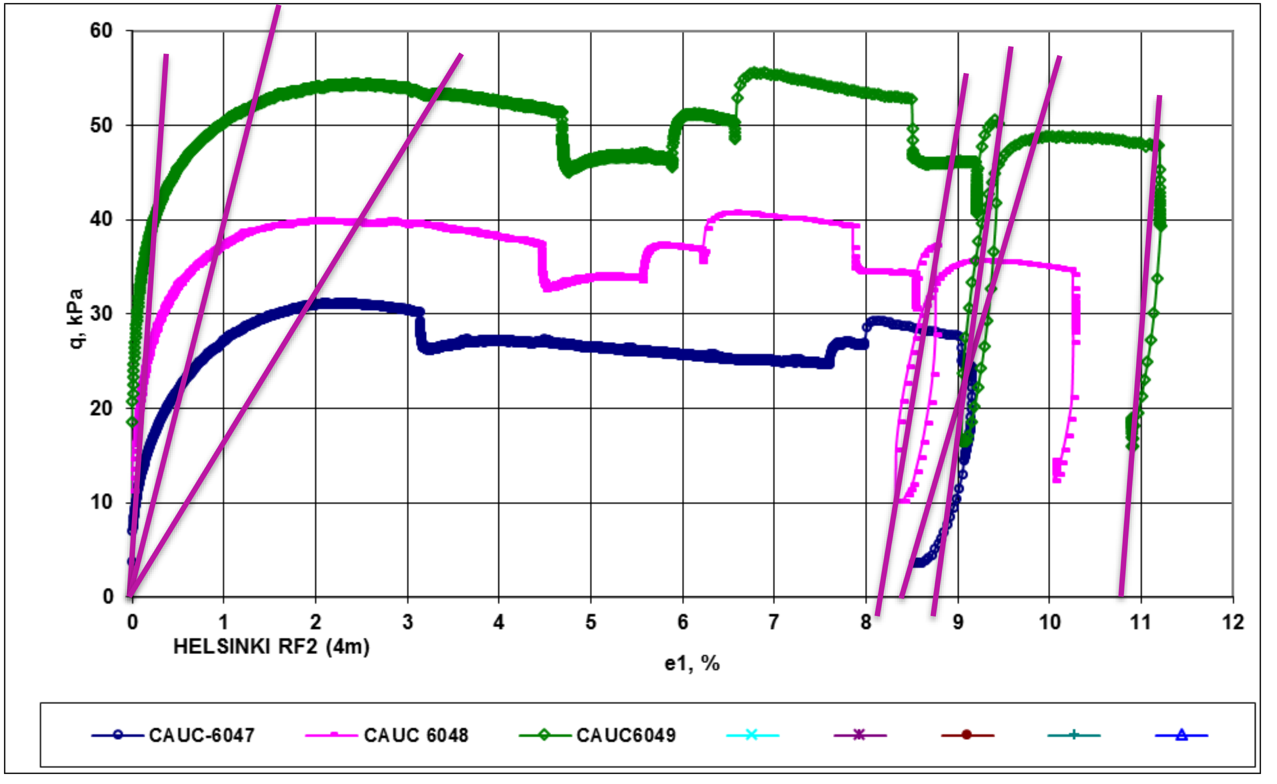
- assumes that after removing the stress the material returns back to the initial state

  - behaviour is reversible

    1) no matter the level of stress, the material will not break
    2) no matter the level of stress, after removing it, the material will be back in the original state before the loading, no change…

Elasticity

Linear elasticity – useful?



Real soil behaviour

Elasticity

Can be useful… as long as you know the stress level, deformations and have data on how the soil behaves at that stress level and deformations level…

- than you just choose the exactly right elastic parameter for given circumstances… and calculate the strains from the stresses – or stresses from strains…

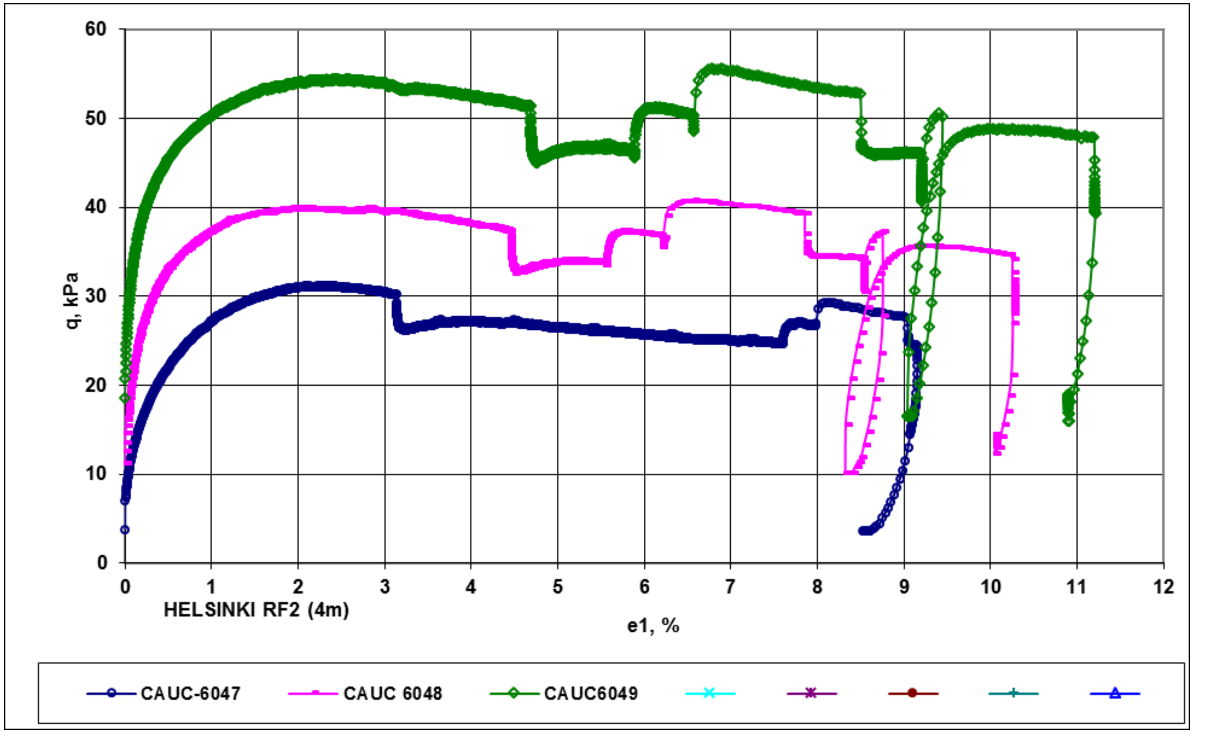
- used for example in calculations of settlements – we know the stress levels from some simple solutions – than we compute strains using right set of elastic parameters for given small layer of soil… Can be pretty accurate…

Elasticity

$$\left\{ \overbrace{\begin{bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \\ \Delta\sigma_{13} \\ \Delta\sigma_{23} \end{bmatrix}}^{\Delta\boldsymbol{\sigma}} = \overbrace{\frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ & & & 1-2v & 0 & 0 \\ & & & & 1-2v & 0 \\ . & & & & & 1-2v \end{bmatrix}}^{\mathbf{D}^{el}} \overbrace{\begin{bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{33} \\ \Delta\varepsilon_{12} \\ \Delta\varepsilon_{13} \\ \Delta\varepsilon_{23} \end{bmatrix}}^{\Delta\boldsymbol{\varepsilon}} \right.$$

$$\Delta\boldsymbol{\sigma} = \mathbf{D}^{el}\Delta\boldsymbol{\varepsilon}$$

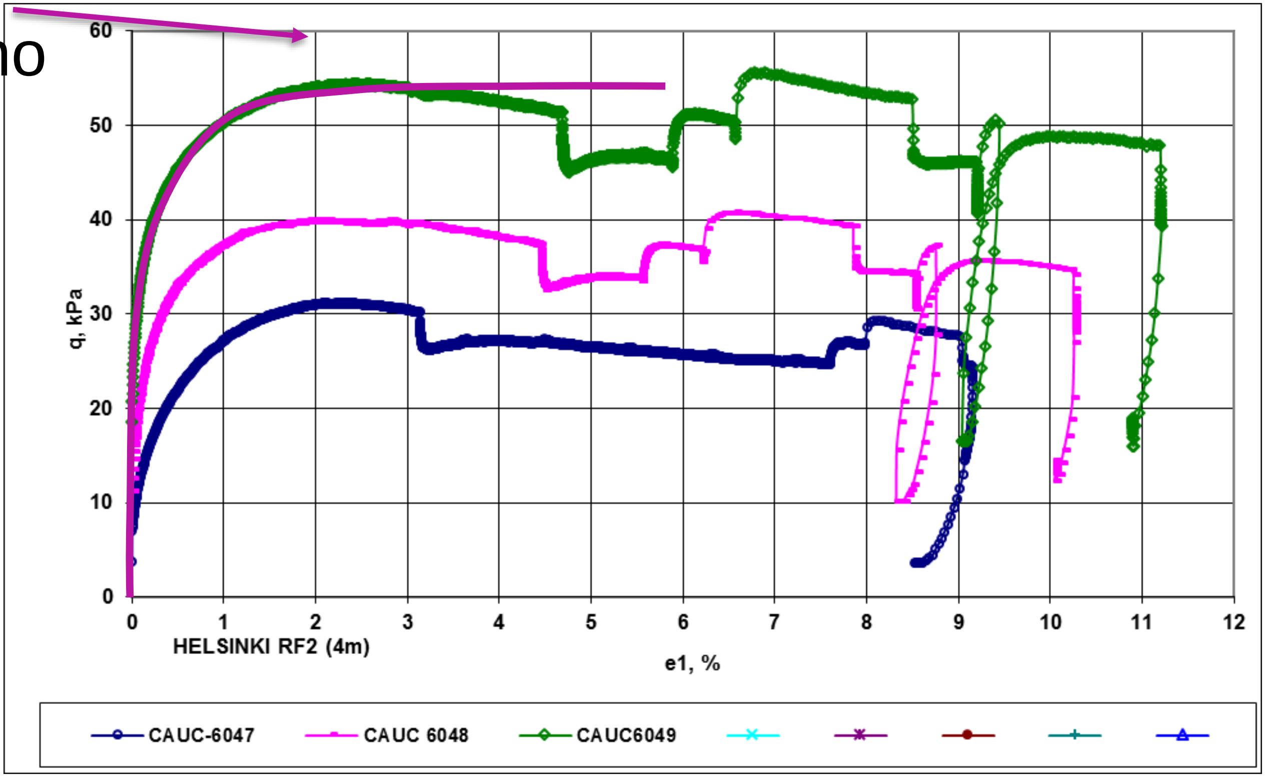
Non-linear elasticity



Real soil behaviour…

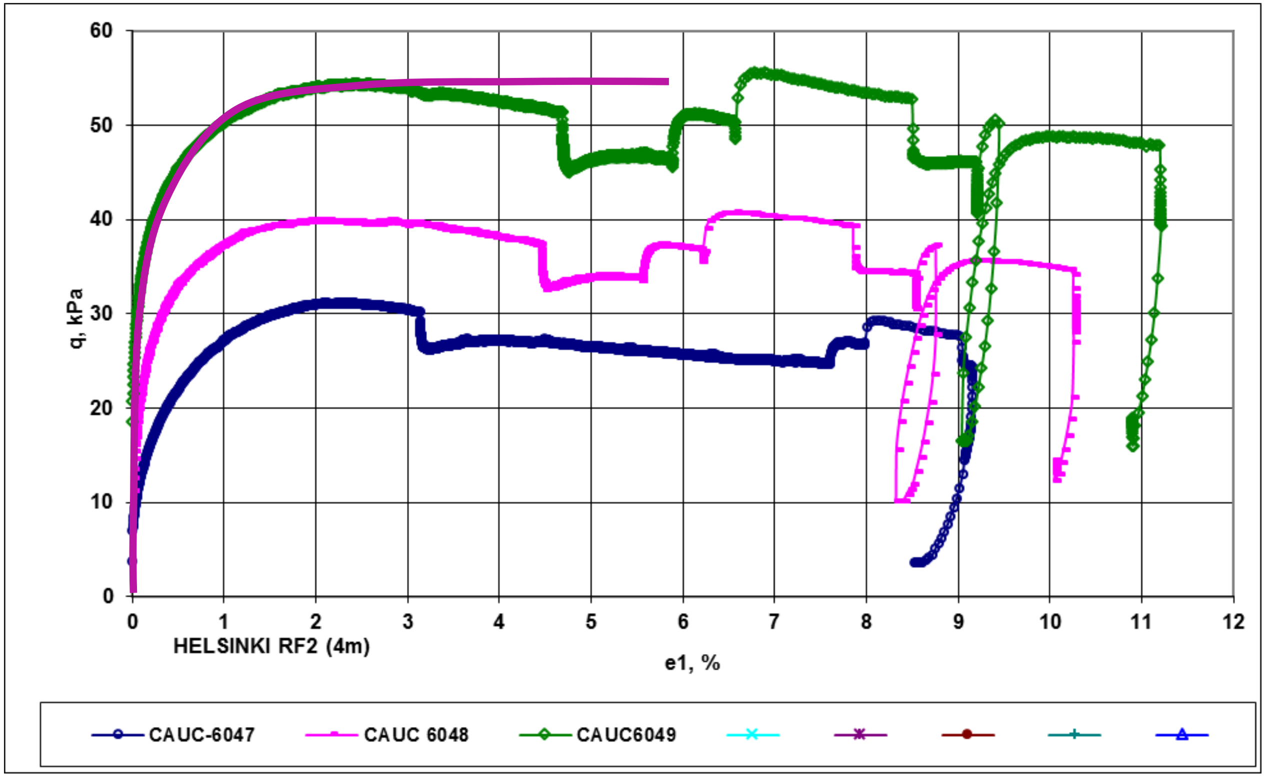
Non-linear elasticity

We can only match single line – as no dependency on stress level



Real soil behaviour…

Non-linear elasticity



$$\Delta\boldsymbol{\sigma} = \mathbf{D}^{el}(\boldsymbol{\sigma}_0)\Delta\boldsymbol{\varepsilon}$$

Elasto-plasticity

Idea:

- let's divide strains into these which are elastic (which are fully recoverable) and plastic (which are generally not)

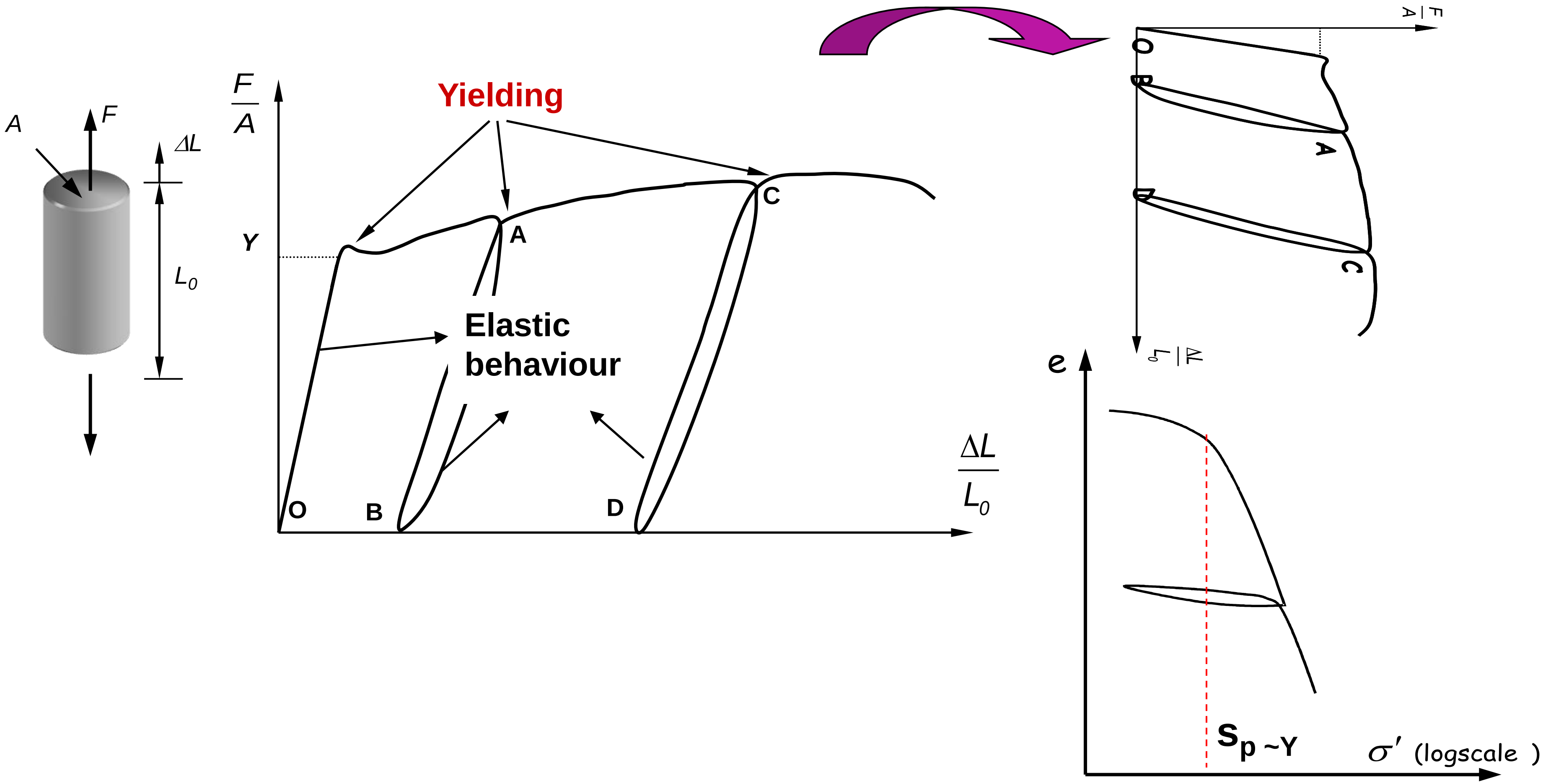
- assumes that after removing the stress the material does not necessarily returns back to the initial state (only the elastic part of strains is reversible)

- plastic behaviour (in simpler models) unrecoverable
    - in more advanced models, the plastic strains are connected to the model parameter called hardening parameter
        - in such case if we have hardening parameter decrease, the plastic strain will decrease too…

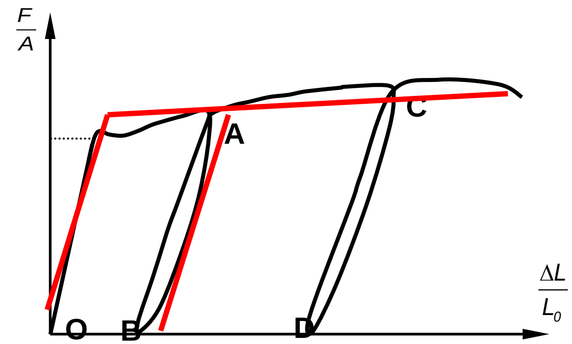
Basic Concepts of Perfect Plasticity

Plastic Behaviour

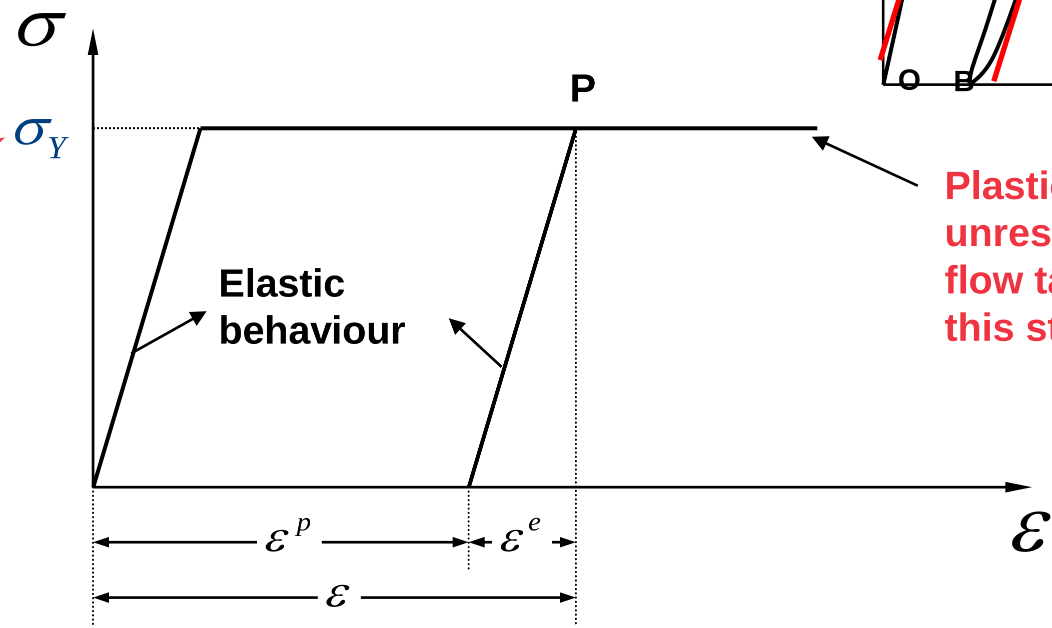
Example of elasto-plastic behaviour: traction test (1D) in metals



Plastic Behaviour



Idealization of elasto-plastic behaviour



Yield point, the stress cannot be higher than this value

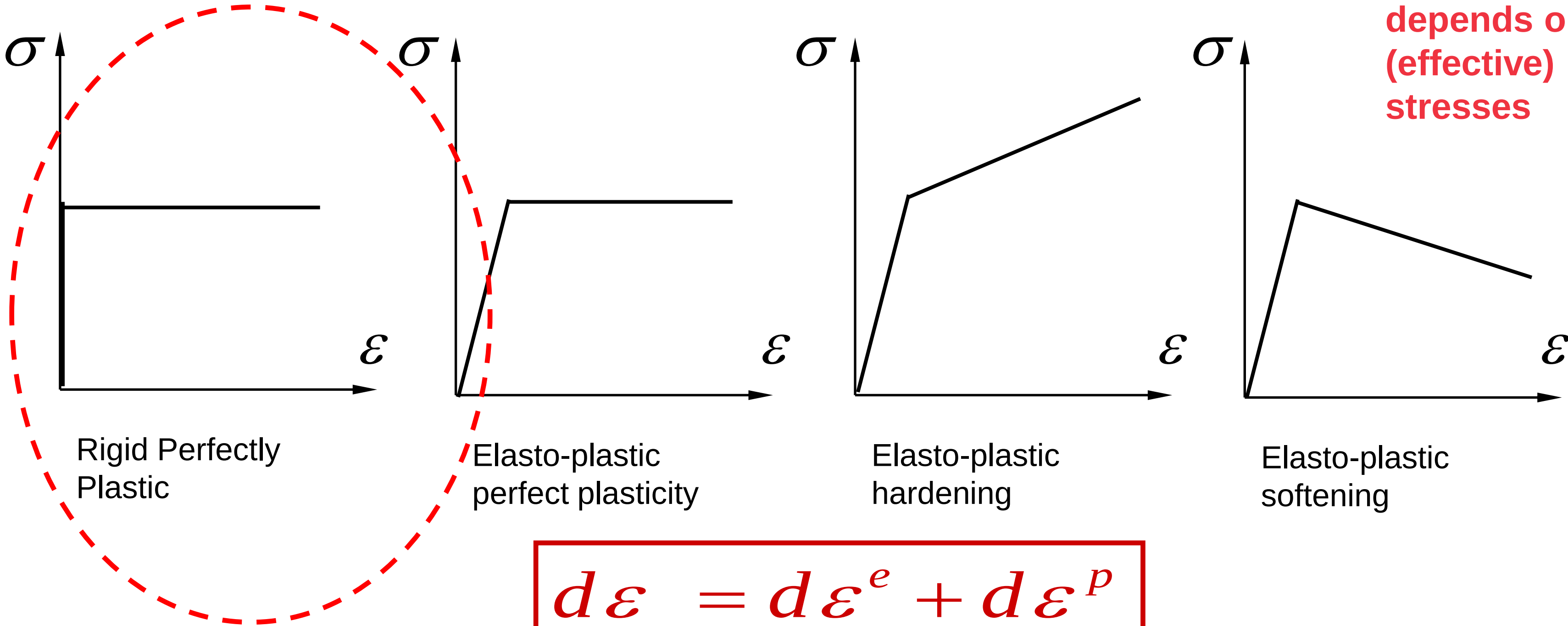
Plastic behaviour: unrestricted plastic flow takes place at this stress level.

$$\varepsilon = \varepsilon^{e} + \varepsilon^{p}$$

Plastic Behaviour

✓ **Idealization of elasto-plastic behaviour, different models**

Yield limit depends on (effective) stresses



$$\underset{\text{total}}{d\varepsilon} = \underset{\text{elastic}}{d\varepsilon^{e}} + \underset{\text{plastic}}{d\varepsilon^{p}}$$

Some Basic Concepts

Plastic models allow

- ✓ to determine in a direct way the ultimate states and failure
- ✓ to model irrecoverable strains
- ✓ to model changes in material behaviour
- ✓ to model a more proper way the behaviour of fragile or quasi-fragile materials

Some Basic Concepts

- Strains (ε)
  - Total strains
  - Elastic strains (recoverable on unloading)
  - Plastic strains (not recoverable on unloading)

Total strains = Elastic strains + Plastic strains

- Stresses (σ)

Total stresses = Effective stresses + Pore Pressures

Theory of Plasticity - Summary

1. Elastic Strain

$$\begin{bmatrix} d\varepsilon_p^e \\ d\varepsilon_q^e \end{bmatrix} = \begin{bmatrix} 1/K' & 0 \\ 0 & 1/3G' \end{bmatrix} \begin{bmatrix} dp' \\ dq' \end{bmatrix}$$

2. Yield surface

$$f(p', q, p_0') = 0$$

3. Plastic Potential

$$g(p', q, \zeta) = 0$$

4. Flow rule

$$d\varepsilon_p^p = d\lambda \frac{\partial g}{\partial p'}; \quad d\varepsilon_q^p = d\lambda \frac{\partial g}{\partial q}$$

Theory of Plasticity - Summary

5. Hardening law

$$p_0' = p_0'(\varepsilon_p^p, \varepsilon_q^p)$$

$$dp_0' = \frac{\partial p_0'}{\partial \varepsilon_p^p} d\varepsilon_p^p + \frac{\partial p_0'}{\partial \varepsilon_q^p} d\varepsilon_q^p$$

6. Plastic deformations

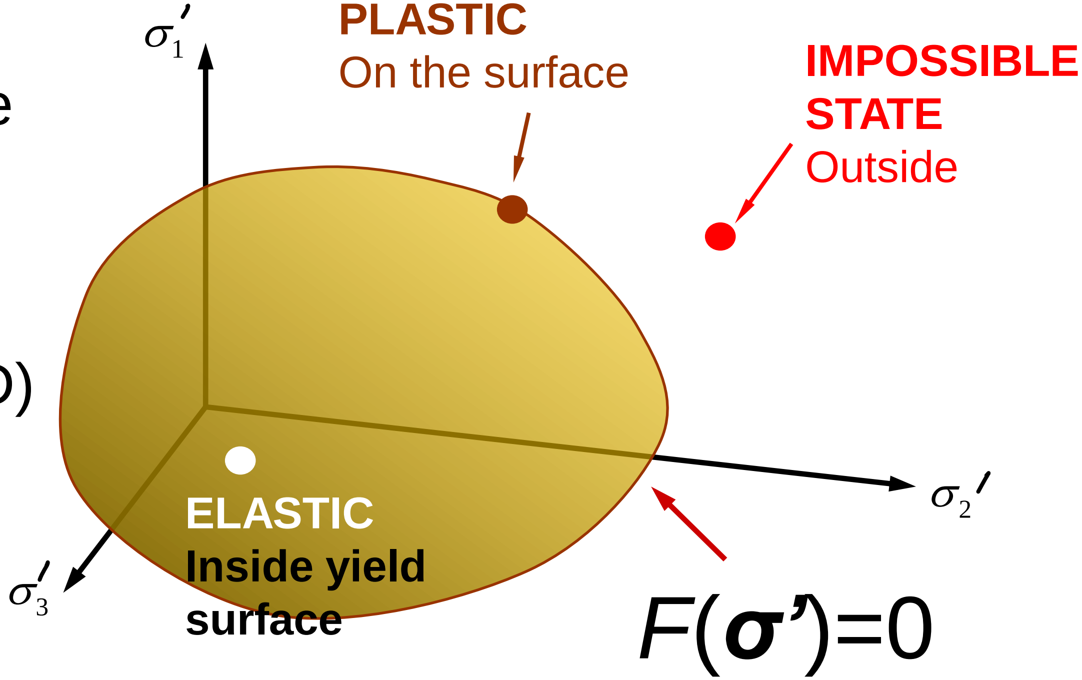
$$\begin{bmatrix} d\varepsilon_p^p \\ d\varepsilon_q^p \end{bmatrix} = \frac{-1}{\left[\frac{\partial f}{\partial p_0'} \left(\frac{\partial p_0'}{\partial \varepsilon_p^p} \frac{\partial g}{\partial p'} + \frac{\partial p_0'}{\partial \varepsilon_q^p} \frac{\partial g}{\partial q} \right) \right]} \begin{bmatrix} \frac{\partial f}{\partial p'} \frac{\partial g}{\partial p'} & \frac{\partial f}{\partial q} \frac{\partial g}{\partial p'} \\ \frac{\partial f}{\partial p'} \frac{\partial g}{\partial q} & \frac{\partial f}{\partial q} \frac{\partial g}{\partial q} \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

7. Total deformations

$$\begin{bmatrix} d\varepsilon_p \\ d\varepsilon_q \end{bmatrix} = \begin{bmatrix} d\varepsilon_p^e \\ d\varepsilon_q^e \end{bmatrix} + \begin{bmatrix} d\varepsilon_p^p \\ d\varepsilon_q^p \end{bmatrix}$$

Yield Surface F (or f)

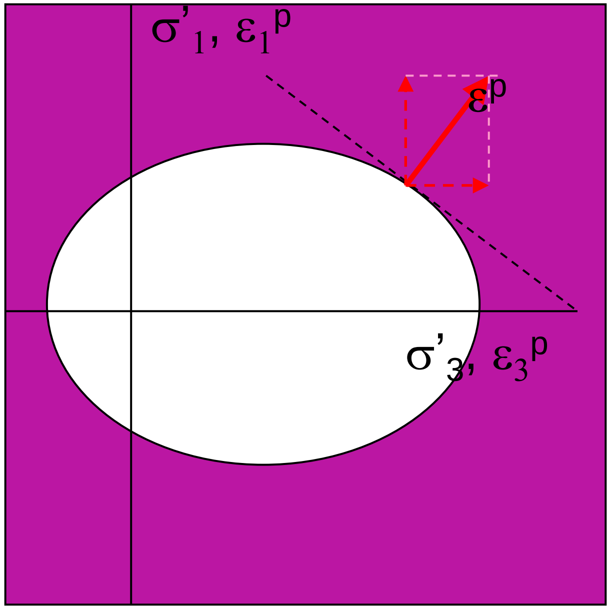
- Delimits the elastic domain
- Generalization of the 1D case
- Yield limit (1-D) ⇢ Yield surface (2D-3D)



Flow Rule

- **In one-dimensional problem, it is clear that plastic strains take place along the direction of applied stress**

- **In 2D or 3D we need to make a hypothesis regarding the direction of plastic flow (relative magnitude of plastic strain increments)**



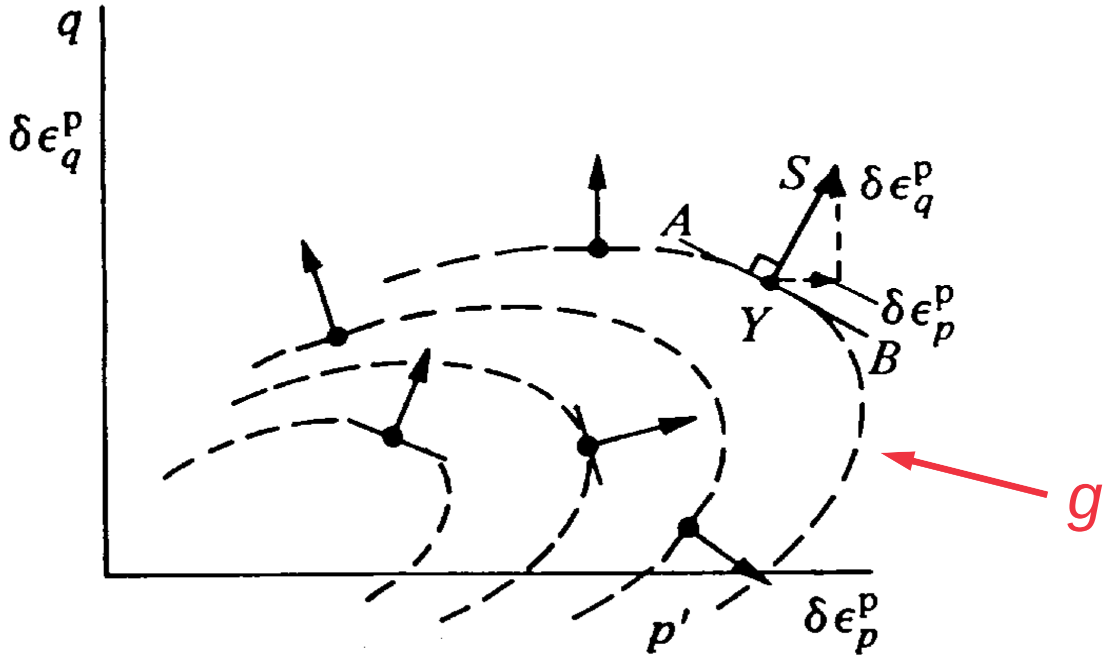
Plastic Potential and Flow Rule

Plastic Deformations – direction governed by the plastic potential surface g (or Q)

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial g}{\partial \sigma_{ij}}$$

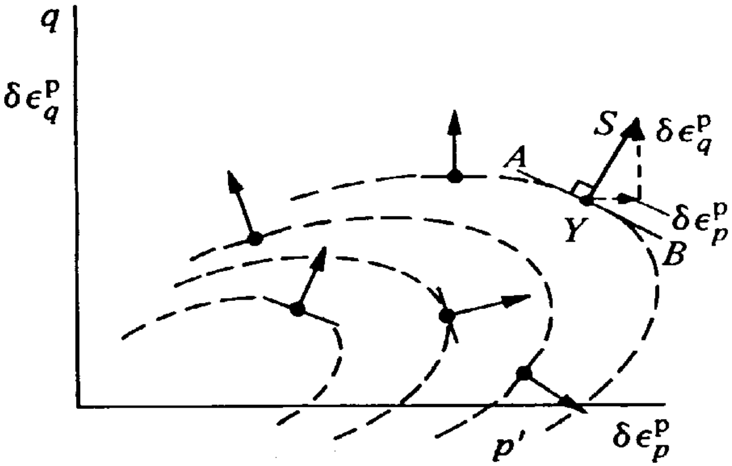
control the **magnitude** of plastic deformation

control the “**direction**” of the plastic deformations: the “vector” of the plastic deformations is normal to the *g* = constant surfaces



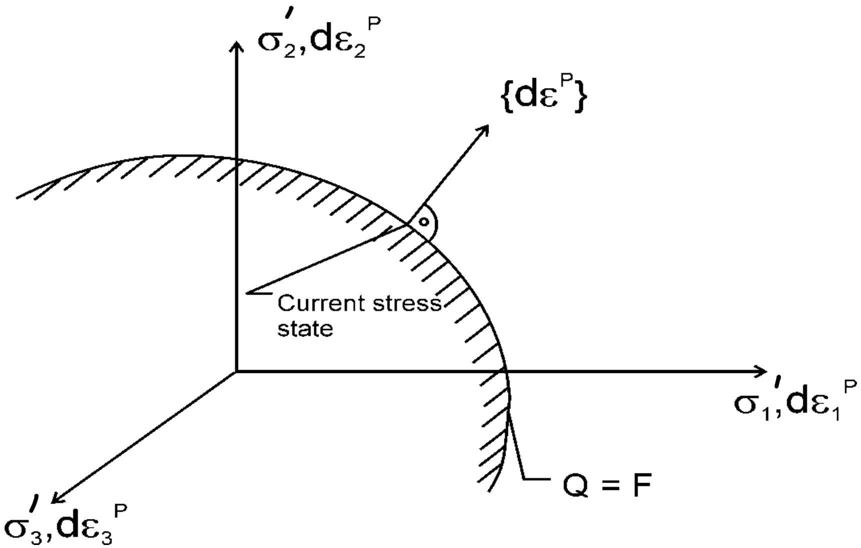
Plastic Potential and Flow Rule

- Yield Surface (f) and Plastic Potential (g) are generally different functions
- If f ≡ *g* => **associated** plasticity
- The components of the plastic deformations are related, i.e. there is a coupling, which is defined by the flow rule
- The plastic deformations depend on the stress state rather than the increment of the stresses applied

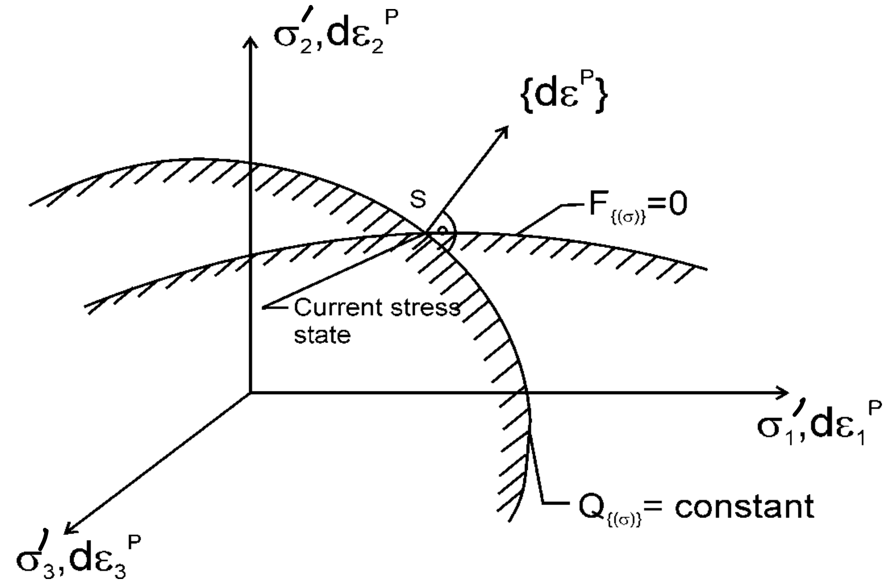


Plastic Potential and Flow Rule

FLOW RULE ASSOCIATED



FLOW RULE – NON ASSOCIATED



- The flow rule defines direction of plastic strain increment
- So, we know the plastic-strain direction, but how we can determine the magnitude?

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7. Total deformations

$$\begin{bmatrix} d\varepsilon_p \\ d\varepsilon_q \end{bmatrix} = \begin{bmatrix} d\varepsilon_p^e \\ d\varepsilon_q^e \end{bmatrix} + \begin{bmatrix} d\varepsilon_p^p \\ d\varepsilon_q^p \end{bmatrix}$$

Thank you

Literature

Hard to find a single book…

Check:

Briaud JL. Geotechnical Engineering : Unsaturated and Saturated Soils. (linked in the materials section, good introduction)

Potts & Zdravkovic, Finite element analysis in geotechnical engineering. Theory. Thomas Telford, 1999.

Muir Wood D., Soil Behaviour and Critical State Soil Mechanics. Cambridge University Press. 1990.

Books on theory of plasticity can be helpful