Aalto University
School of Engineering

GEO – E1050
Finite Element Method in Geoengineering
Mohr-Coulomb and other constitutive models

Wojciech Sołowski

To learn today…

1. Reminder – elasticity

2. Elastic – perfectly plastic theory: concept

3. Specific elastic-perfectly plastic models:
    - Mohr – Coulomb (important)
    - Hoek – Brown (important for rock mechanics)
    - Tresca
    - von Mises

More derivations, Boundary Element Method, Finite Difference Method moved to next lecture: we need models for exercises

Philosophy

Geomaterials behaviour – COMPLEX

Constitutive models – **huge simplifications**

usually the simplification designer is making is mainly about choice of model and its parameters

Model and choice of its parameters must be clearly selected for given problem. No constitutive model for geomaterials is good for solving all problems.

(these which are trying to be good for everything, like the MIT constitutive model for geomaterials are extremely complex and difficult to use, validate and calibrate)

Need to balance simplicity, accuracy and problem needs

Philosophy

Challenge:

- understand the simplifications

- use them to your benefit

- get accurate results from crude tools

- no results is ‘right’ - each result is because of the assumptions made

- if you do good job with modelling, you will get result which is close to reality…

also because you have chosen the right model…

yet primarily because **you have chosen the right set of parameters** for the model and the problem

Elasticity

This lecture covers the very basics of constitutive modelling…

You know elasticity, we will build on that…

$$\left\{\overbrace{\begin{bmatrix}\Delta\sigma_{11}\\ \Delta\sigma_{22}\\ \Delta\sigma_{33}\\ \Delta\sigma_{12}\end{bmatrix}}^{\Delta\boldsymbol{\sigma}} = \overbrace{\frac{E}{(1+v)(1-2v)}\begin{bmatrix}1-v & v & 0\\ v & 1-v & 0\\ v & v & 0\\ 0 & 0 & 1-2v\end{bmatrix}}^{\mathbf{D}}\overbrace{\begin{bmatrix}\Delta\varepsilon_{11}\\ \Delta\varepsilon_{22}\\ \Delta\varepsilon_{12}\end{bmatrix}}^{\Delta\boldsymbol{\varepsilon}}\right.$$

More on modelling in Advanced Soil Mechanics…
Even more in the Numerical Methods in Geotechnics…

Outline

1. Constitutive modelling: Elasticity
   a. Linear elasticity
   b. Non-linear elasticity
   c. Hypoelasticity

2. Constitutive modelling: Elasto-plasticity
   a. What is elasto-plasticity, basic concepts and theory
   b. Mohr-Coulomb
   c. Other elastic-perfectly plastic constitutive models

More on modelling in Advanced Soil Mechanics…
Even more in the Numerical Methods in Geotechnics…

Elasticity

Idea:

- one of the oldest approaches to describe material behaviour

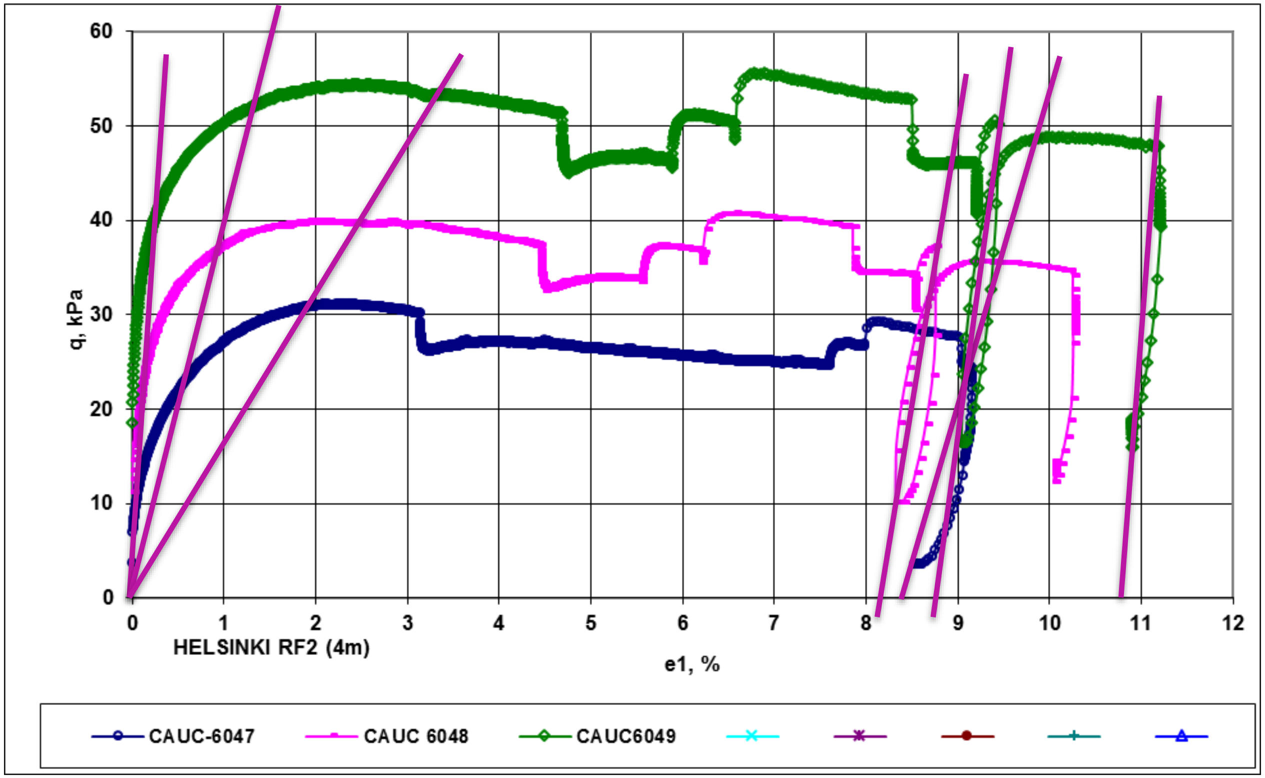
- assumes that after removing the stress the material returns back to the initial state

  - behaviour is reversible

    1) no matter the level of stress, the material will not break
    2) no matter the level of stress, after removing it, the material will be back in the original state before the loading, no change…

Elasticity

Linear elasticity – useful?



Real soil behaviour

Elasticity

Can be useful… as long as you know the stress level, deformations and have data on how the soil behaves at that stress level and deformations level…

- than you just choose the exactly right elastic parameter for given circumstances… and calculate the strains from the stresses – or stresses from strains…

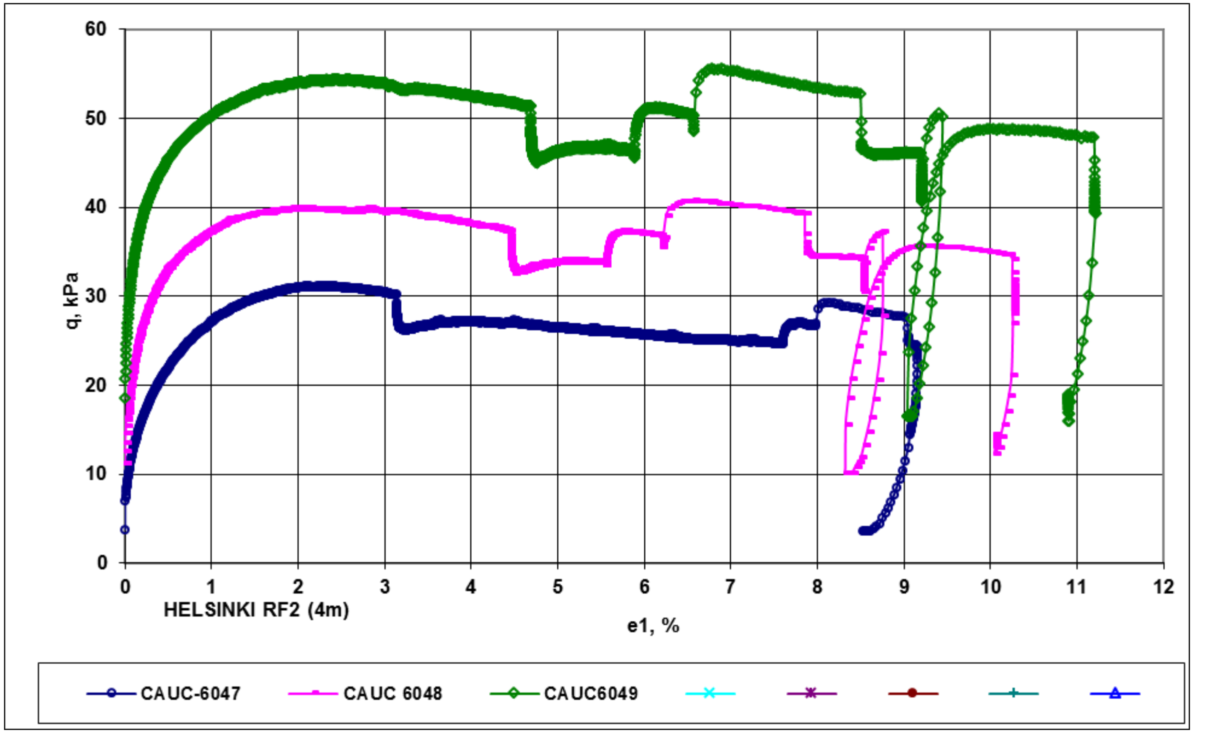
- used for example in calculations of settlements – we know the stress levels from some simple solutions – than we compute strains using right set of elastic parameters for given small layer of soil… Can be pretty accurate…

Elasticity

$$
\overbrace{\begin{bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \\ \Delta\sigma_{13} \\ \Delta\sigma_{23} \end{bmatrix}}^{\Delta\boldsymbol{\sigma}} = \overbrace{\frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ & & & 1-2v & 0 & 0 \\ & & & & 1-2v & 0 \\ . & & & & & 1-2v \end{bmatrix}}^{\mathbf{D}^{el}} \overbrace{\begin{bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{33} \\ \Delta\varepsilon_{12} \\ \Delta\varepsilon_{13} \\ \Delta\varepsilon_{23} \end{bmatrix}}^{\Delta\boldsymbol{\varepsilon}}
$$

$$
\Delta\boldsymbol{\sigma} = \mathbf{D}^{el} \Delta\boldsymbol{\varepsilon}
$$

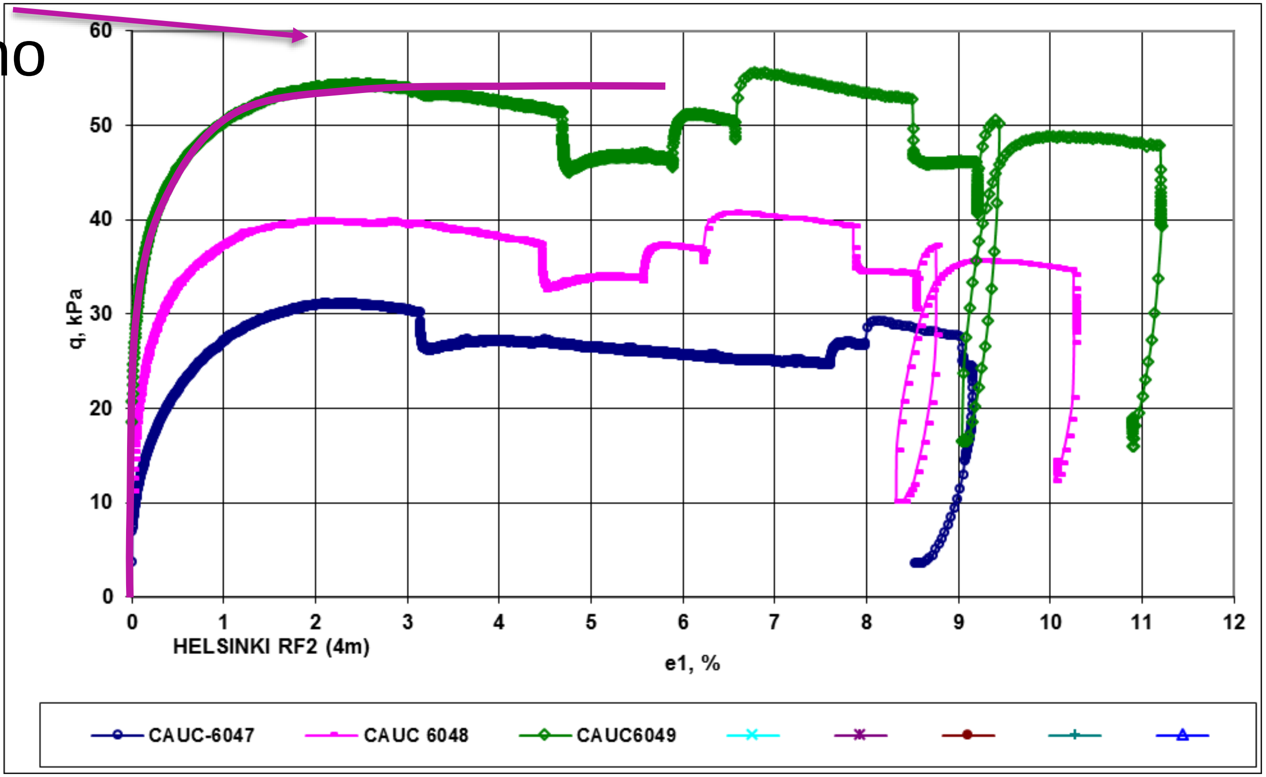
Non-linear elasticity



Real soil behaviour…

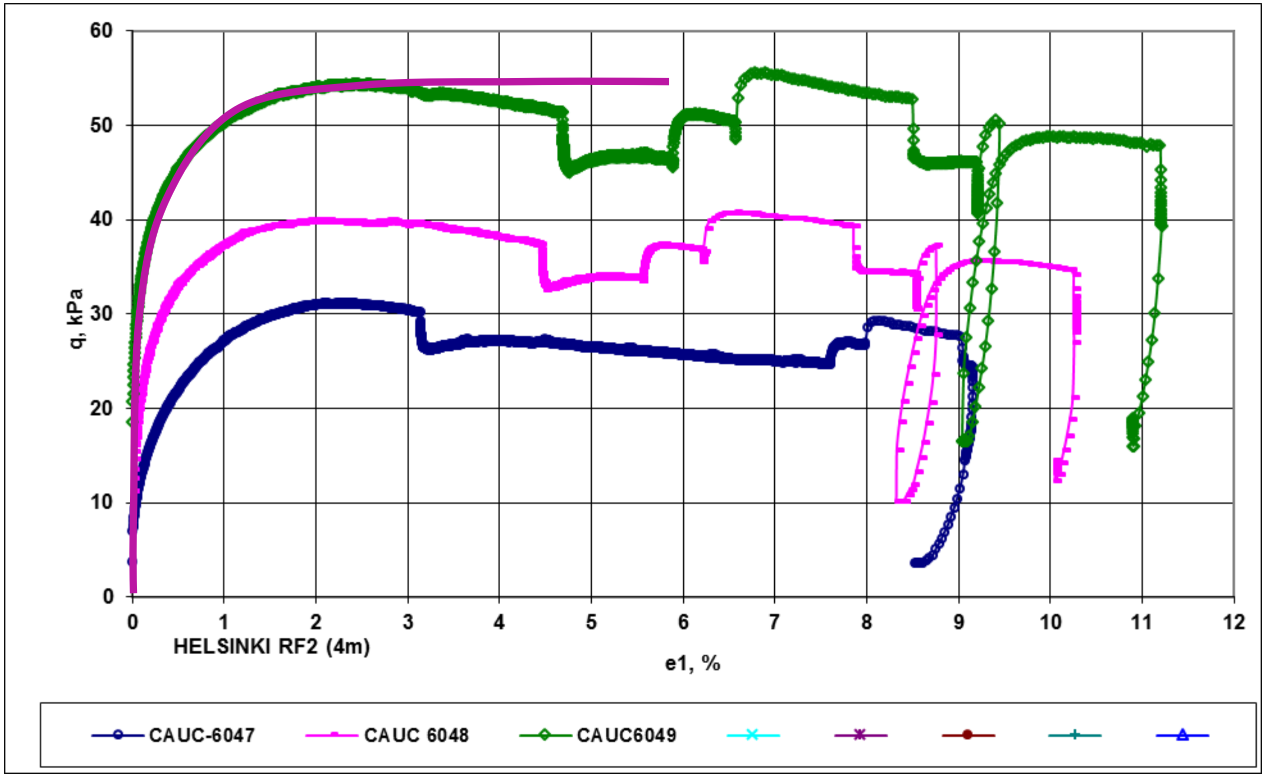
Non-linear elasticity

We can only match single line – as no dependency on stress level



Real soil behaviour…

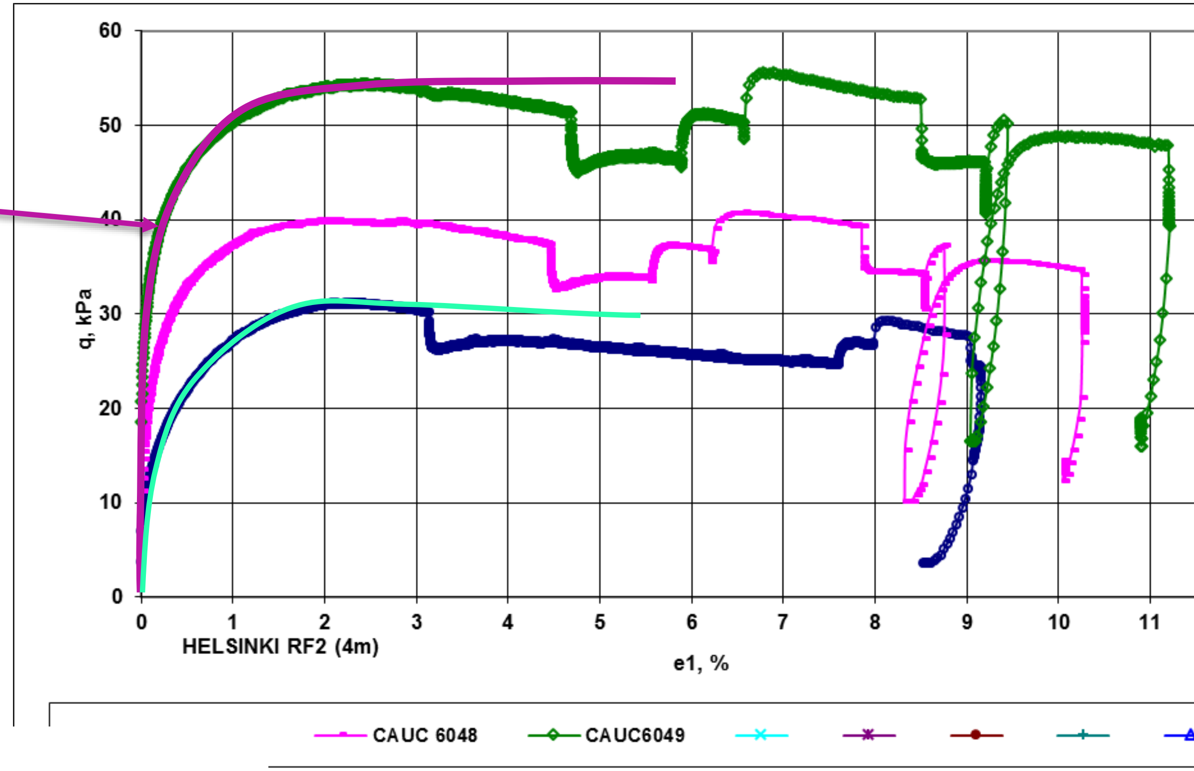
Non-linear elasticity



$$\Delta\boldsymbol{\sigma} = \mathbf{D}^{el}(\boldsymbol{\sigma}_0)\Delta\boldsymbol{\varepsilon}$$

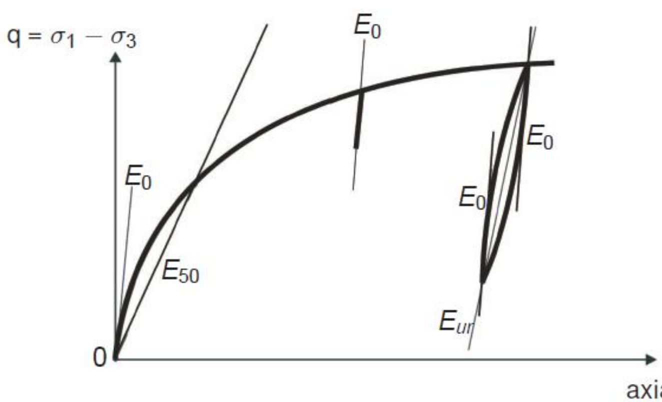
Hypoelasticity

Introduces dependence on stress level etc; adds complexity but does not solve all problems; still there are some successful models based on these concepts



Real soil behaviour…

Non-linear elasticity:



Stiffness parameters E_{50}, E_{ur}, and $E_0 = 2G_0(1 + \nu_{ur})$ of the Hardening Soil model with small-strain stiffness in a triaxial test

Elasto-plasticity

Idea:

- let's divide strains into these which are elastic (which are fully recoverable) and plastic (which are generally not)

- assumes that after removing the stress the material does not necessarily returns back to the initial state (only the elastic part of strains is reversible)

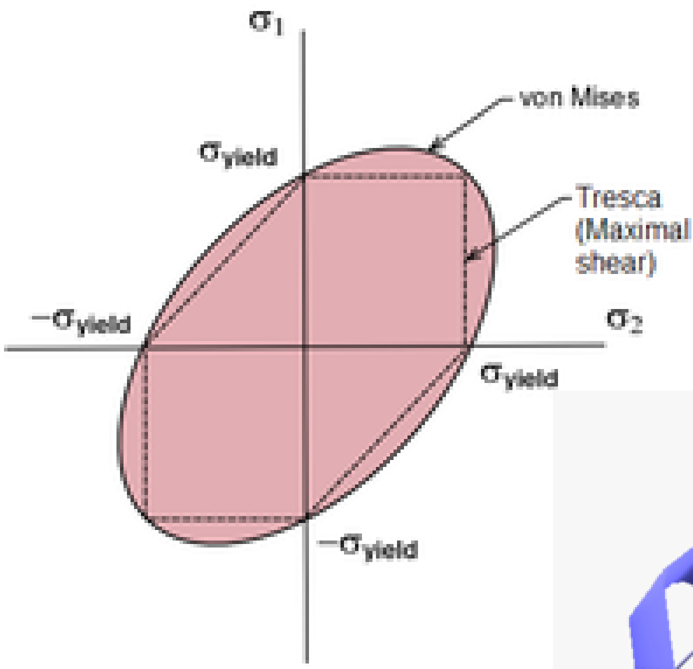
- plastic behaviour (in simpler models) unrecoverable
  - in more advanced models, the plastic strains are connected to the model parameter called hardening parameter
    - in such case if we have hardening parameter decrease, the plastic strain will decrease too…

Basic Concepts of Perfect Plasticity

Elasto-plasticity: Tresca

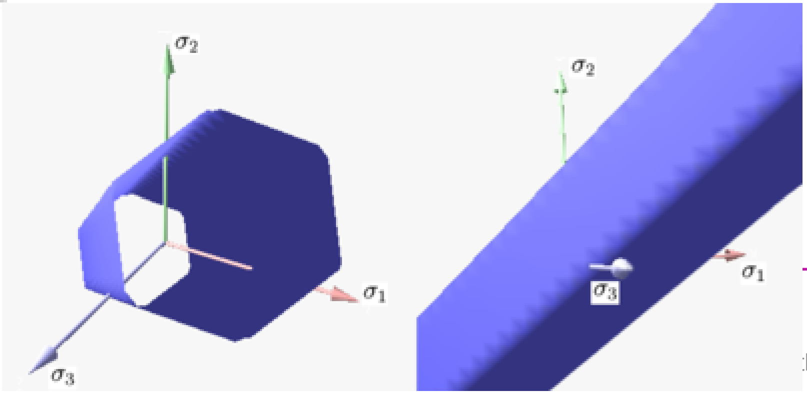
Simplest:

- perfect plasticity (concept dated back to Roman times) – yield criterion: anything in compression is safe…
- currently used for soils – Tresca model (elastic – perfectly plastic)



Size of the yield surface does not change – is independent from the mean stress level:

$$\sigma_{tresca} = \sigma_1 - \sigma_3 > \sigma_{max}$$



Elasto-plasticity: Tresca

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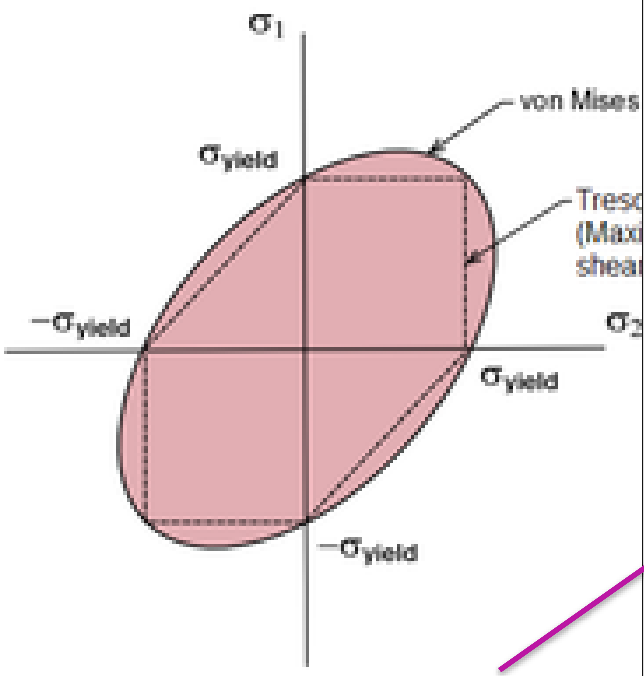
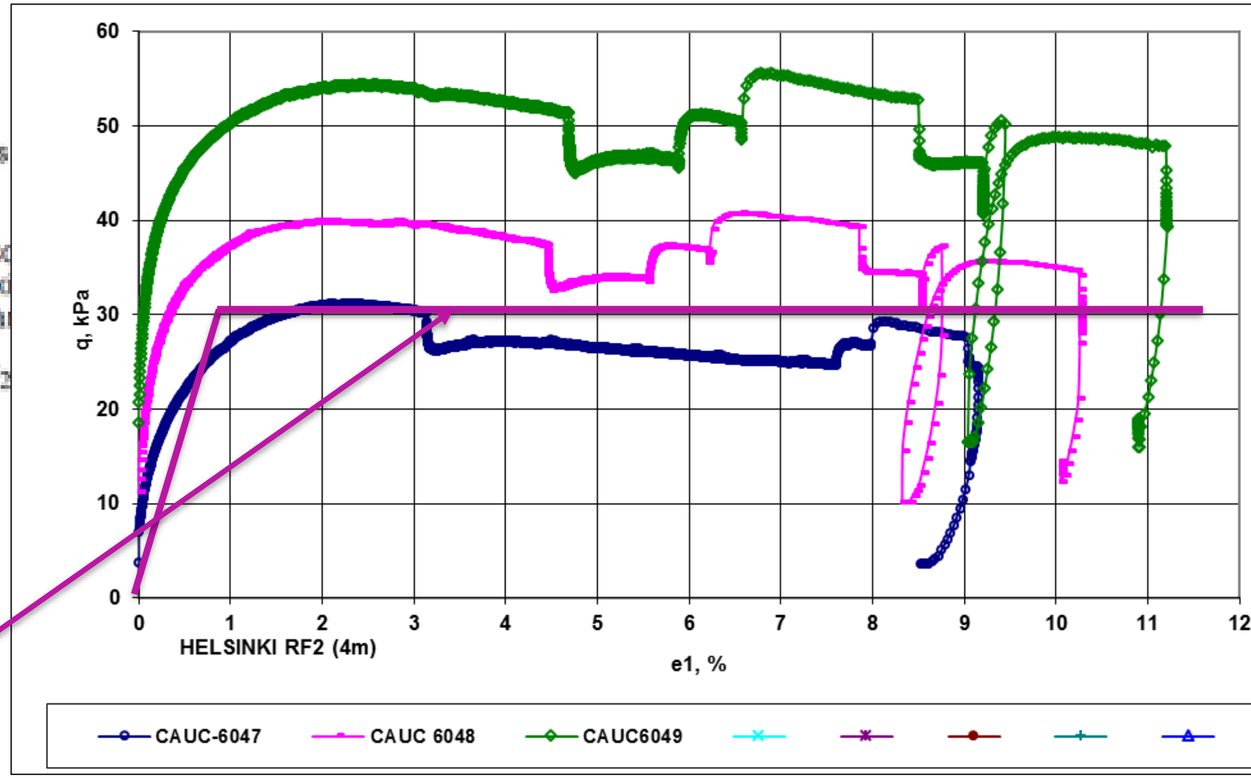




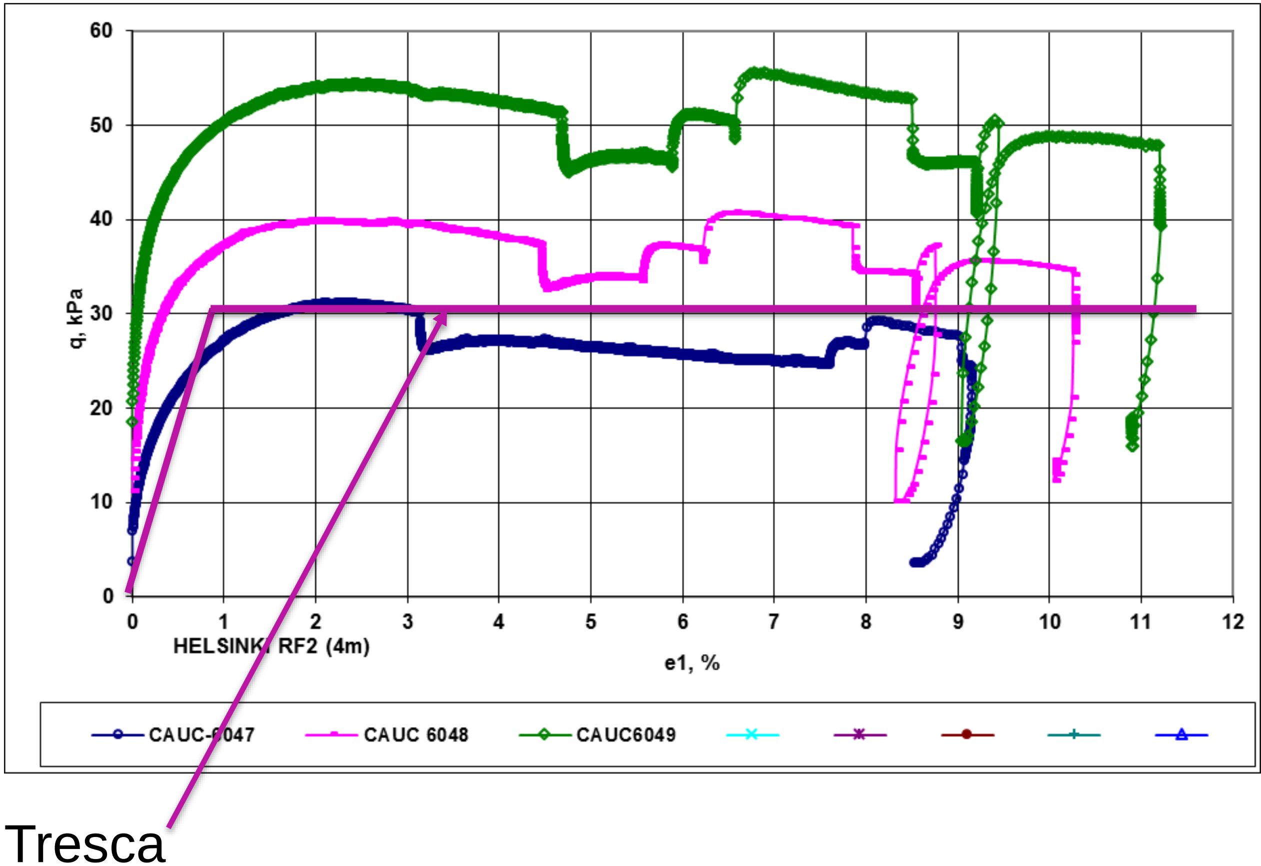
Tresca

Useful?

Elasto-plasticity: Tresca

Useful when we assume that soil has undrained shear strength s_u which is independent from the stress state…

Used (when strain rate dependency added e.g. for simulation of pile installation)



Tresca

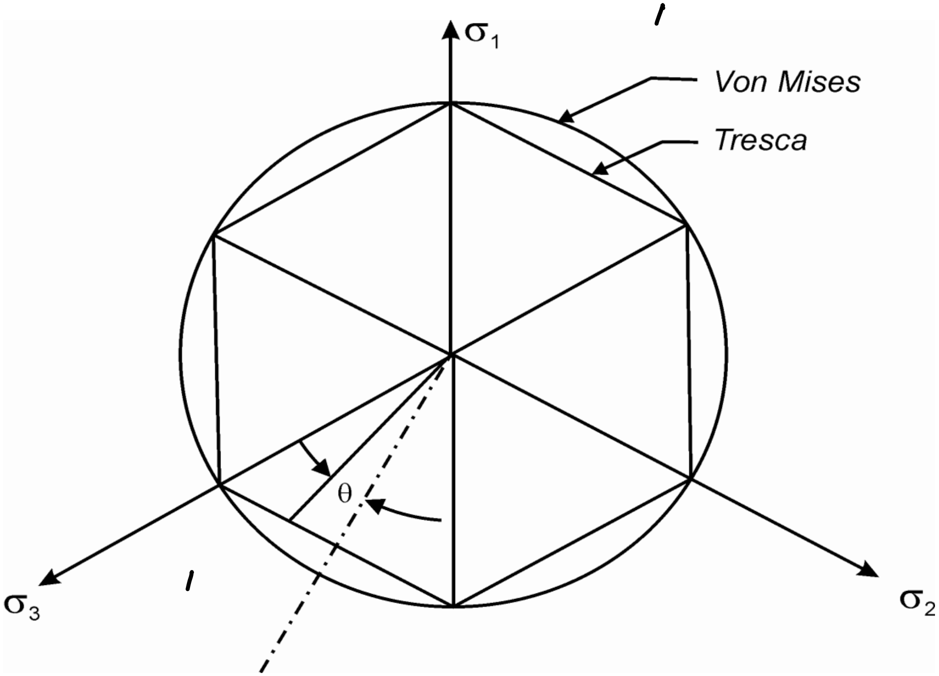
$$p' = \tfrac{1}{3}\left(\sigma_1' + \sigma_2' + \sigma_3'\right)$$

$$J = \frac{1}{\sqrt{6}}\sqrt{\left(\sigma_1' - \sigma_2'\right)^2 + \left(\sigma_2' - \sigma_3'\right)^2 + \left(\sigma_3' - \sigma_1'\right)^2}$$

$$\theta = \tan^{-1}\left[\frac{1}{\sqrt{3}}\left(2\frac{(\sigma_2' - \sigma_3')}{(\sigma_1' - \sigma_3')} - 1\right)\right]$$

$$F = \sigma_1' - \sigma_3' - 2c_u = 0$$

$$\begin{Bmatrix}\sigma_1' \\ \sigma_2' \\ \sigma_3'\end{Bmatrix} = p'\begin{Bmatrix}1 \\ 1 \\ 1\end{Bmatrix} + \frac{2}{\sqrt{3}}J\begin{Bmatrix}\sin\left(\theta + \frac{2\pi}{3}\right) \\ \sin\theta \\ \sin\left(\theta - \frac{2\pi}{3}\right)\end{Bmatrix}$$



in invariant formulation

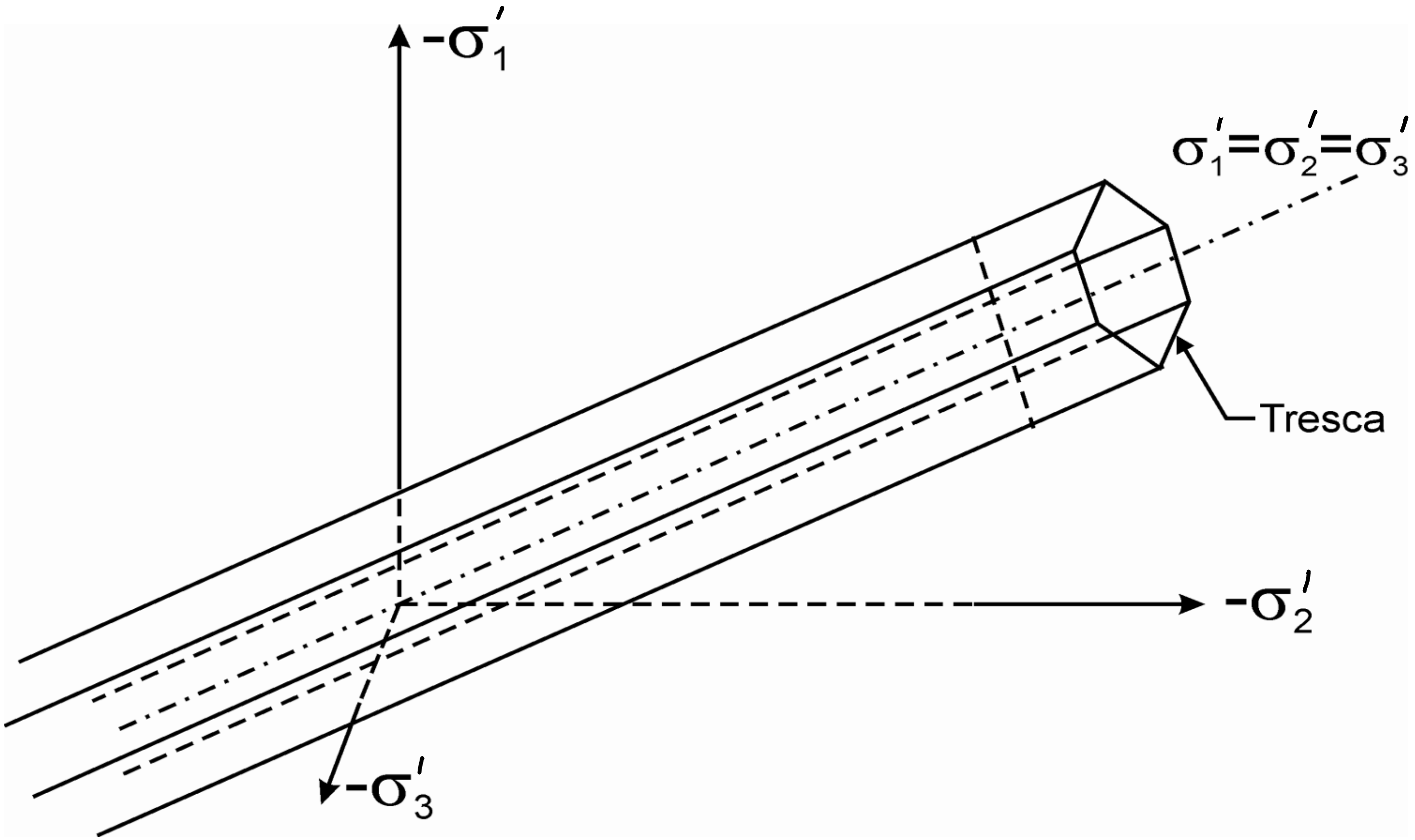
$$F = \bar{\sigma}\cdot\cos\theta - c_u = 0$$

> dependent on Lode angle

Tresca in 3D

$$F = \bar{\sigma} \cdot \cos\theta - c_u = 0$$

> dependent on Lode angle



Tresca – Applications in geotechnics

Total stress analysis for undrained behaviour (φ = 0)
c_u ….. undrained shear strength

No volume change > ν = 0.5 (for numerical reasons 0.49)

Plastic volumetric strains ?
assume Lode angle = 0

$$F = 2 \cdot \overline{\sigma} - 2 \cdot c_u = 0$$

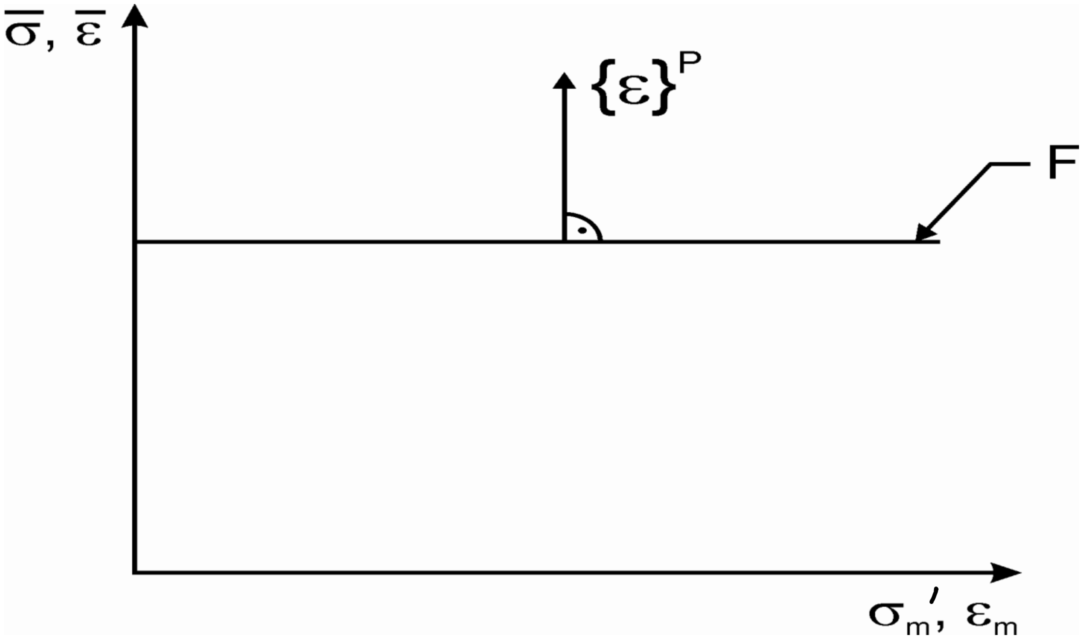
Assume *associated* flow rule

$$d\varepsilon_M^P = d\lambda \left\{ \frac{\partial F}{\partial \sigma_M} \right\} = 0$$

$$d\overline{\varepsilon}^P = d\lambda \left\{ \frac{\partial F}{\partial \overline{\sigma}} \right\} = 2 d\lambda$$

$$d\varepsilon_{vol}^P = d\varepsilon_M^P = 0$$

Tresca – Applications in geotechnics

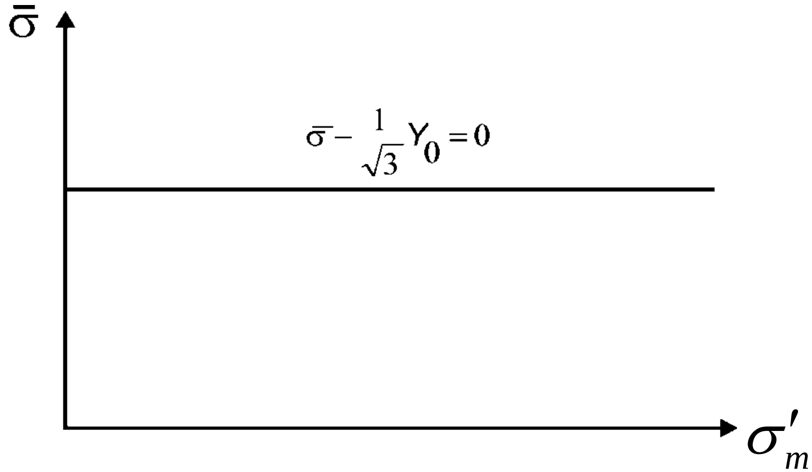


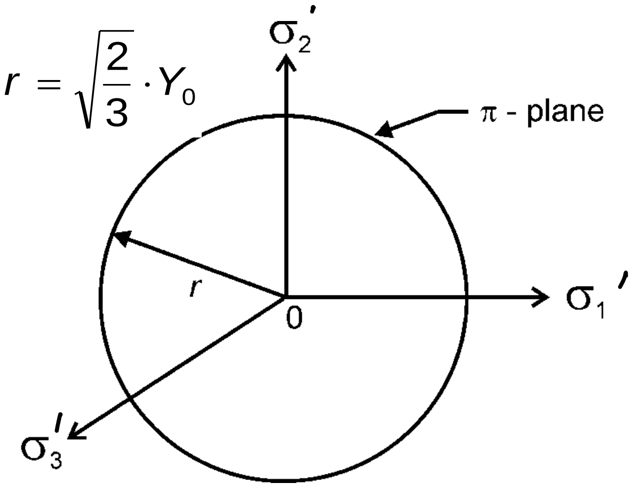
$$d\varepsilon_{vol}^{P} = d\varepsilon_{M}^{P} = 0$$

Von Mises

$$F = \sqrt{3} \cdot \bar{\sigma} - 2c_u = 0$$ **> not dependent on Lode angle**

Y_0 ….. yield stress for axial compression or tension





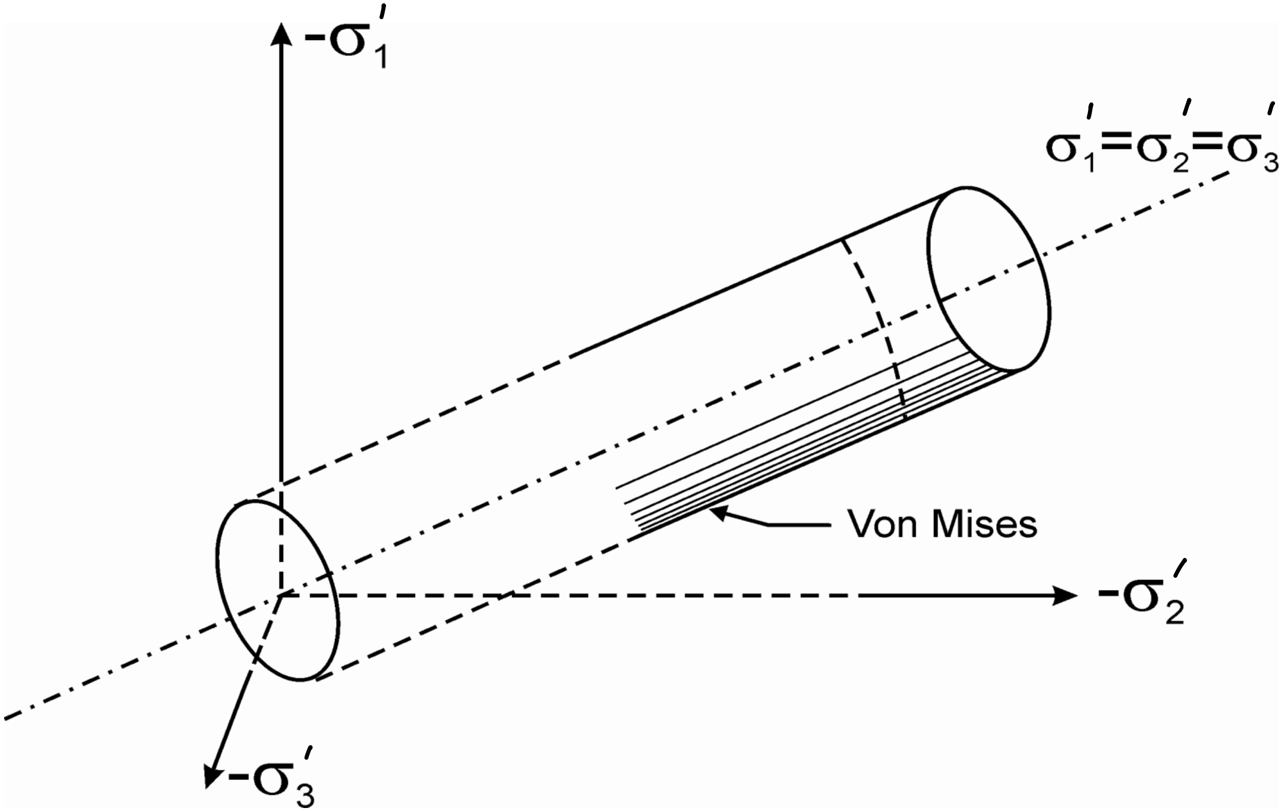
Von Mises criterion in $\bar{\sigma} - \sigma'_m -$ space

Von Mises criterion in π - plane

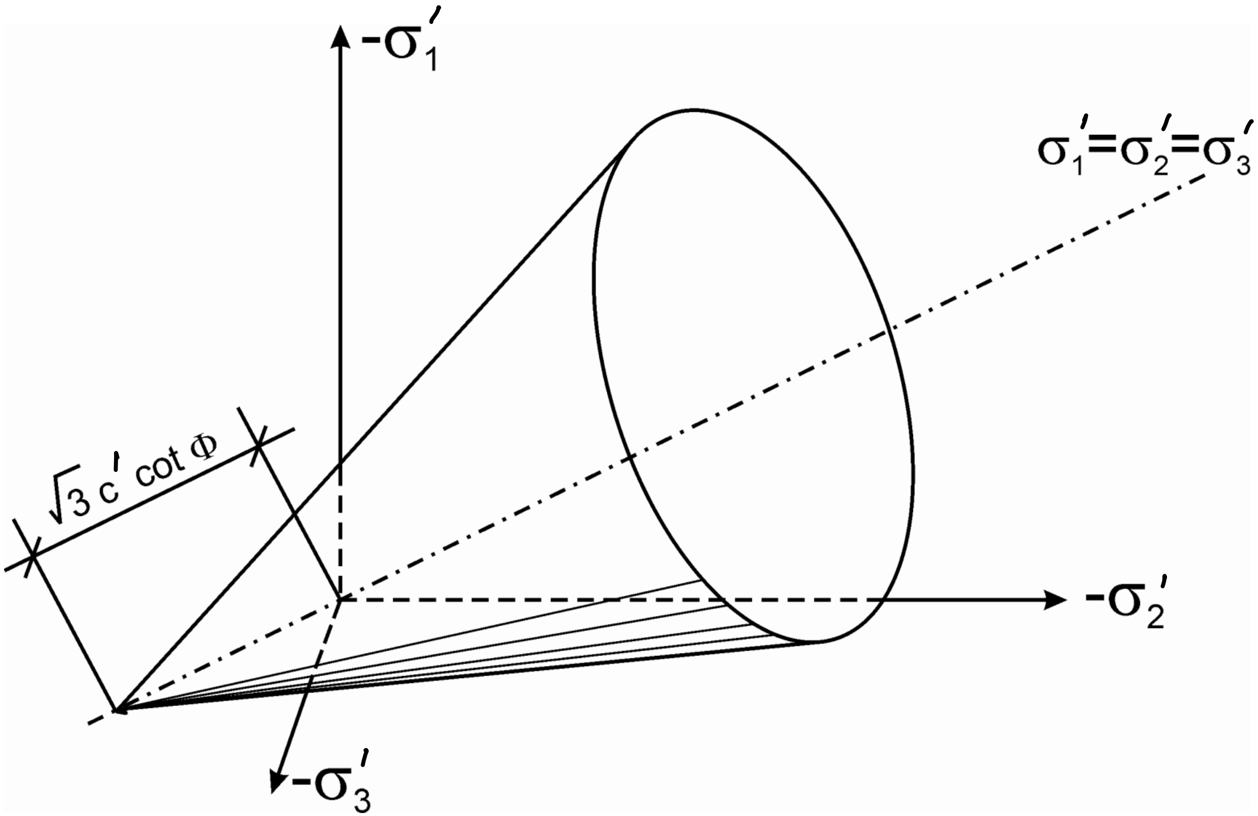
Von Mises in 3D

$$F = \sqrt{3} \cdot \bar{\sigma} - 2c_u = 0$$ **> not dependent on Lode angle**

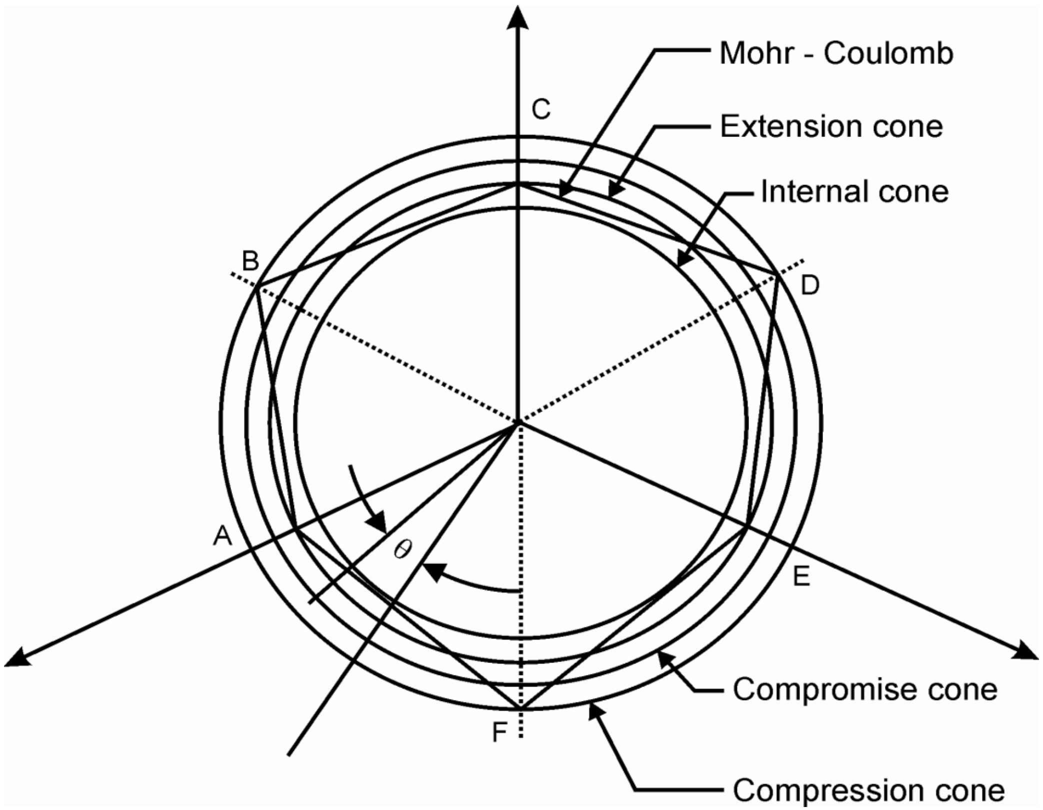
Y_0 ….. yield stress for axial compression or tension



Drucker-Prager



Drucker-Prager vs MC

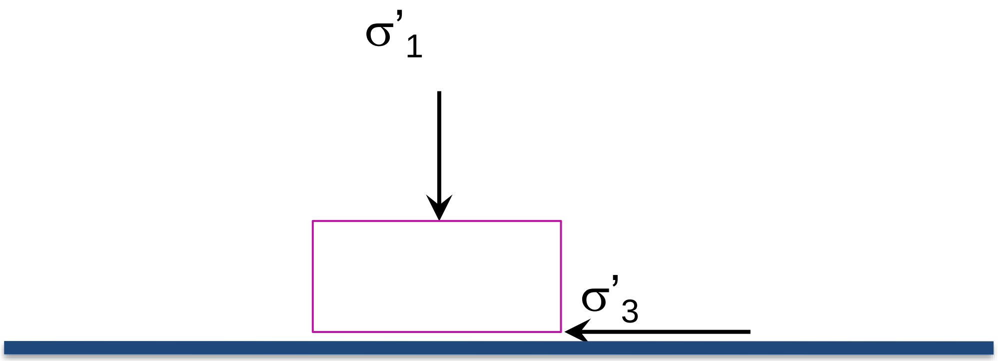


Drucker-Prager and Mohr-Coulomb criteria in π - plane

Mohr Coulomb Model

Mohr-Coulomb Idealisation of Geomaterials

Comes from the Coulomb friction:



$$\sigma'_3 = \sigma'_1 * \tan \phi$$

Cohesion: adds extra strength…

$$\tau = \sigma'_1 * \tan \phi + c$$

Mohr-Coulomb Idealisation of Geomaterials

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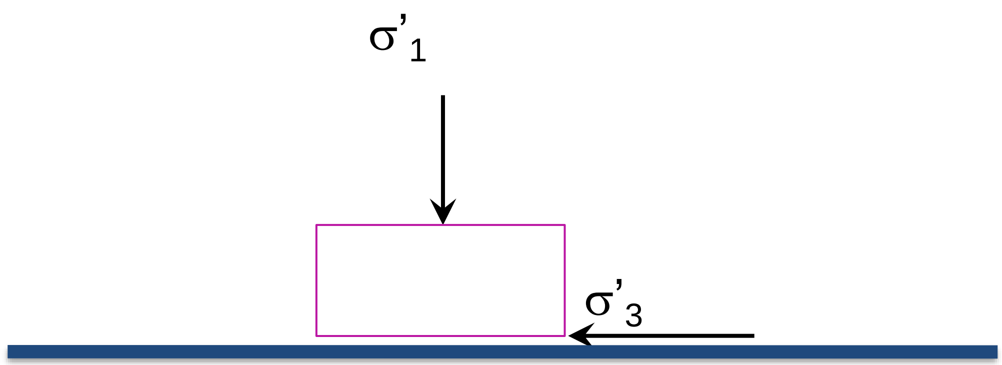

$$\tau = \sigma'_1 * \tan \phi + c$$

When the block moves, strains increase, but not the stress

Mohr-Coulomb Idealisation of Geomaterials

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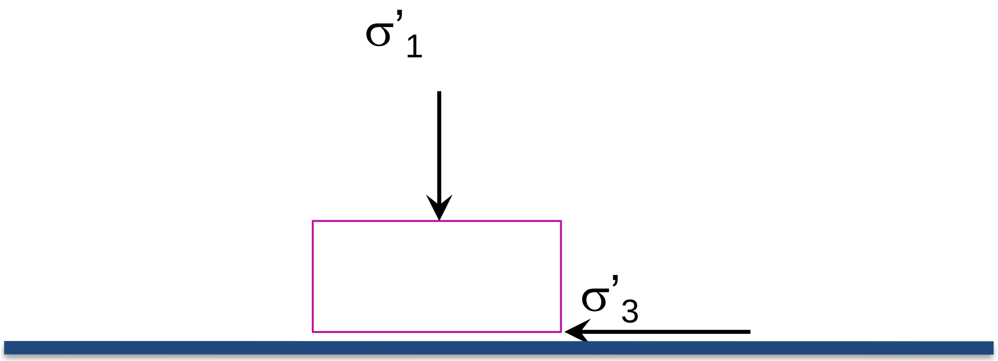

$$\tau = \sigma'_1 * \tan \phi + c$$

Before the block moves, strains are zero, but the stress increases (rigid - perfectly plastic – still useful)

Mohr-Coulomb Idealisation of Geomaterials

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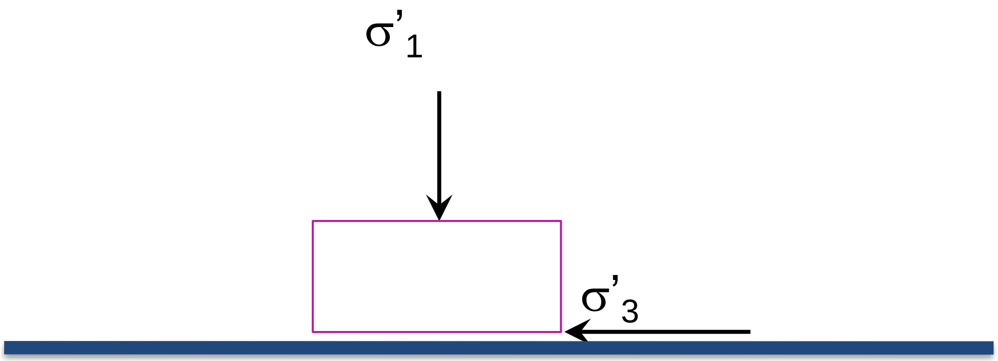

$$\tau = \sigma'_1 * \tan \phi + c$$

Elasto – plasticity = some elastic strain (and corresponding stress), some plastic strain (corresponding to the block moving, but no stress change – in reality it is slippage of grains, crystals, etc.)

Mohr-Coulomb Idealisation of Geomaterials

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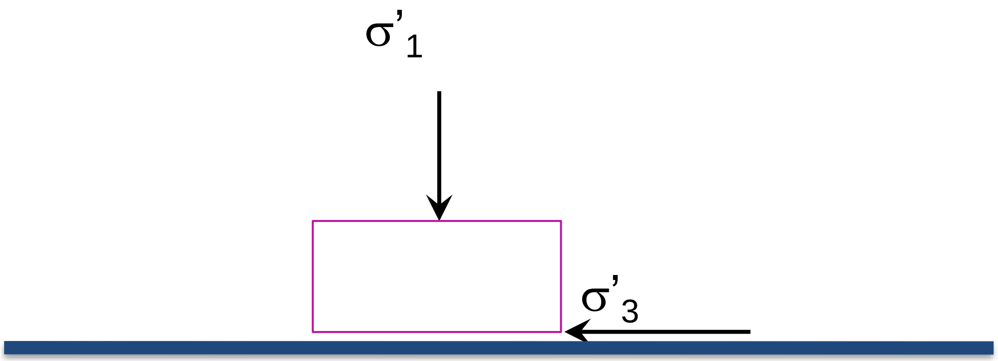

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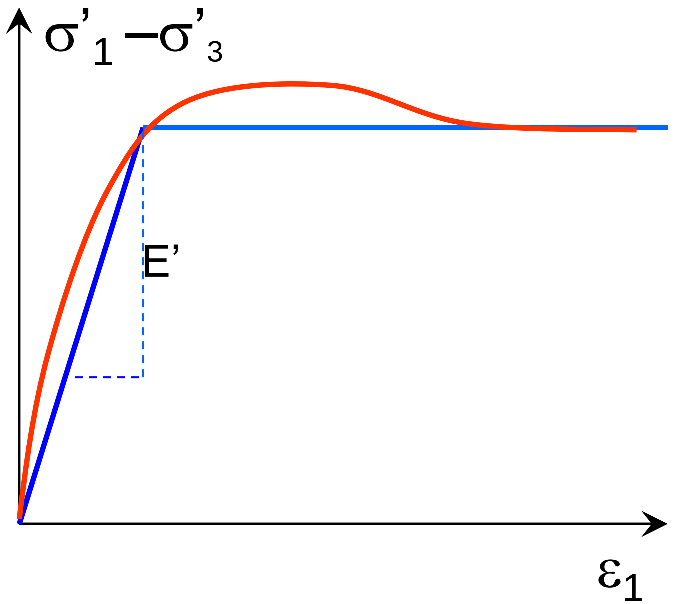

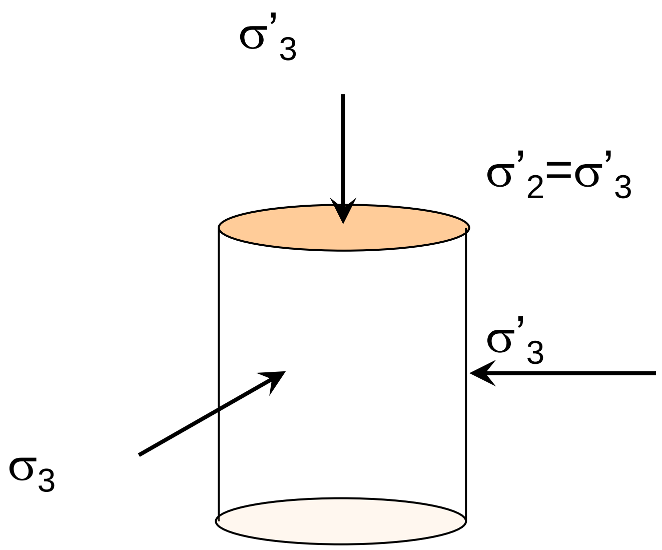

Mohr-Coulomb Idealisation of Geomaterials

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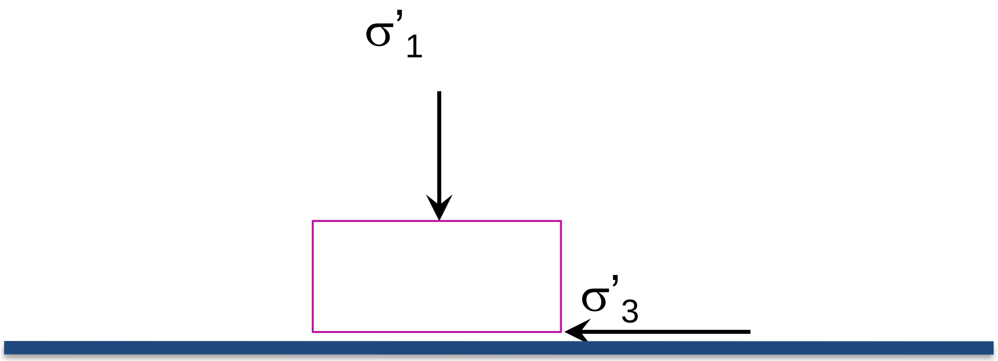


$$\sigma'_3 = \sigma'_1 * \tan \phi$$

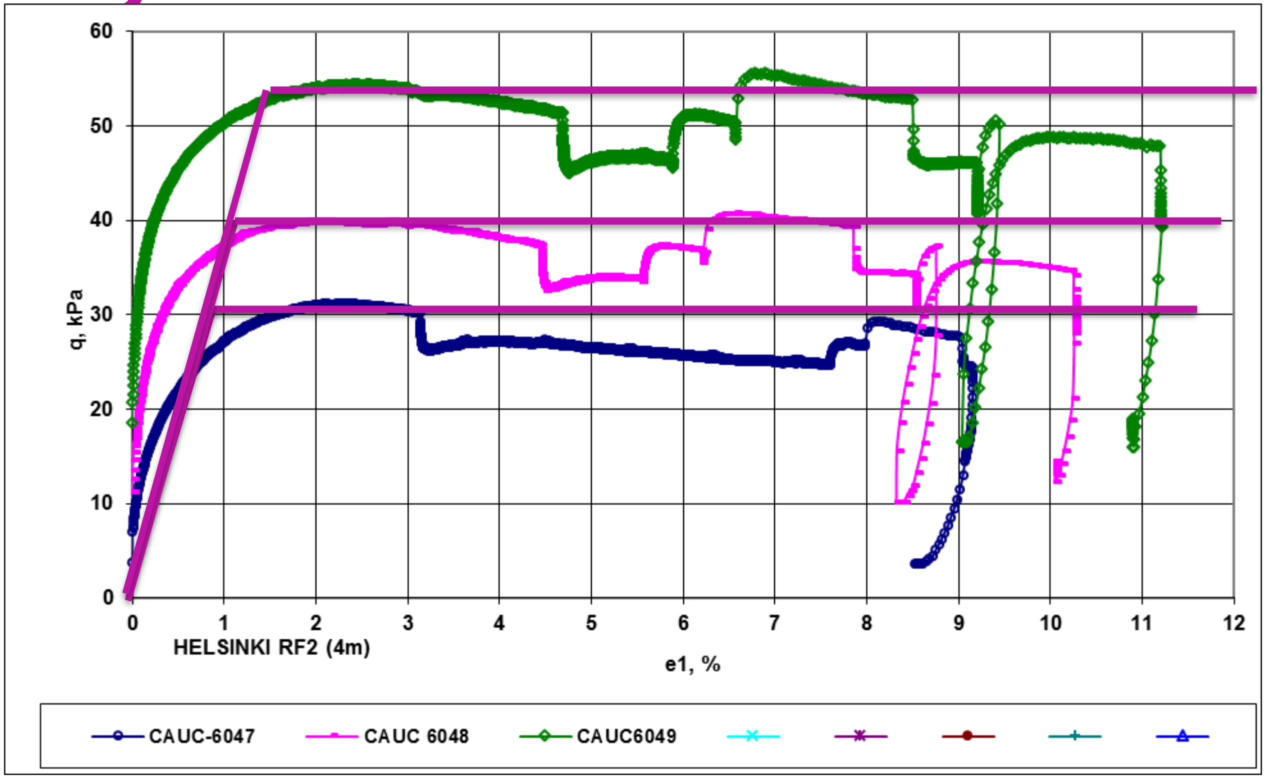
Cohesion: adds extra strength…

$$\tau = \sigma'_1 * \tan \phi + c$$

Elasto-plasticity: Mohr-Coulomb

Reasonably good for:
- determining soil strength
- especially when soil is ***not*** overconsolidated

Bad for:
- deformations
- inaccurate both in elastic and plastic regions



$$F = (\sigma_1' - \sigma_3') - (\sigma_1' + \sigma_3')\sin\phi' - 2c'\cos\phi' = 0$$

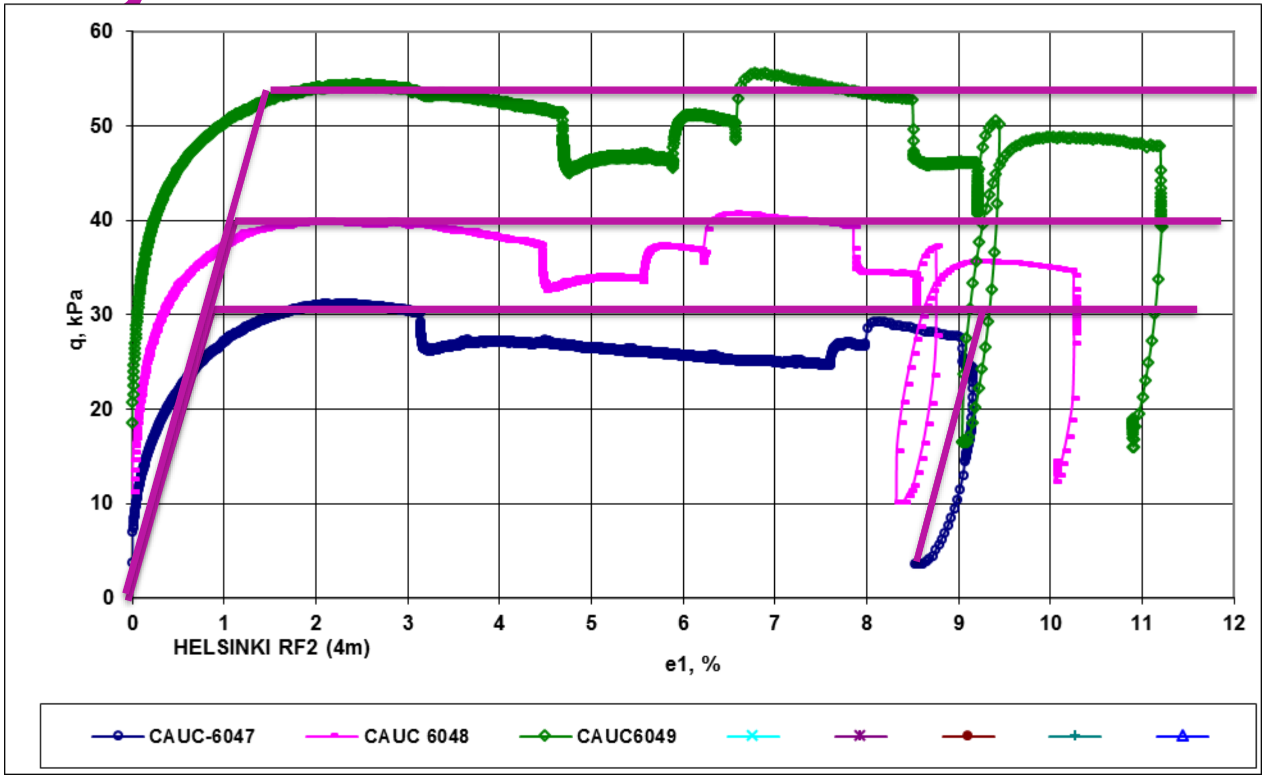
Problem: cohesion is not constant.

Friction angle is also not constant (but changes usually little in the usable range)

Elasto-plasticity: Mohr-Coulomb

Engineers are used to MC model – so it is widely used…

Many modifications to improve its prediction of deformations…



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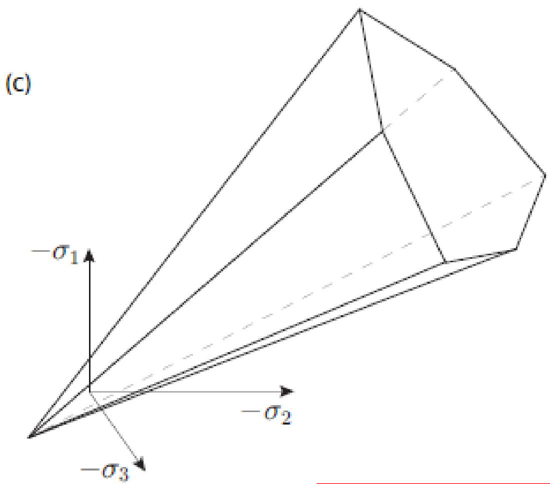
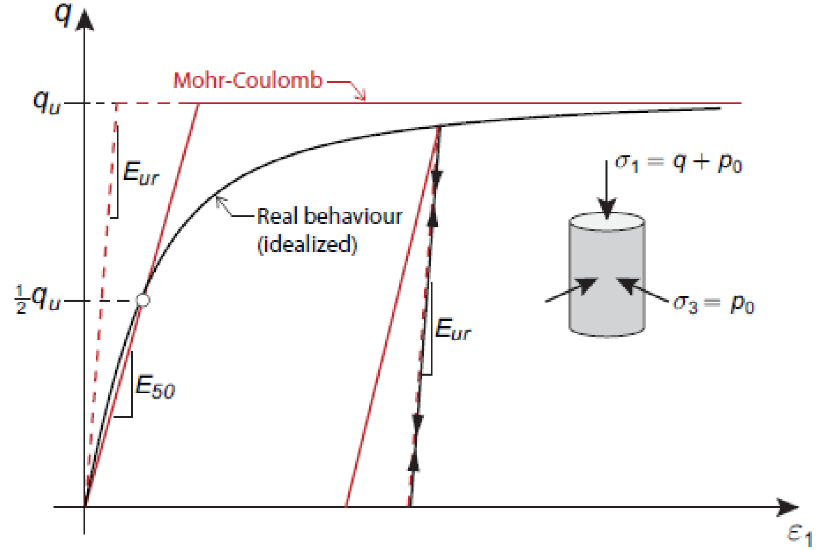
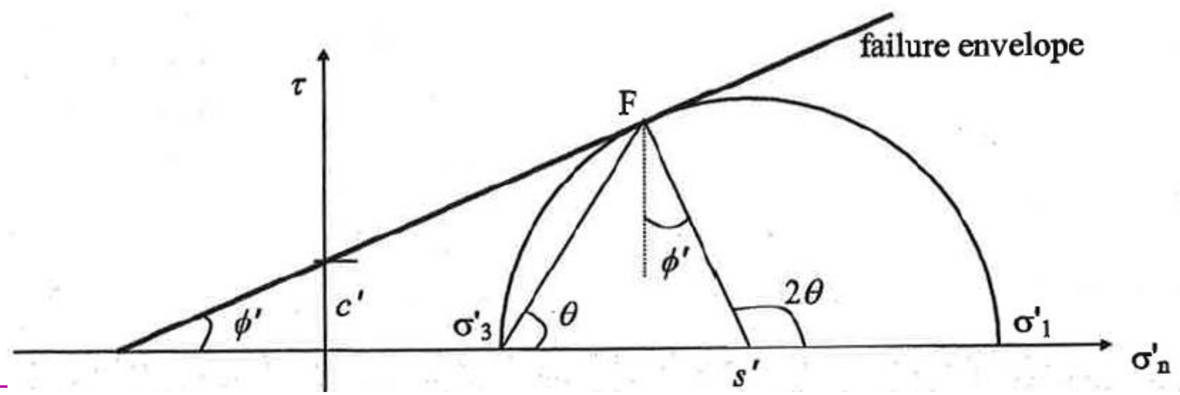
Elasto-plasticity: Mohr-Coulomb





$$F = (\sigma_1' - \sigma_3') - (\sigma_1' + \sigma_3')\sin\phi' - 2c'\cos\phi' = 0$$



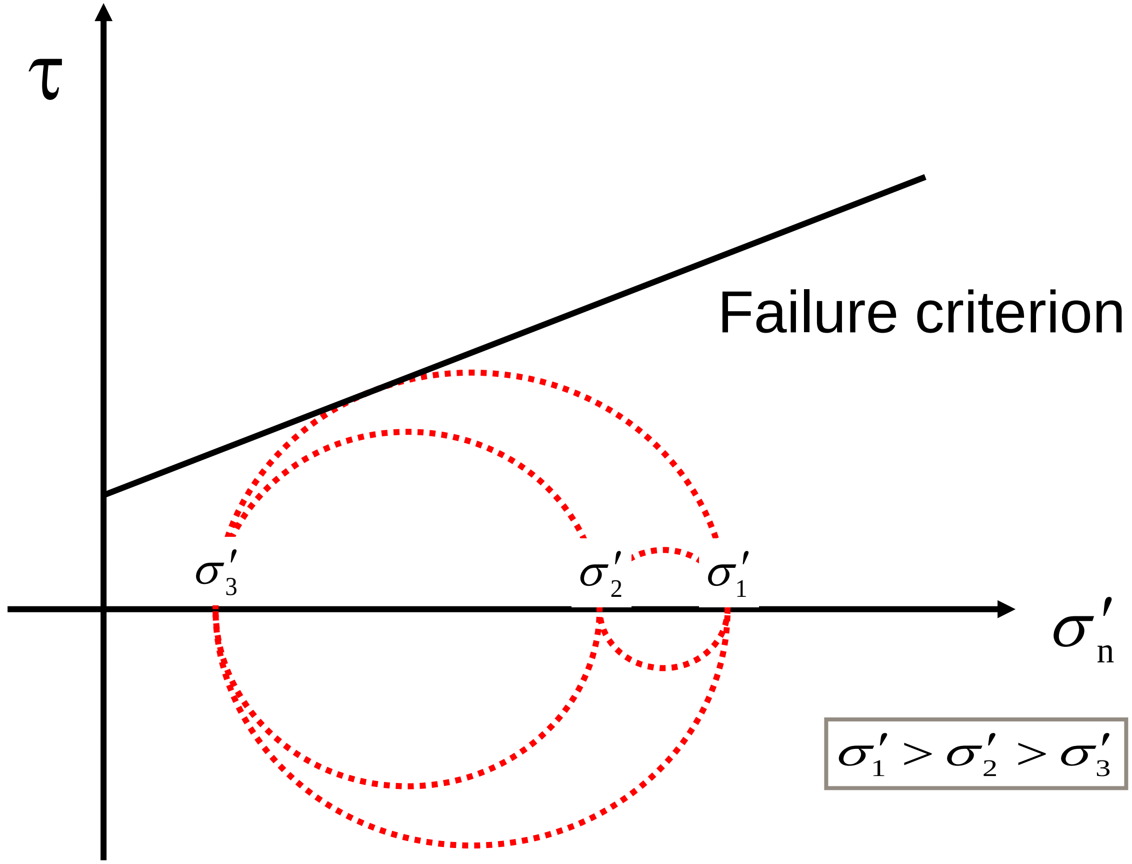
Mohr Coulomb Yield/Failure Condition

Yielding (and failure) takes place in the soil mass when mobilised (actual) shear stress at any plane (τ_m) becomes equal to shear strength (τ_f) which is given by:

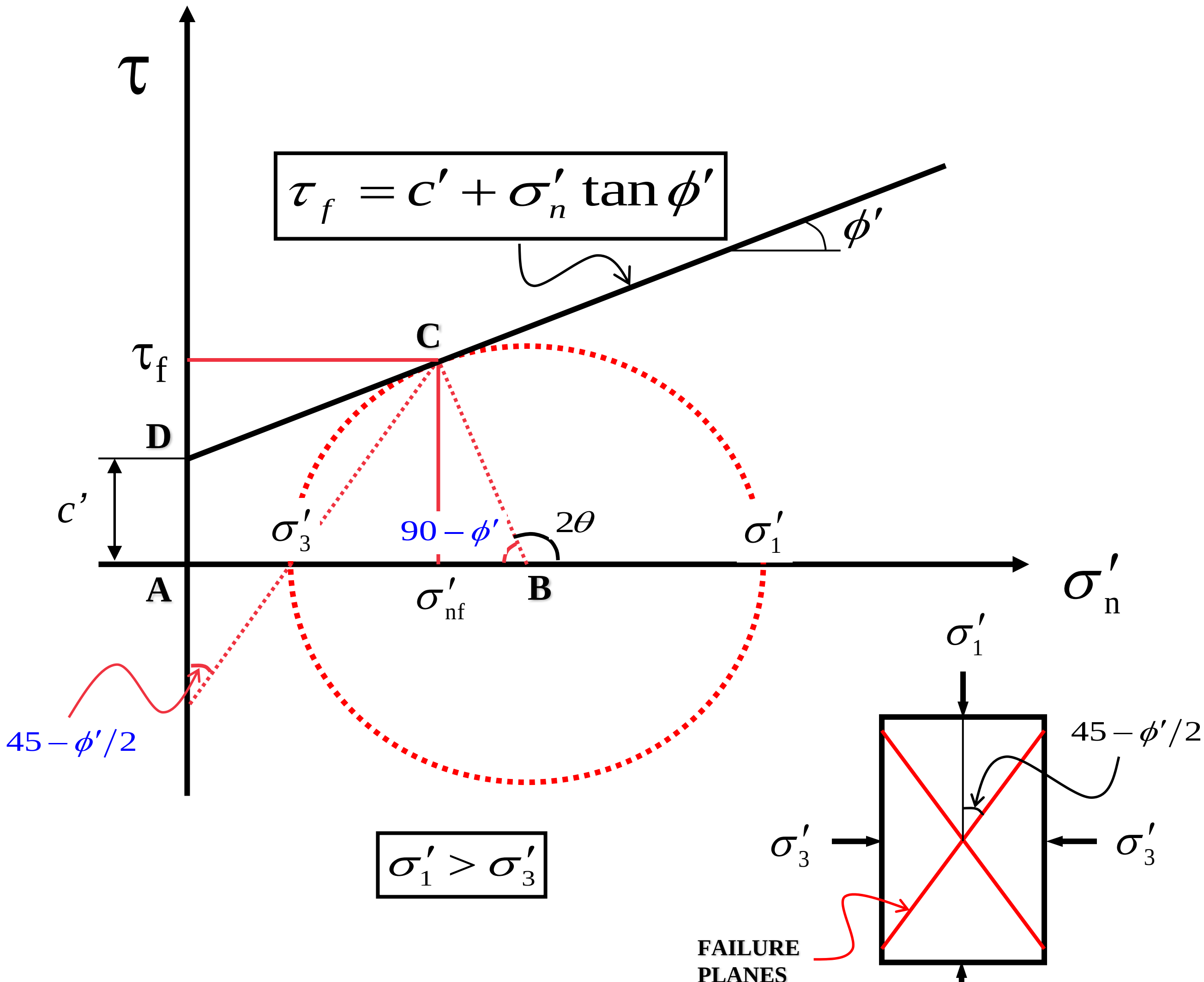
$$\tau_m = c' + \sigma'_n \tan\phi' = \tau_f$$

where c' and ϕ' are strength parameters.

$$f(\sigma') = \tau - \sigma'_n \tan\phi' - c' = 0$$

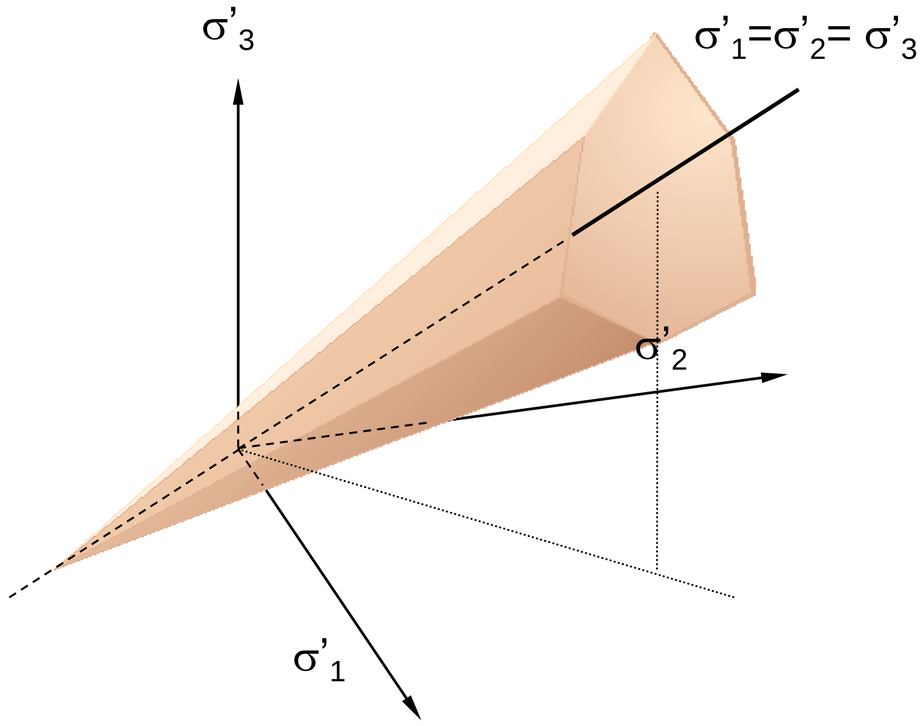


Note that the value of intermediate stress (σ_2') does not influence failure



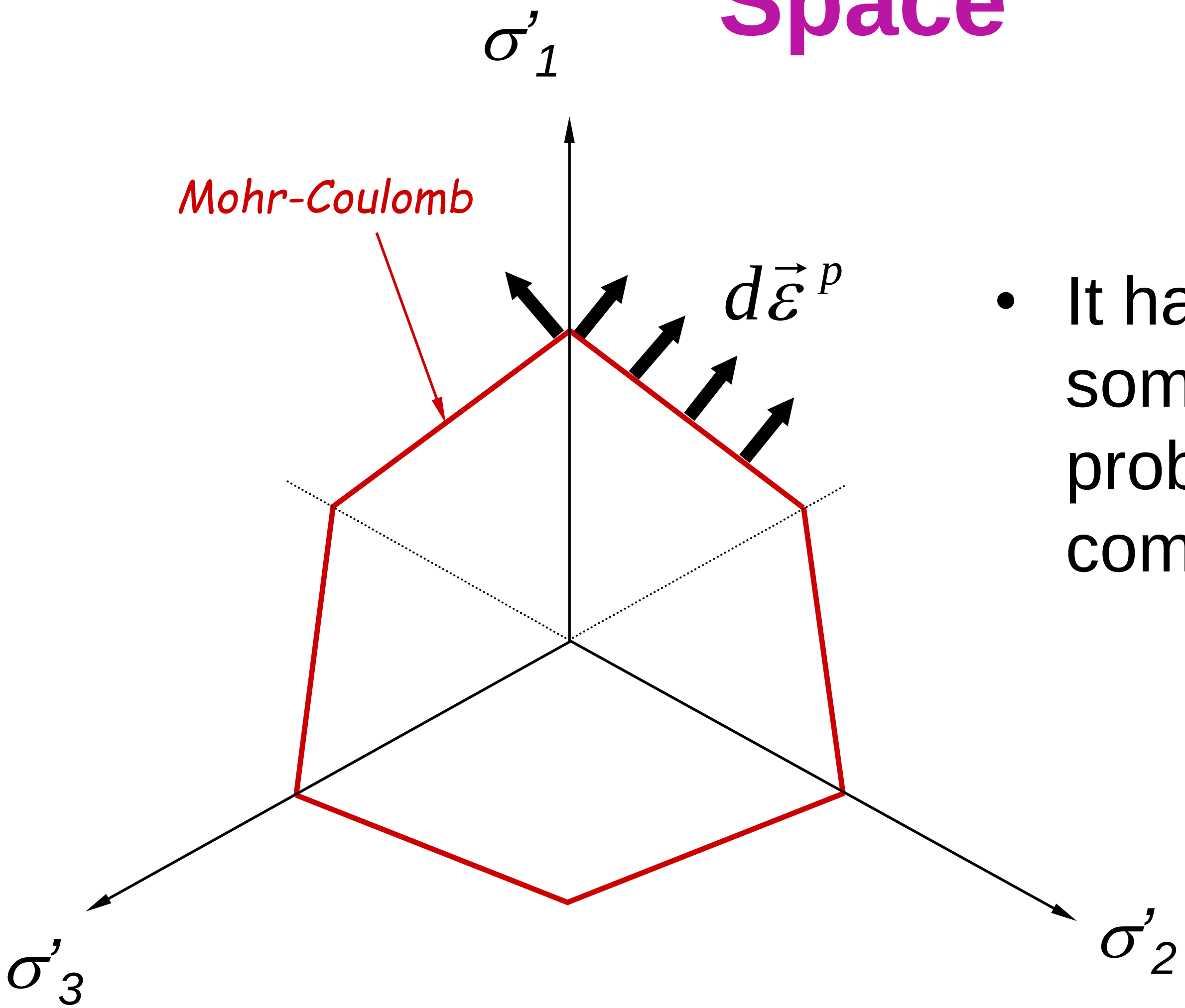
$$F = (\sigma'_1 - \sigma'_3) - (\sigma'_1 + \sigma'_3)\sin\phi' - 2c'\cos\phi' = 0$$

Mohr-Coulomb in Principal Stress Space



- Mohr – Coulomb failure surface is a irregular hexagon in the principal stress space

Mohr-Coulomb in Principal Stress Space



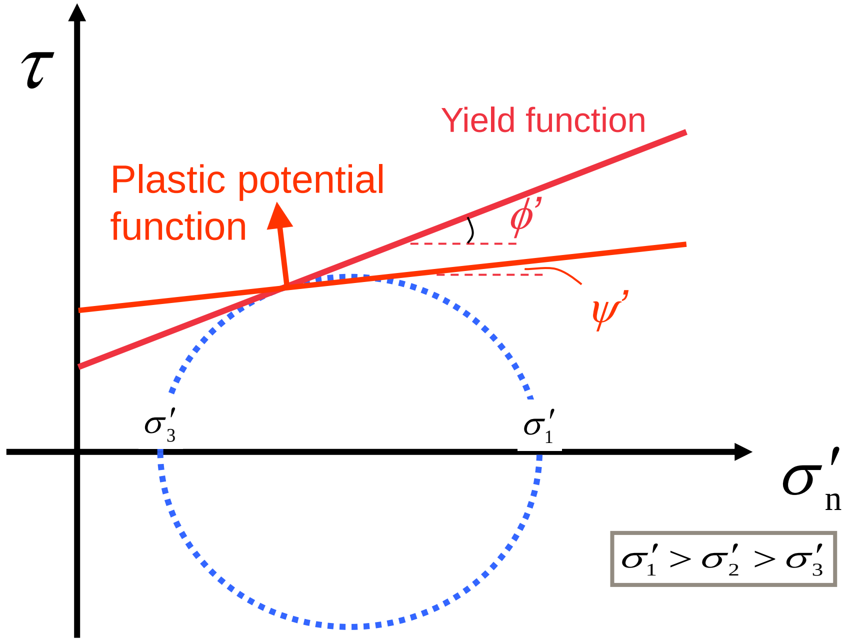
- It has corners that may sometimes create problems in computations

Flow Rule for Mohr Coulomb

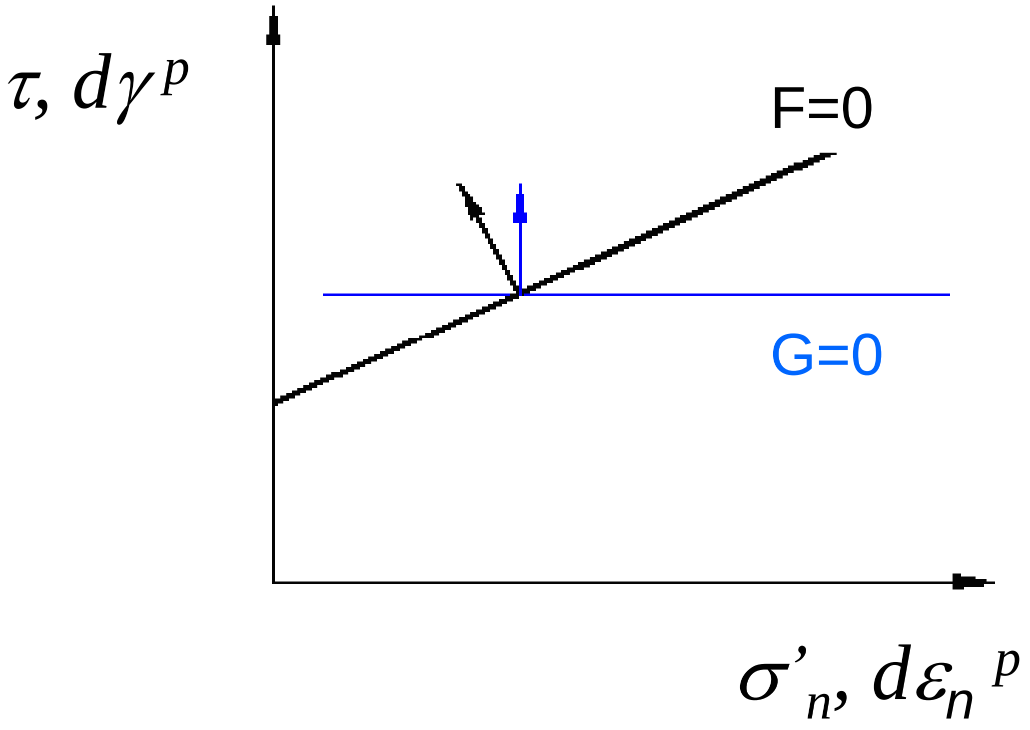
For Mohr-Coulomb flow rule is defined through the 'dilatancy angle' of the soil.

$G(\sigma') = \tau - \sigma'_n \tan\psi' - \text{const.} = 0$

where ψ' is the dilatancy angle and $\psi' \leq \phi'$.

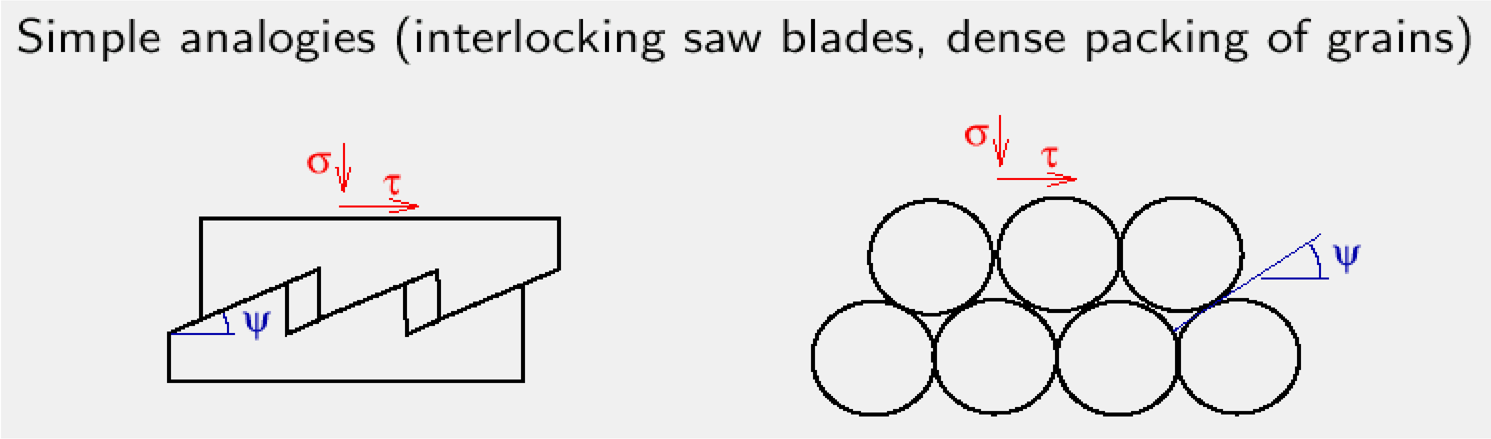


Associated and Non-Associated Flow Rule



How to understand dilatancy

i.e., why do we get volume changes when applying shear stresses?

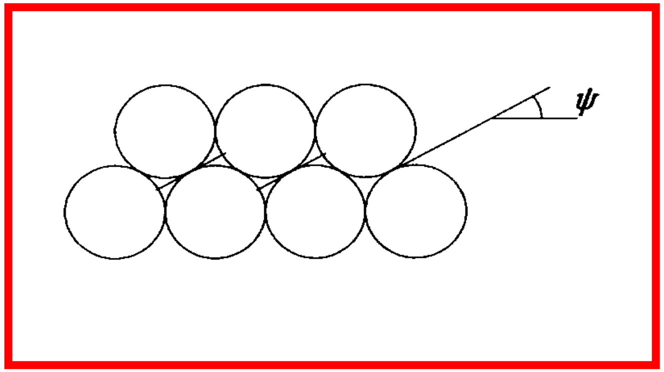


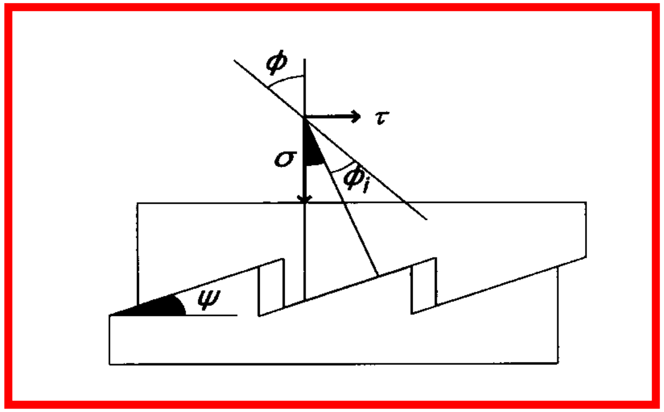
$$\varphi = \psi + \varphi_i$$

The apparent externally mobilized angle of friction on horizontal planes (φ) is larger than the angle of friction resisting sliding on the inclined planes (φ_i)

strength = friction + dilatancy

How to understand dilatancy



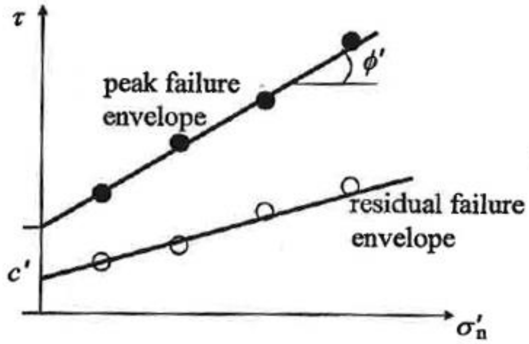


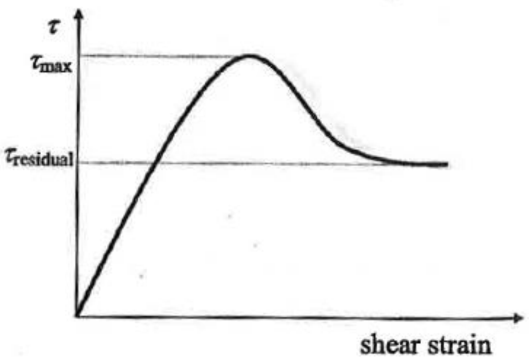
- When dense sands or overconsolidated clays are sheared they dilate
- Larger the particle size, greater the dilation
- Mohr-Coulomb idealisation implies dilation at a constant rate when soil is sheared. This is unrealistic.

Mohr Coulomb Model

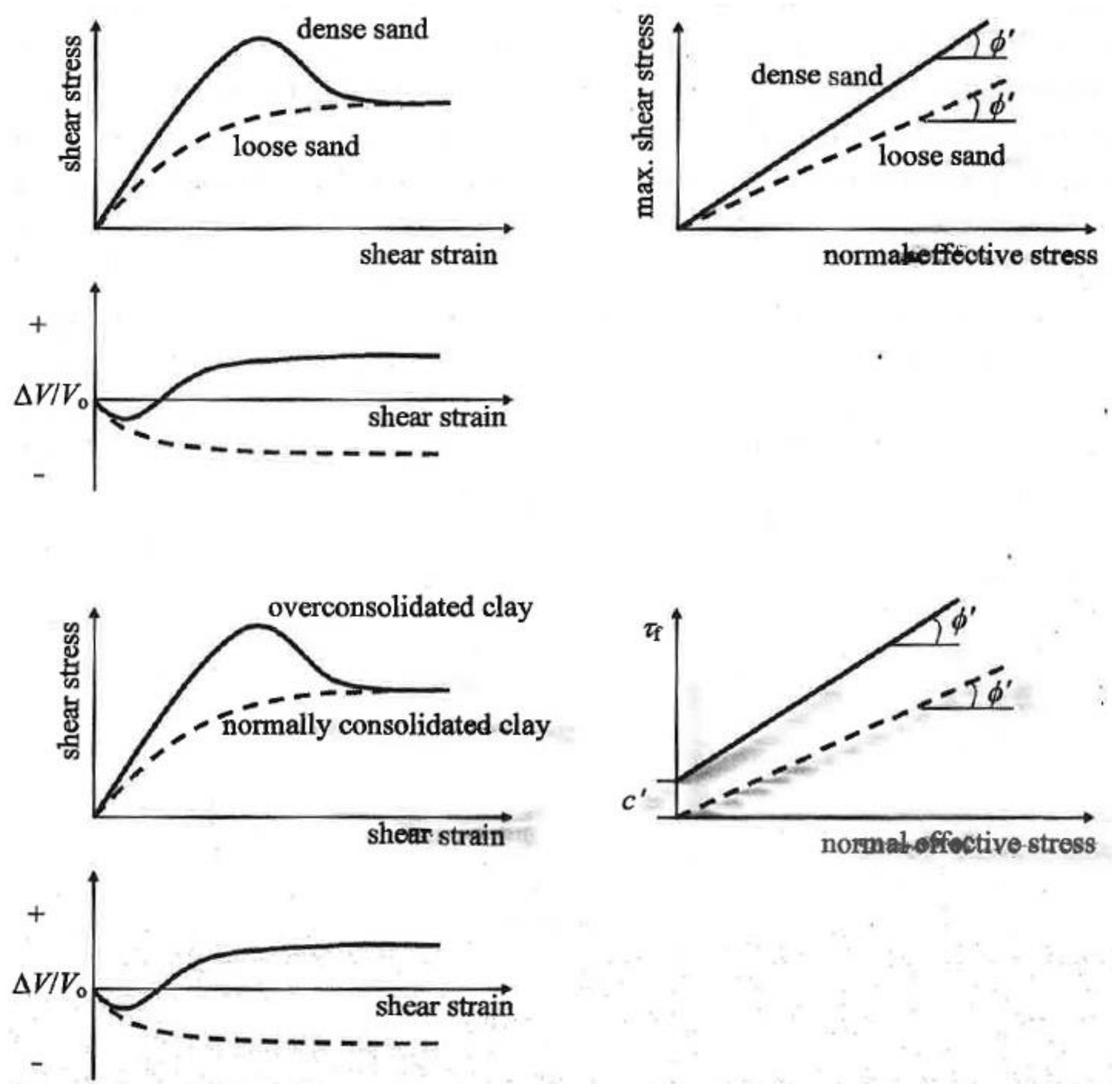
Limitations & comments

Reality of friction angle:





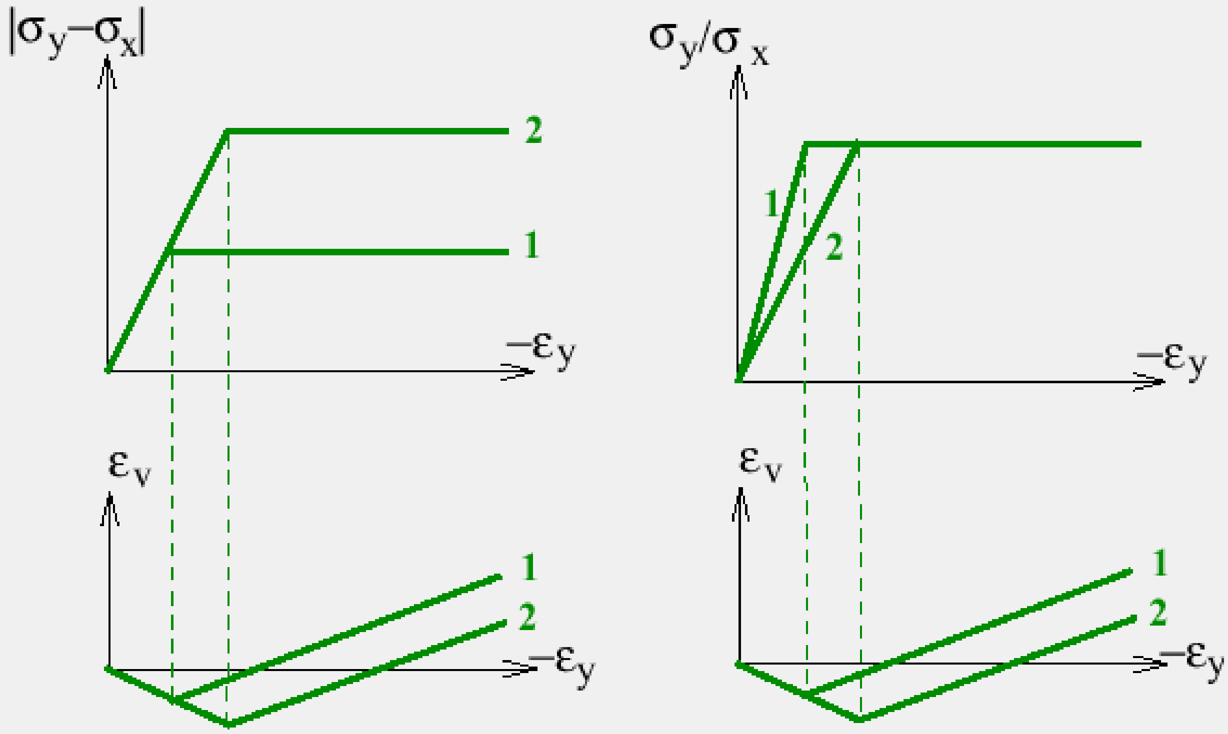
Reality of dilation angle:



MC model for element tests

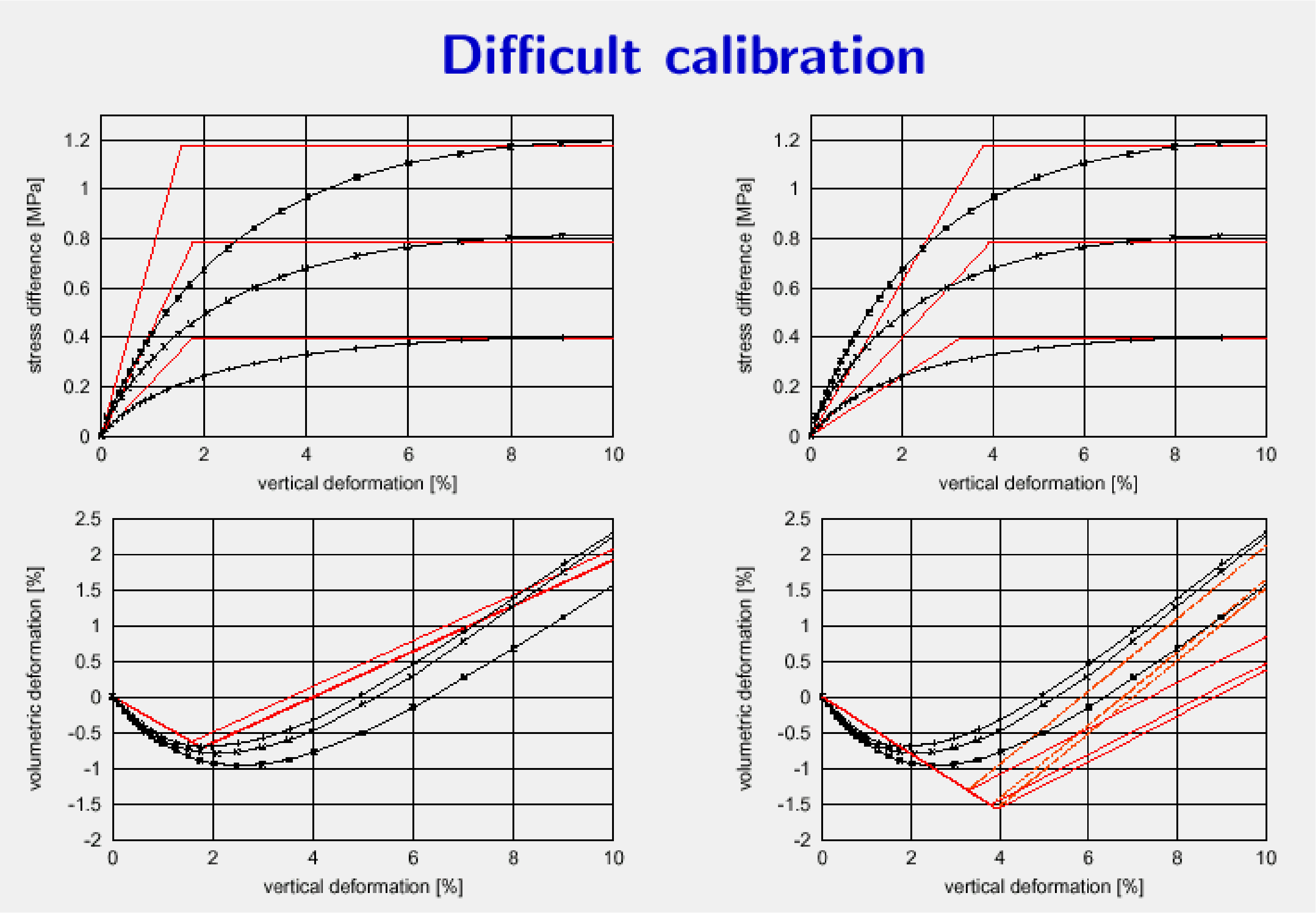
Triaxial tests at different pressures

1: $\sigma_x = 100$ kPa, 2: $\sigma_x = 200$ kPa



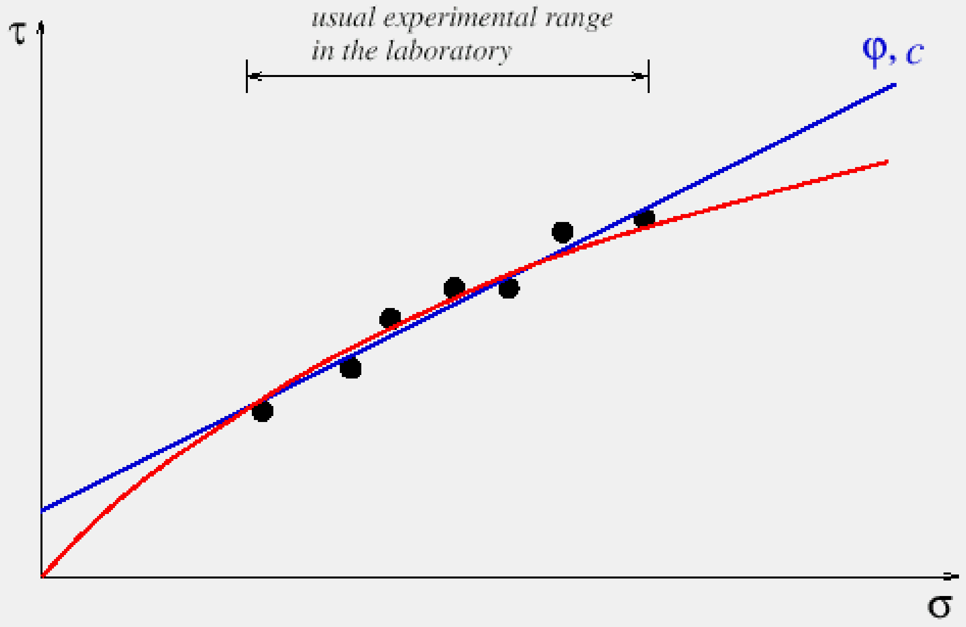
$$\frac{\sigma_y}{\sigma_x} = \frac{1+\sin\varphi}{1-\sin\varphi}, \quad E\text{=const., } \varphi\text{=const., } \psi\text{=const. (c=0)}$$

Limitations of MC model (1)



Limitations of MC model (2)

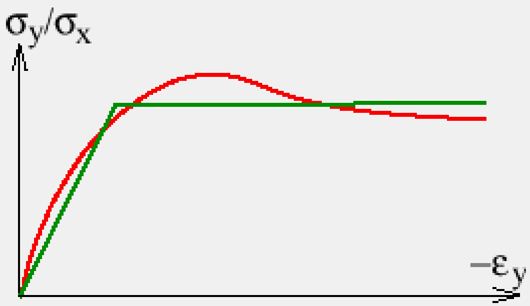
Linearization of the limit stress envelope



possible overestimation of the safety factor in slope stability calculations!

Warning for dense sands

Post-peak softening (dense sand)



Do not rely on high φ !

Post-peak softening
$\rightsquigarrow$ progressive failure

Be conservative: $\varphi \leq 35°$

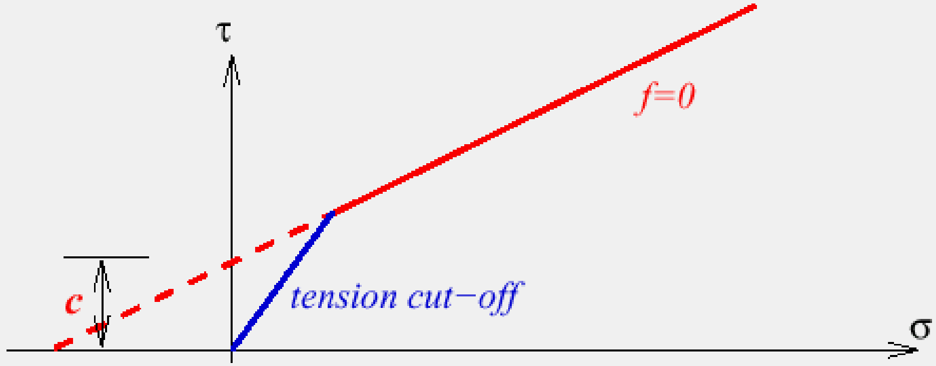
Dilatancy: $\psi \approx \varphi - 30° \quad (\psi \geq 0)$

Tips for fine-grained soils

Cohesive soils (clay)

experimental observation: c is proportional to w
(c is proportional to OCR)

$w \approx w_l$ . . . c is very small
$w \approx w_p$ . . . c is significant



Small tensile strength can be considered in special cases
(needed for rock and concrete!)

Drawbacks of MC

- Mohr-Coulomb failure criterion is well proven through experiments for most geomaterials, but data for clays is still contradictory!

- An associated flow rule implies continuous dilation at a constant rate upon shearing; this is unrealistic and leads to negative pore pressures in undrained conditions. In an non-associated flow rule with $\psi' < \phi'$, the rate of dilation is less, but it is still constant. If $\psi' = 0$ then the rate of dilation is zero.

- Care must be taken in applying the model for undrained loading.

Drawbacks of MC

- **Soils on shearing exhibit variable volume change characteristics depending on pre-consolidation pressure which cannot be accounted for with MC**

- **In soft soils volumetric plastic strains on shearing are compressive (negative dilation) whilst Mohr-Coulomb model will predict continuous dilation**

To summarize the limitations of MC are:

- **bi-linearity (const. E')**
- **unlimited dilation**
- **isotropy**
- **elastic response far from the limit state**
- **…**

→ more advanced approximation of soil behavior:
Hardening Soil Model (sand), Soft Soil Model (clay)

Other elastic-perfectly plastic models

Elasto-plasticity: Hoek - Brown

Used in rock mechanics…

$$\sigma_1' = \sigma_3' + \sigma_{ci}\left(m_b \sigma_3' / \sigma_{ci} + s\right)^a$$

$$m_b = m_i \exp\left(GSI - 100/28 - 14D\right)$$

$$s = \exp\left(GSI - 100/9 - 3D\right)$$

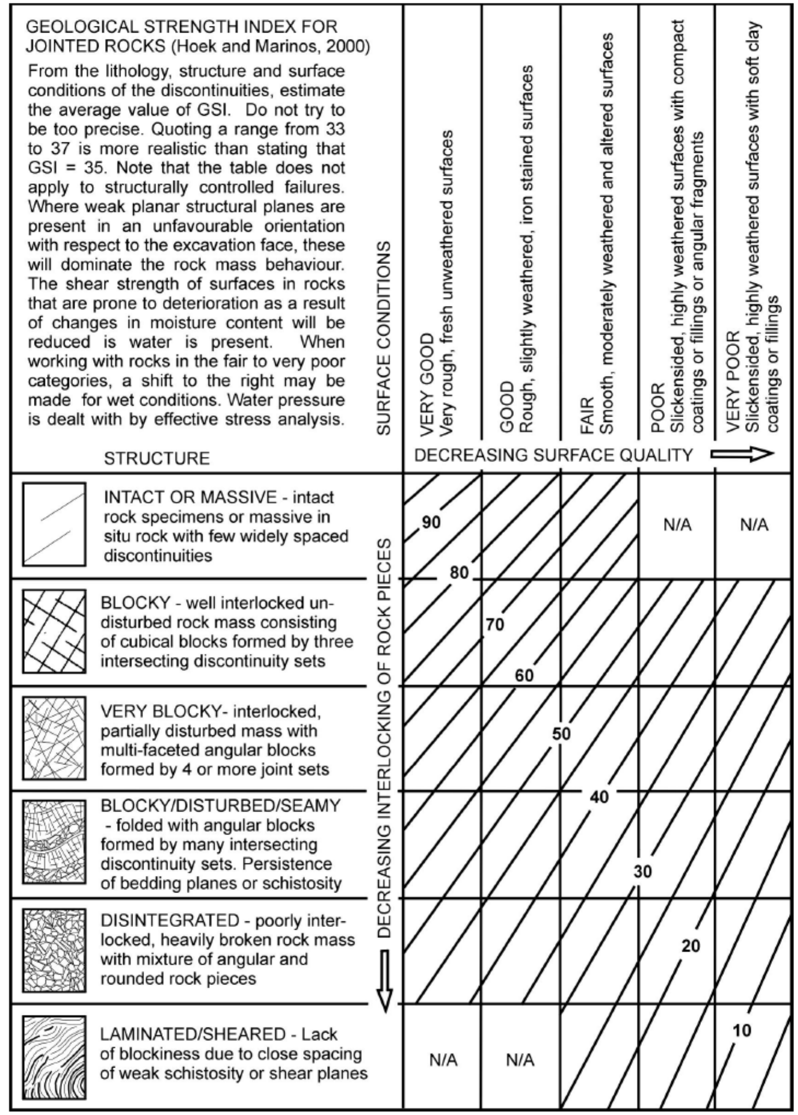
$$a = \frac{1}{2} + \frac{1}{6}\left(e^{-GSI/15} - e^{-20/3}\right)$$

$$E_m(GPa) = \left(1 - \frac{D}{2}\right)\sqrt{\frac{\sigma_{ci}}{100}} \cdot 10^{((GSI-10)/40)}$$

GSI

Geological strength index

Geological Strength Index Chart



Elasto-plasticity: Hoek - Brown

Used in rock mechanics…

$$\sigma_1^{'} = \sigma_3^{'} + \sigma_{ci}\left(m_b\,\sigma_3^{'}/\sigma_{ci} + s\right)^a$$

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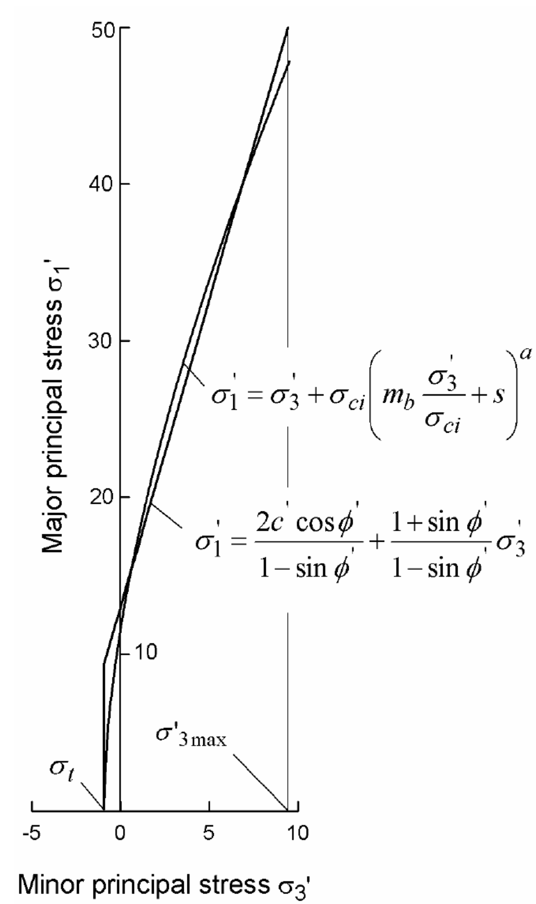


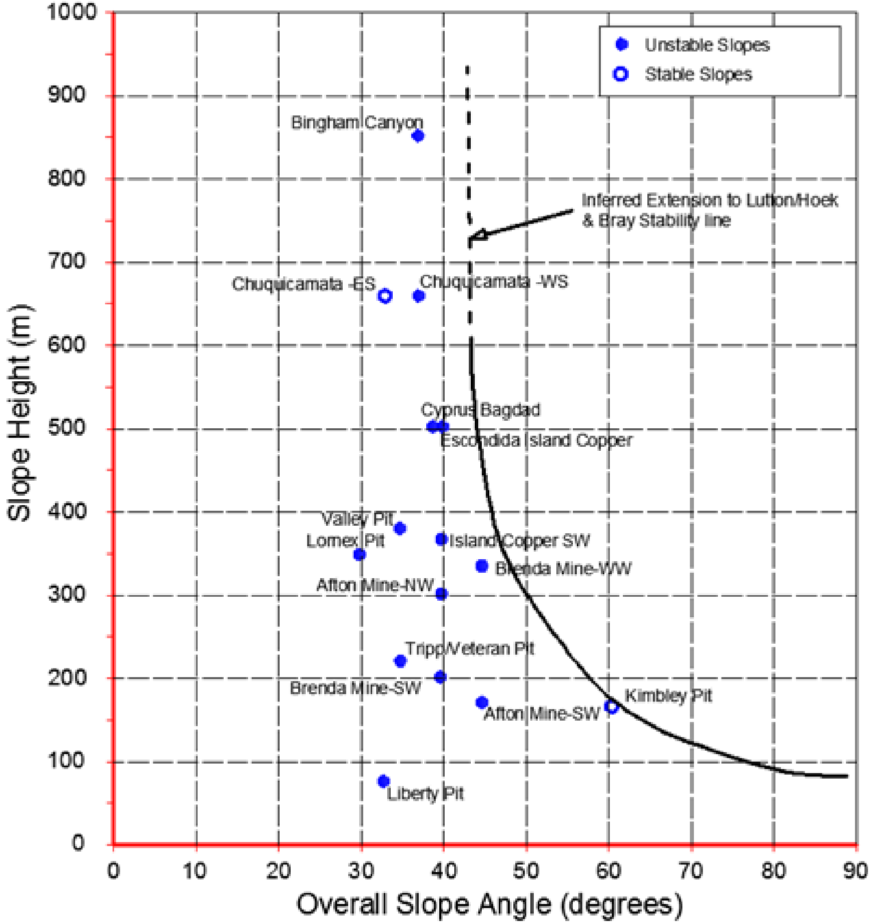
Elasto-plasticity: Hoek - Brown

Designed for:

- relatively low-height walls (generally 10-20m, less than 50m)
- tunnels
- generally rocks

If you use it outside of the intended use… you have problems.

Hoek-Brown into Mohr-Coulomb – very tricky



Mohr Coulomb Model

some more advanced subjects

MC model p'- q- space

$$p' = \frac{\sigma'_1 + 2\sigma'_3}{3} \qquad q = \sigma'_1 - \sigma'_3$$

$$\boxed{F = (\sigma'_1 - \sigma'_3) - (\sigma'_1 + \sigma'_3)\sin\phi' - 2c'\cos\phi' = 0}$$

$$\sigma'_1 = 3p' - 2\sigma'_3 \qquad \sigma'_3 = \sigma'_1 - q$$

$$\sigma'_1 = 3p' - 2\sigma'_1 + 2q \quad = \quad \frac{3p' + 2q}{3}$$

$$\sigma'_3 = 3p' - 2\sigma'_3 - q \quad = \quad \frac{3p' - q}{3}$$

$$\sigma'_1 + \sigma'_3 = \frac{6p' + q}{3}$$

MC model p'- q- space

$$or \quad q = \sin\phi\left(\frac{6p' + q}{3}\right) + 2c\cos\phi$$

$$3q = 6p'\sin\phi + q\sin\phi + 6c\cos\phi$$

$$q = \frac{6\sin\phi'}{3 - \sin\phi'}p' + \frac{6c\cos\phi'}{3 - \sin\phi'}$$

$$q = \eta p' + c^*$$

$$where \quad \eta = \frac{6\sin\phi'}{3 - \sin\phi'}, c^* = \frac{6c\cos\phi'}{3 - \sin\phi'}$$

$$F = q - \eta p' - c^* = 0$$

MC - Model formulated in p' - q

MC model p'-q- space

$$\begin{Bmatrix} \Delta p' \\ \Delta q \end{Bmatrix} = \begin{bmatrix} K' & 0 \\ 0 & 3G' \end{bmatrix} \begin{Bmatrix} \Delta \varepsilon_v^e \\ \Delta \varepsilon_q^e \end{Bmatrix}$$

Assuming associated flow rule and ideal plasticity

$$F\left(p', q\right) = q - \eta p' - c^* = 0$$

$$\frac{\partial F}{\partial p'} dp' + \frac{\partial F}{\partial q} dq = 0 \quad :$$ **consistency condition**

Formulation of D^{ep} for MC

$$dp' = K'\left(d\varepsilon_v - d\varepsilon_v^p\right)$$

$$dq = 3G'\left(d\varepsilon_q - d\varepsilon_q^p\right)$$

$$d\varepsilon_v^p = d\lambda \frac{\partial F}{\partial p'}$$

$$d\varepsilon_q^p = d\lambda \frac{\partial F}{\partial q}$$

Substituting into consistency condition leads to:

$$\frac{\partial F}{\partial p'} K' d\varepsilon_v - \frac{\partial F}{\partial p'} K' d\lambda \frac{\partial F}{\partial p'} + \frac{\partial F}{\underset{q}{\partial Q}} 3G' d\varepsilon_q - \frac{\partial F}{\partial q} 3G' d\lambda \frac{\partial F}{\partial q} = 0$$

Formulation of D^{ep} for MC

$$d\lambda = \frac{\dfrac{\partial F}{\partial p'} K' d\varepsilon_v + \dfrac{\partial F}{\partial q'} 3G' d\varepsilon_q}{\dfrac{\partial F}{\partial p'} K' \dfrac{\partial F}{\partial p'} + \dfrac{\partial F}{\partial q} 3G' \dfrac{\partial F}{\partial q}} = \frac{\begin{bmatrix} \dfrac{\partial F}{\partial p'}, \dfrac{\partial F}{\partial q} \end{bmatrix} \begin{bmatrix} K' & 0 \\ 0 & 3G' \end{bmatrix} \begin{Bmatrix} d\varepsilon_v \\ d\varepsilon_q \end{Bmatrix}}{\begin{bmatrix} \dfrac{\partial F}{\partial p'}, \dfrac{\partial F}{\partial q} \end{bmatrix} \begin{bmatrix} K' & 0 \\ 0 & 3G' \end{bmatrix} \begin{Bmatrix} \dfrac{\partial F}{\partial p'} \\ \dfrac{\partial F}{\partial q} \end{Bmatrix}}$$

Formulation of D^{ep} for MC

$$\frac{\partial F}{\partial p'} = -\eta, \quad \frac{\partial F}{\partial q} = 1, \quad d\lambda = \frac{-\eta K' d\varepsilon_v + 3G' d\varepsilon_q}{\eta^2 K' + 3G'}$$

$$d\varepsilon_v^p = -d\lambda\eta, \quad d\varepsilon_q^p = d\lambda$$

$$F = q - \eta p' - c^* = 0$$

$$\begin{Bmatrix} dp' \\ dq \end{Bmatrix} = \begin{bmatrix} K' & 0 \\ 0 & 3G' \end{bmatrix} \begin{Bmatrix} d\varepsilon_v - d\varepsilon_v^p \\ d\varepsilon_q - d\varepsilon_q^p \end{Bmatrix}$$

$$= \begin{bmatrix} K' & 0 \\ 0 & 3G' \end{bmatrix} \begin{Bmatrix} d\varepsilon_v \\ d\varepsilon_q \end{Bmatrix} - \begin{bmatrix} K' & 0 \\ 0 & 3G' \end{bmatrix} d\lambda \begin{Bmatrix} -\eta \\ 1 \end{Bmatrix}$$

$$= \begin{bmatrix} K' & 0 \\ 0 & 3G' \end{bmatrix} \begin{Bmatrix} d\varepsilon_v \\ d\varepsilon_q \end{Bmatrix} - \begin{bmatrix} K' & 0 \\ 0 & 3G' \end{bmatrix} \begin{Bmatrix} -\eta \\ 1 \end{Bmatrix} \frac{\left[-\eta K' \quad 3G'\right]}{\eta^2 K' + 3G'} \begin{Bmatrix} d\varepsilon_v \\ d\varepsilon_q \end{Bmatrix}$$

Formulation of D^ep for MC

$$
= \left[\begin{bmatrix} K' & 0 \\ 0 & 3G' \end{bmatrix} - \frac{1}{\eta^2 K' + 3G} \begin{bmatrix} K' & 0 \\ 0 & 3G \end{bmatrix} \begin{bmatrix} \eta^2 K' & -3G\eta \\ -\eta K' & 3G \end{bmatrix} \right] \begin{Bmatrix} d\varepsilon_v \\ d\varepsilon_q \end{Bmatrix}
$$

$$
= \left[\begin{bmatrix} K' & 0 \\ 0 & 3G' \end{bmatrix} - \frac{1}{\eta^2 K' + 3G} \begin{bmatrix} \eta^2 K'^2 & -3G'K'\eta \\ -3GK'\eta & 9G'^2 \end{bmatrix} \right] \begin{Bmatrix} d\varepsilon_v \\ d\varepsilon_q \end{Bmatrix}
$$

$$
= \begin{bmatrix} K' - \dfrac{\eta^2 K'^2}{\eta^2 K' + 3G'} & \dfrac{3G'K'\eta}{\eta^2 K' + 3G'} \\ \dfrac{3G'K'\eta}{\eta^2 K' + 3G'} & 3G' - \dfrac{9G'^2}{\eta^2 K' + 3G'} \end{bmatrix} \begin{Bmatrix} d\varepsilon_v \\ d\varepsilon_q \end{Bmatrix}
$$

Formulation of D^{ep} for MC

$$D^{ep} = \begin{bmatrix} K' - \dfrac{\eta^2 K'^2}{\eta^2 K' + 3G'} & \dfrac{3G'K'\eta}{\eta^2 K' + 3G'} \\ \dfrac{3G'K'\eta}{\eta^2 K' + 3G'} & 3G' - \dfrac{9G'^2}{\eta^2 K' + 3G'} \end{bmatrix}$$

$$\det\left|D^{ep}\right| = 0$$ ➡ **perfect plasticity**

In general form :

for non associated flow:

$$\frac{\partial Q}{\partial \sigma'}$$

$$D^{ep} = \frac{D^e - D^e \dfrac{\partial F}{\partial \underset{\sim}{\sigma}'} \left(\dfrac{\partial F}{\partial \underset{\sim}{\sigma}'}\right)^T D^{eT}}{\left(\dfrac{\partial F}{\partial \underset{\sim}{\sigma}'}\right)^T D^e \dfrac{\partial F}{\partial \underset{\sim}{\sigma}'}}$$

Stress invariants

Mean effective stress

$$\sigma'_M = \frac{1}{3}\left(\sigma'_1 + \sigma'_2 + \sigma'_3\right) = \frac{1}{3}\left(\sigma'_x + \sigma'_y + \sigma'_z\right)$$

Deviatoric stress (in general form)

$$\bar{\sigma} = \sqrt{J_2}$$

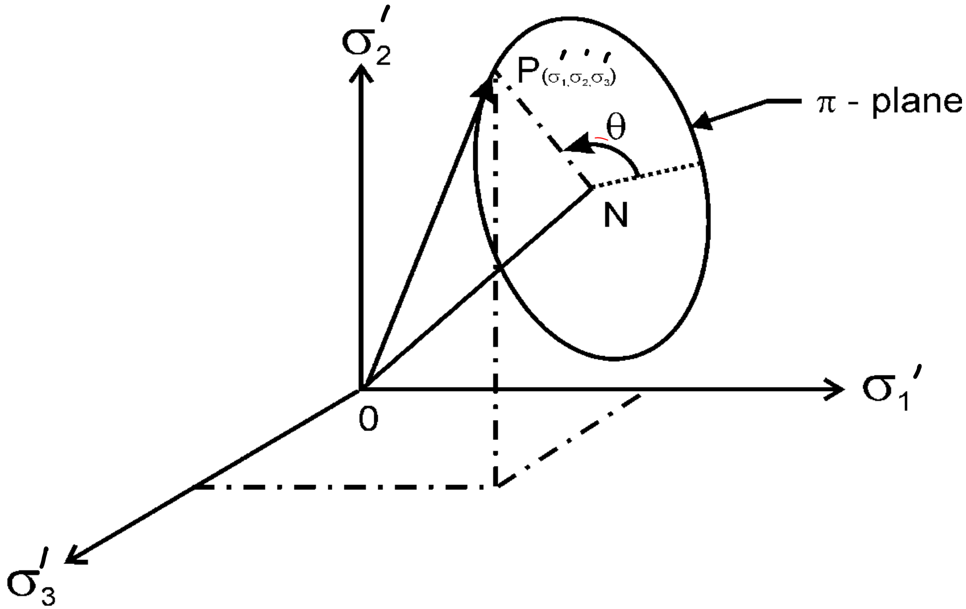
$$J_2 = \frac{1}{6}\left[\left(\sigma'_1 - \sigma'_2\right)^2 + \left(\sigma'_2 - \sigma'_3\right)^2 + \left(\sigma'_3 - \sigma'_1\right)^2\right] =$$

$$\frac{1}{2}\left[\left(\sigma'_1 - \sigma'_M\right)^2 + \left(\sigma'_2 - \sigma'_M\right)^2 + \left(\sigma'_3 - \sigma'_M\right)^2\right]$$

$$J_3 = \left(\sigma'_1 - \sigma'_M\right)\left(\sigma'_2 - \sigma'_M\right)\left(\sigma'_3 - \sigma'_M\right)$$

Lode angle $\theta = \frac{1}{3}\sin^{-1}\left(\frac{-3\sqrt{3}}{2}\frac{J_3}{\bar{\sigma}^3}\right)$ $-30° < \theta < 30°$

Meaning of stress invariants



$$\overline{ON} = \sqrt{3} \cdot \sigma_M'$$

$$\overline{PN}^2 = \overline{OP}^2 - \overline{ON}^2 =$$

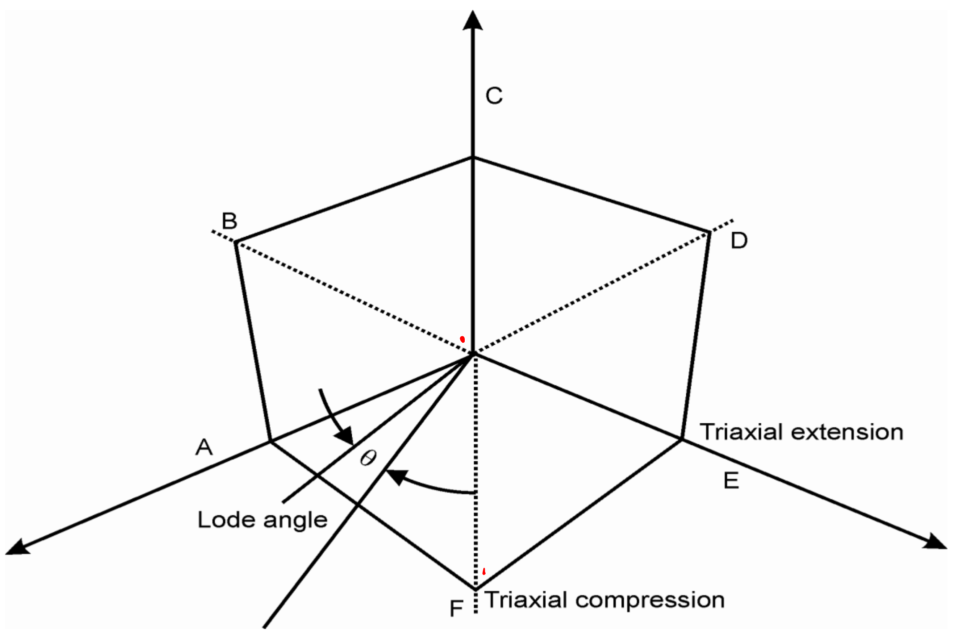
$$= (\sigma_1' - \sigma_M')^2 + (\sigma_2' - \sigma_M')^2 + (\sigma_3' - \sigma_M')^2 =$$

$$= 2J_2$$

$$\overline{PN} = \sqrt{2} \cdot \sqrt{J_2} = \sqrt{2} \cdot \overline{\sigma}$$

Mohr Coulomb – Invariant formulation

$$F = \sigma'_M \cdot \sin\varphi + \bar{\sigma} \cdot \cos\theta - \frac{1}{\sqrt{3}} \cdot \bar{\sigma} \cdot \sin\varphi \sin\theta - c \cdot \cos\varphi = 0$$



Thank you

Literature

Hard to find a single book…

Check:

Briaud JL. Geotechnical Engineering : Unsaturated and Saturated Soils. (linked in the materials section, good introduction)

Potts & Zdravkovic, Finite element analysis in geotechnical engineering. Theory. Thomas Telford, 1999.

Muir Wood D., Soil Behaviour and Critical State Soil Mechanics. Cambridge University Press. 1990.

Books on theory of plasticity can be helpful