Aalto University
School of Engineering

GEO – E1050
Finite Element Method in Geoengineering

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Lecture 4. More general derivation of FEM: strong forms, weak forms and functionals

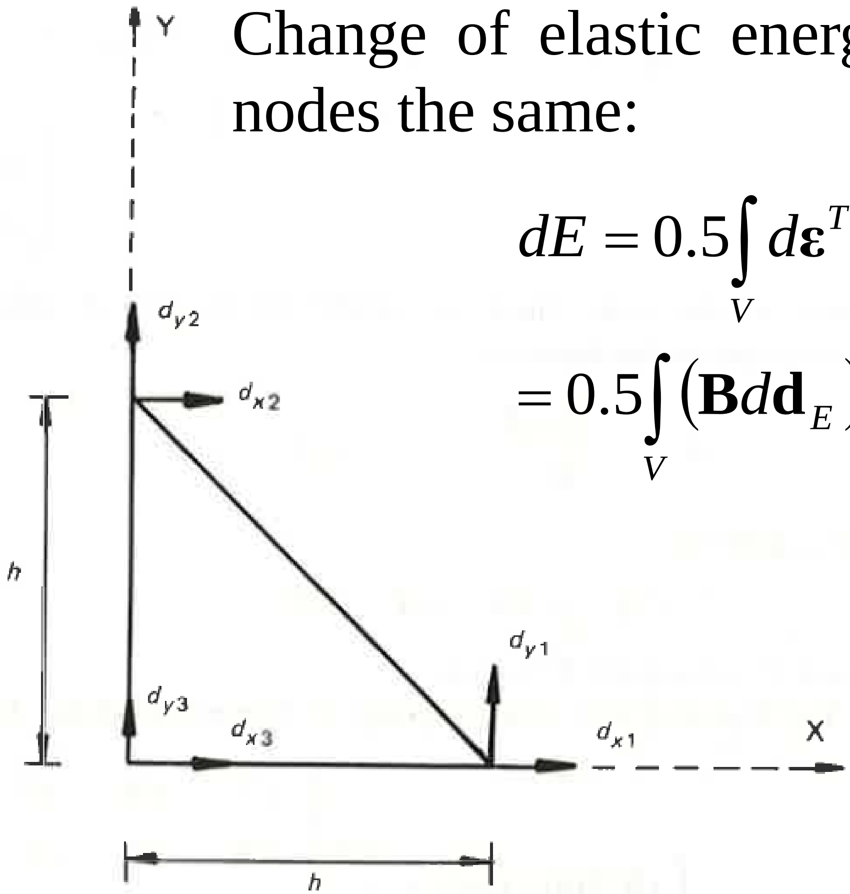
To learn today…

The lecture should give you overview of how FEM is derived and link the basic differential equations to integral forms and Finite Element discretisation. You should:

1. Understand the idea behind the **strong form**, **variational form**, **weak form** of the differential equation we are solving
2. Can give an example of **transformation between strong form and weak form** (probably the Poisson equation easiest)
3. Can give an example of transformation between **weak form and variational form**
4. Understand **why variational form is so useful…**

Note: 1-4 will be also touched upon during next lecture.

In lecture 2 we were deriving FE stiffness matrix…



Change of elastic energy inside the element and on the nodes the same:

$$dE = 0.5\int_V d\boldsymbol{\varepsilon}^T d\boldsymbol{\sigma} dV = 0.5\int_V d\boldsymbol{\varepsilon}^T \mathbf{D} d\boldsymbol{\varepsilon} dV =$$

$$= 0.5\int_V \left(\mathbf{B} d\mathbf{d}_E\right)^T \mathbf{D}\mathbf{B} d\mathbf{d}_E dV = 0.5\int_V d\mathbf{d}_E^T \mathbf{B}^T \mathbf{D}\mathbf{B} d\mathbf{d}_E dV$$

work at nodes:

$$dL = \int_S d\mathbf{d}_E^T \mathbf{T} dS = d\mathbf{d}_E^T \Delta\mathbf{R}$$

In lecture 2 we were deriving FE stiffness matrix…

We can write in general:

$$dE - dL = 0 = 0.5\int_V d\boldsymbol{\varepsilon}^T d\boldsymbol{\sigma} dV - \int_S d\mathbf{d}_{out}\mathbf{T}dS$$

Where $\mathbf{d}_{out}$ are the displacements on the outside of the body we compute the energy due to stress/strain change and $\mathbf{T}$ are corresponding external tractions. After discretisation that reduces to forces on the nodes of the discretised domain.

Variational form

We can write in general – the variation of the functional is zero:

$$dE - dL = 0 = 0.5\int_V d\boldsymbol{\varepsilon}^T d\boldsymbol{\sigma} dV - \int_S \mathbf{d}_{out}\mathbf{T} dS$$

In fully variational form, including the body forces the functional is

$$\Pi_{\text{TPE}}[u_i] = \frac{1}{2}\int_V \sigma_{ij}^u\, e_{ij}^u\, dV - \int_V b_i\, u_i\, dV - \int_{S_t} \hat{t}_i\, u_i\, dS.$$

Functional stationary condition 'generate' **strong** and **weak** forms of the differential equations.

Variational form

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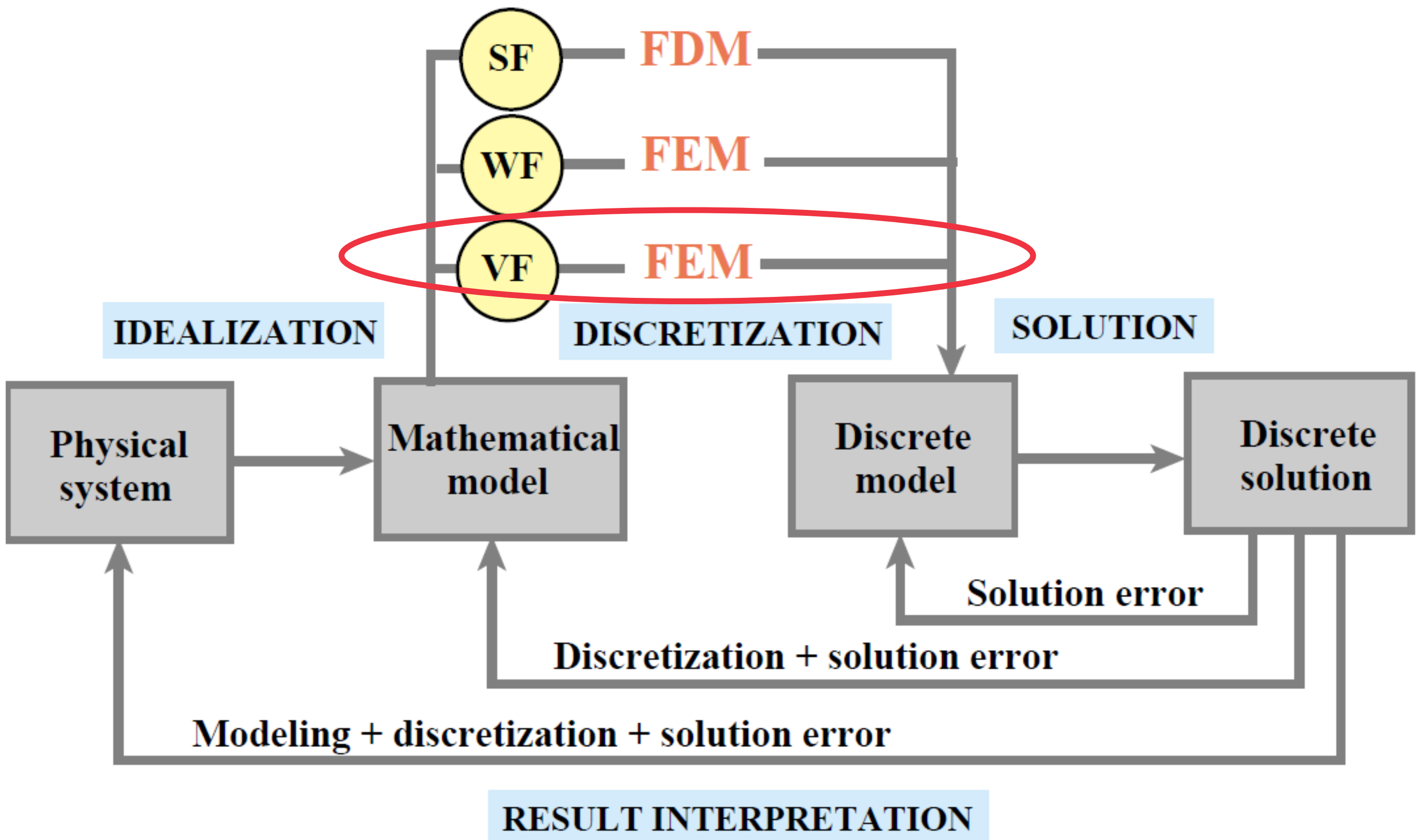
$$\Pi_{\text{TPE}}[u_i] = \frac{1}{2}\int_V \sigma_{ij}^u \, e_{ij}^u \, dV - \int_V b_i \, u_i \, dV - \int_{S_t} \hat{t}_i \, u_i \, dS.$$

Functional stationary condition ‘generate’ **strong** and **weak** forms of the differential equations.

We can generate weak form by finding the variation (similar to a derivative) of the functional

On the other hand, having the variation, it can be very difficult to find out the functional (similar to integration)

Variational form



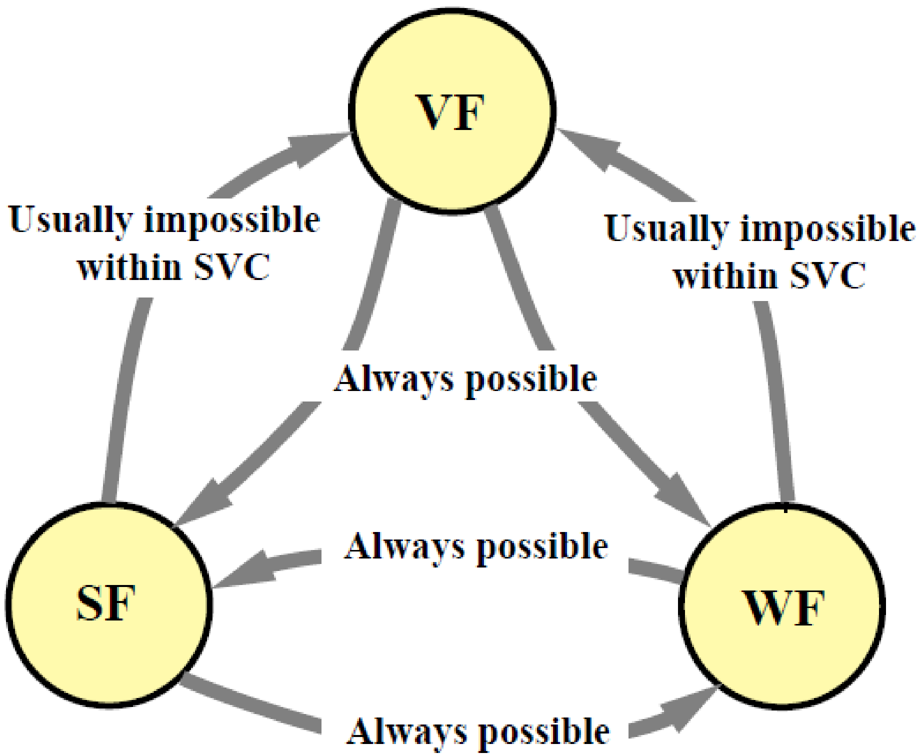
Forms of equation for discretisation

SF Strong Form. Presented as a system of *ordinary or partial differential equations* in space and/or time, complemented by appropriate boundary conditions. Occasionally this form may be presented in integral or differential form, or reduce to algebraic equations

WF Weak Form. Presented as a *weighted integral equation* that "relaxes" the strong form into a domain-averaging statement.

VF Variational Form. Presented as a *functional* whose stationary conditions generate the weak and strong forms.

Equation form: what is possible



Variational Calculus deals with transformations!!!

Elasticity

Strong form

The Strain-Displacement Equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

Constitutive Equations

$$\sigma_{ij} = E_{ijk\ell}\, e_{k\ell} \quad \text{in } V.$$

Internal Equilibrium Equations

$$\sigma_{ij,j} + b_i = \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \quad \text{in } V.$$

THE BOUNDARY CONDITIONS

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on } S_u.$$

Surface Equilibrium Equations

$$\sigma_{ij}\, n_j = \hat{t}_i \quad \text{on } S_t,$$

Weak form

A solution which satisfies strong form (that is the differential equations), will also satisfy weak form. In principle there may be a solution which satisfies weak form, but not the strong form

Obtaining strong form can be made in many ways:

- test functions
- weighted residual
- Lagrange multipliers

Mathematically **we can always move from strong form to weak and vice versa**. Which route is chosen does not matter, though some may have better physical meaning.

Weak form to variational form

§5.8.1. Step 1: Choose Master Field(s)

One or more of the unknown internal fields

$$u_i, \quad e_{ij}, \quad \sigma_{ij}, \tag{5.33}$$

are chosen as masters. A master (also called *primary*, *varied* or *parent*) field is one that is subjected to the δ-variation process of the calculus of variations. Fields that are not masters, *i.e.* not subject to variation, are called *slave*, *secondary* or *derived*. The *owner* (also called *parent* or *source*) of a slave field is the master from which it comes from.

If only one master field is chosen, the resulting variational principle (obtained after going through Steps 2, 3 and 4) is called *single-field*, and *multifield* otherwise.

A known or data field (for example: body forces or surface tractions in elastostatics) cannot be a master field because it is not subject to variation, and is not a secondary field because it does not derive from others. Hence we see that *fields can only be of three types*: master, slave, or data.

§5.8.2. Step 2: Choose Weak Connections

Given a master field, consider the equations that link it to other known and unknown fields. These are called the *connections* of that field. Classify these connections into two types:

Strong connection. The connecting relation is enforced *point by point* in its original form. For example if the connection is a PDE or an algebraic equation we use it as such. Also called *a priori* enforcement. When applied to a boundary condition, a strong connection is also referred to as an *essential constraint* or *essential B.C.*

Weak connection. The connection relationship is enforced only in an *average* or *mean* sense through the use of a weight or test function, or of a distributed Lagrange multiplier. Also called *a-posteriori* enforcement. When applied to a boundary condition, a weak connection is also referred to as a *natural constraint* or *natural B.C.*

A general rule to keep in mind is that *a slave field must be reachable from its owner through strong connections*.

If there is more than one master field (*i.e.* we are constructing a multifield principle), the foregoing definitions *must be applied to each master field in turn*. In other words, we must consider the connections that "emanate" from each of the master fields. The end result is that the same field may appear more than once. For example in elasticity the strain field **e** may appear up to three times: (1) as a master field, (2) as a slave field derived from displacements, and (3) as a slave field derived from stresses. These complications cannot occur with single-field principles.

Weak form to variational form

§5.8.3. Step 3: Construct a First Variation

Once all choices of Steps 1 and 2 have been made, the remaining manipulations are technical in nature, and essentially consist of applying the tools and techniques of vector, tensor and variational calculus: Lagrange multipliers, integration by parts, homogenization of variations, surface integral splitting, and so on. Since the number of operational combinations is huge, the techniques are best illustrated through specific examples.

The end result of these gyrations should be a variational statement

$$\boxed{\delta\Pi = 0,} \tag{5.34}$$

where the symbol δ here embodies variations with respect to *all master fields*.

Weak form to variational form

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Once all choices of Steps 1 and 2 have been made, the remaining manipulations are technical in nature, and essentially consist of applying the tools and techniques of vector, tensor and variational calculus: Lagrange multipliers, integration by parts, homogenization of variations, surface integral splitting, and so on. Since the number of operational combinations is huge, the techniques are best illustrated through specific examples.

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§5.8.4. Step 4: Functionalize

With luck, the variational statement (5.34) will be recognized as the *exact variation* of a functional Π, whence the variational statement becomes a true variational principle. If so, Π represents the Variational Form we were looking for, and the search is successful.

We now illustrate the foregoing steps with the detailed derivation of the most important single-field VF in elastostatics: the principle of Total Potential Energy or TPE.

Weak form to variational form

Here, we will use Lagrange multipliers to get the variational form… and we will guess them… to avoid mathematical troubles. We will also assume that we are **discretising displacement field**. So we use equations without displacements!

$$\sigma_{ij,j} + b_i = \frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \quad \text{in} \;\; V.$$

$$\int_V (\sigma^u_{ij,j} + b_i)\, \lambda_i \, dV = 0.$$

λ_i piecewise differentiable Lagrange multiplier vector field

Weak form

$$\int_V (\sigma^u_{ij,j} + b_i)\,\lambda_i\,dV = 0.$$

Apply the divergence theorem to the first term

$$\int_V \sigma^u_{ij,j}\,\lambda_i\,dV = -\int_V \sigma^u_{ij}\lambda_{i,j}\,dV + \int_S \sigma^u_{ij}\,n_j\lambda_i\,dS.$$

For a symmetric stress tensor $\sigma^u_{ij} = \sigma^u_{ji}$ this formula may be transformed[8] to

$$\int_V \sigma^u_{ij,j}\lambda_i\,dV = -\int_V \sigma^u_{ij}\,\tfrac{1}{2}(\lambda_{i,j} + \lambda_{j,i})\,dV + \int_S \sigma^u_{ij}\,n_j\lambda_i\,dS.$$

λ_i piecewise differentiable Lagrange multiplier vector field

Weak form

… and we will guess them… what are those Lagrange multipliers ???

$$\int_V \sigma^u_{ij,j} \lambda_i \, dV = -\int_V \sigma^u_{ij} \, \tfrac{1}{2}(\lambda_{i,j} + \lambda_{j,i}) \, dV + \int_S \sigma^u_{ij} \, n_j \lambda_i \, dS.$$

λ_i piecewise differentiable Lagrange multiplier vector field

Weak form

… yep, they look like displacements…

$$\int_V \sigma^u_{ij,j} \lambda_i \, dV = -\int_V \sigma^u_{ij} \, \tfrac{1}{2}(\lambda_{i,j} + \lambda_{j,i}) \, dV + \int_S \sigma^u_{ij} \, n_j \lambda_i \, dS.$$

$$\int_V \sigma^u_{ij,j} \, \delta u_i \, dV = -\int_V \sigma^u_{ij} \; \delta e^u_{ij} \, dV + \int_S \sigma^u_{ij} \, n_j \, \delta u_i \, dS,$$

$$\int_V \sigma^u_{ij} \, \delta e^u_{ij} \, dV - \int_V b_i \, \delta u_i \, dV - \int_S \sigma^u_{ij} \, n_j \, \delta u_i \, dS = 0.$$

λ_i piecewise differentiable Lagrange multiplier vector field

Variational form

$$\int_V \sigma_{ij}^u \, \delta e_{ij}^u \, dV - \int_V b_i \, \delta u_i \, dV - \int_S \sigma_{ij}^u \, n_j \, \delta u_i \, dS = 0.$$

$$\int_S \sigma_{ij}^u \, n_j \, \delta u_i \, dS = \int_{S_t} \sigma_{ij}^u \, n_j \, \delta u_i \, dS + \int_{S_u} \sigma_{ij}^u \, n_j \, \cancelto{0}{\delta u_i} \, dS = \int_{S_t} \sigma_{ij}^u \, n_j \, \delta u_i \, dS.$$

$$\int_{S_t} (\sigma_{ij}^u n_j - \hat{t}_i) \, \delta u_i \, dS = 0, \quad \text{whence} \quad \int_{S_t} \sigma_{ij}^u n_j \, \delta u_i \, dS = \int_{S_t} \hat{t}_i \, \delta u_i \, dS.$$

$$\delta \Pi_{\mathrm{TPE}} = \int_V \sigma_{ij}^u \; \delta e_{ij}^u \, dV - \int_V b_i \, \delta u_i \, dV - \int_{S_t} \hat{t}_i \; \delta u_i \, dS = 0.$$

$$\Pi_{\mathrm{TPE}}[u_i] = \frac{1}{2} \int_V \sigma_{ij}^u \, e_{ij}^u \, dV - \int_V b_i \, u_i \, dV - \int_{S_t} \hat{t}_i \, u_i \, dS.$$

Poisson equation

Poisson equation…

u – not displacements ! can be almost any variable…

$$\nabla \cdot (\rho \nabla u) = s,$$ Generalised Poisson eq.

Constant ρ: $$\rho \nabla^2 u = s.$$ Poisson equation

No source term: $$\nabla^2 u = 0.$$ Laplace equation

It describes:

- heat transfer (u=T)
- steady potential flow (e.g. water flow)
- electrostatics
- magnetostatics

Solutions of Laplace equation are harmonic functions…

Poisson equation…

$$\nabla \cdot (\rho \nabla u) = s,$$

$$\frac{\partial}{\partial x_1}\left(\rho \frac{\partial u}{\partial x_1}\right) = s,$$

$$\frac{\partial}{\partial x_1}\left(\rho \frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\rho \frac{\partial u}{\partial x_2}\right) = s,$$

$$\frac{\partial}{\partial x_1}\left(\rho \frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\rho \frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\rho \frac{\partial u}{\partial x_3}\right) = s.$$

Constant ρ:

$$\rho \frac{\partial^2 u}{\partial x_1^2} = s, \qquad \rho \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = s, \qquad \rho \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) = s.$$

Steady state linear heat conduction

Isotropic body of volume V, Temperature: **primal variable** (like displacements in elasticity)

Thermal equilibrium $T=T(x_i)$=const.

We have thermal gradient: $\mathbf{g} = \nabla T,$ $\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} \partial T/\partial x_1 \\ \partial T/\partial x_2 \\ \partial T/\partial x_3 \end{bmatrix}$

Fourier law of heat conduction $\mathbf{q} = -k\mathbf{g} = -k\nabla T.$ $\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = -k \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}.$

ρ becomes thermal conductivity coefficient k

s – source – becomes heat production $h = h(x_i)$

Steady state linear heat conduction

Balance equation: $\mathbf{div}\,\mathbf{q} + h = 0.$ $\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_2}{\partial x_3} + h = 0.$

s – source – becomes heat production h = h (x_i)

Thermal equilibrium T=T(x_i)=const.

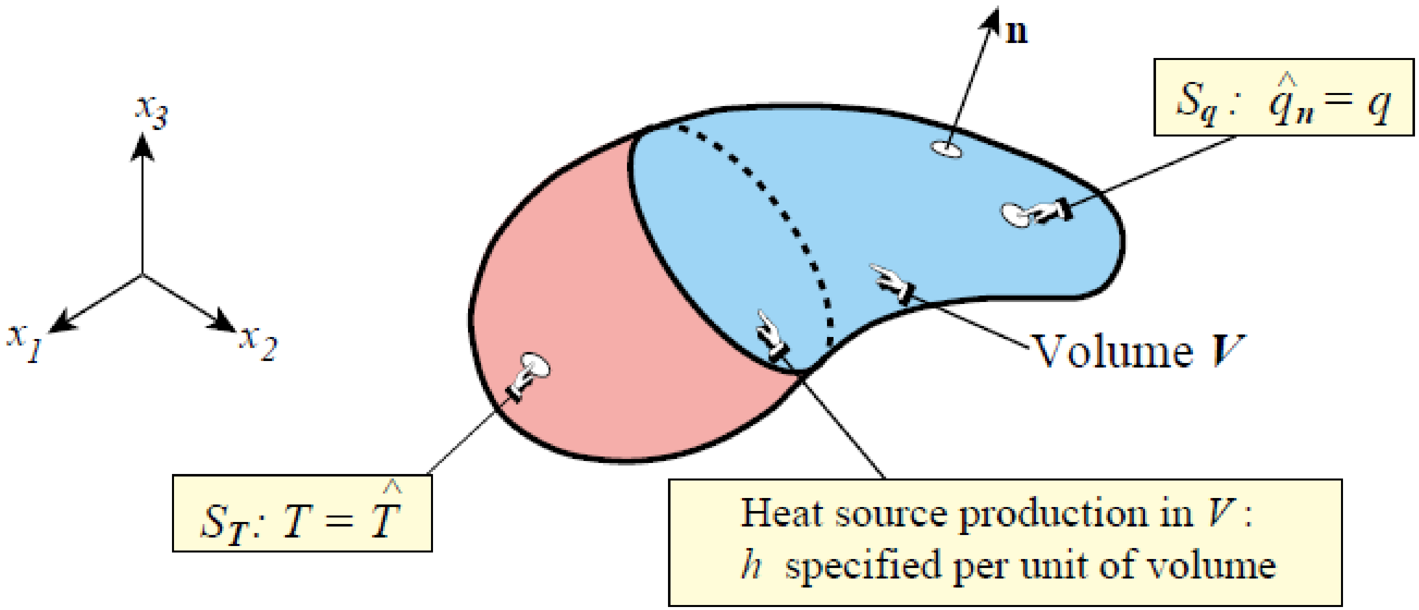
We have thermal gradient: $\mathbf{g} = \nabla T,$ $\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} \partial T/\partial x_1 \\ \partial T/\partial x_2 \\ \partial T/\partial x_3 \end{bmatrix}$

Fourier law of heat conduction $\mathbf{q} = -k\mathbf{g} = -k\nabla T.$ $\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = -k \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}.$

ρ becomes thermal conductivity coefficient k

Steady state linear heat conduction

Boundary conditions: prescribed temperature $\hat{T}$ and prescribed flux q_n



Steady state linear heat conduction

Summary of equations over volume:

$$\mathbf{div}\,\mathbf{q} + h = 0. \qquad \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_2}{\partial x_3} + h = 0.$$

$$\mathbf{g} = \nabla T, \qquad \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} \partial T/\partial x_1 \\ \partial T/\partial x_2 \\ \partial T/\partial x_3 \end{bmatrix}$$

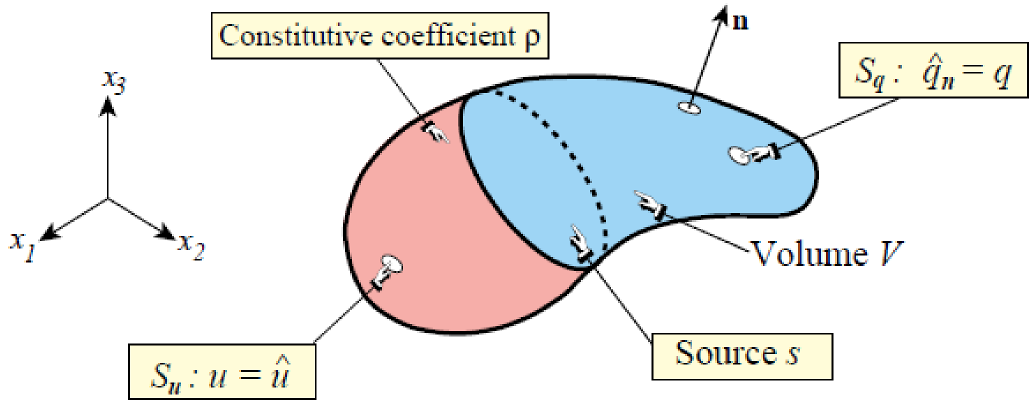
$$\mathbf{q} = -k\mathbf{g} = -k\nabla T. \qquad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = -k \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}.$$

And boundary conditions:

$$T = \hat{T} \qquad \text{on } S_T,$$

$$\mathbf{q}^T\mathbf{n} = q_n = \hat{q}_n \qquad \text{on } S_q.$$

Generally: steady state Poisson equation



$$\nabla u = \mathbf{g} \quad \text{in } V,$$
$$\rho \mathbf{g} = \mathbf{q} \quad \text{in } V,$$
$$\nabla \cdot \mathbf{q} = s \quad \text{in } V.$$

$$u = \hat{u} \quad \text{on } S_u,$$
$$\mathbf{q}.\mathbf{n} = q_n = \hat{q}_n, \quad \text{on } S_q.$$

Steady state Poisson equation
Derivation of functional form

Weighted residual: we take u as primary variable. Multiply the **other** equations by weighted residual and integrate over domain. Note it is zero after integration:

$$\nabla \cdot \mathbf{q} = s \qquad \text{in } V. \qquad\qquad R_{BE} = \int_V (\nabla \cdot \mathbf{q} - s) w_{BE} \, dV = 0,$$

$$\mathbf{q}.\mathbf{n} = q_n = \hat{q}_n, \qquad\qquad R_{FBC} = \int_{S_q} (\mathbf{q} \cdot \mathbf{n} - \hat{q}) \, w_{FBC} \, dS = 0.$$

Does not get us far…

trying to find a black cat in a dark cellar at midnight

Steady state Poisson equation

Derivation of functional form

Well, we just introduce into equations what we know:

$$\mathbf{g}^u = \nabla u, \quad \mathbf{q}^u = \rho \mathbf{g}^u = \rho \nabla u.$$

$$R_{BE} = \int_V (\nabla \cdot \mathbf{q}^u - s) w_{BE} \, dV = \int_V (\nabla \cdot \rho \nabla u - s) w_{BE} \, dV$$

$$R_{FBC} = \int_{S_q} (\mathbf{q}^u \cdot \mathbf{n} - \hat{q}) \, w_{FBC} \, dS = \int_{S_q} (\rho \nabla u) \cdot \mathbf{n} - \hat{q}) \, w_{FBC} \, dS.$$

Here we have **weak form:** same gradient of u unknown in both equations… useful – we can do something if we choose weights…

But we can do better!

Steady state Poisson equation

Derivation of functional form

Replace weights by variations of the primary variable – any functions which are allowed…

$$\delta \Pi_{BE} = \int_V \left(-\nabla \cdot \rho \nabla u + s \right) \delta u \, dV,$$

$$\delta \Pi_{FBC} = \int_{S_q} \left(\rho \nabla u \cdot \mathbf{n} - \hat{q} \right) \delta u \, dS.$$

Also, let's change residuals – which are zero, to variations of the functional we are trying to find…

Steady state Poisson equation

$$\delta\Pi_{BE} = \int_V \left(-\nabla \cdot \rho\nabla u + s\right)\delta u \; dV,$$

Divergence theorem:

$$-\int_V \nabla \cdot (\rho\nabla u)\;\delta u\, dV = \int_V \rho\nabla u \cdot \delta\nabla u \; dV - \int_{S_q} \rho\,\nabla u \cdot \mathbf{n}\,\delta u \; dS.$$

Leading to:

$$\delta\Pi_{BE} = \int_V \left(\rho\nabla u \cdot \delta\,\nabla u + s\,\delta u\right) dV - \int_{S_q} \rho\,\nabla u \cdot \mathbf{n}\;\delta u \; dS.$$

Adding:

$$\delta\Pi = \delta\Pi_{BE} + \delta\Pi_{FBC} = \int_V \left(\rho\nabla u \cdot \delta\,\nabla u + s\,\delta u\right) dV - \int_{S_q} \hat{q}\,\delta u\, dS$$

$$= \delta\int_V \tfrac{1}{2}\rho\,\nabla u \cdot \nabla u \; dV + \delta\int_V s\,u\,dV - \delta\int_{S_q} \hat{q}\,u\,dS.$$

variation symbol δ can be then pulled in front of the integrals:

Steady state Poisson equation

Functional:

$$\Pi_{\mathrm{TPE}}[u] = \frac{1}{2}\int_V \rho \nabla u \cdot \nabla u \, dV + \int_V s u \, dV - \int_{S_q} \hat{q} u \, dS$$

$$= \frac{1}{2}\int_V (\mathbf{q}^u)^T \mathbf{g}^u \, dV + \int_V s\, u \, dV - \int_{S_q} \hat{q} u \, dS$$

$$= \frac{1}{2}\int_V \rho \left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2 \right] dV + \int_V s\, u \, dV - \int_{S_q} \hat{q} u \, dS.$$

The variational principle is

$$\boxed{\delta \Pi_{\mathrm{TPE}} = 0.}$$

$$\delta \Pi = \delta \Pi_{BE} + \delta \Pi_{FBC} = \int_V (\rho \nabla u \cdot \delta \nabla u + s\, \delta u) \, dV - \int_{S_q} \hat{q}\, \delta u \, dS$$

$$= \delta \int_V \frac{1}{2} \rho \nabla u \cdot \nabla u \, dV + \delta \int_V s\, u \, dV - \delta \int_{S_q} \hat{q}\, u \, dS.$$

Steady state Poisson equation

Why variation of functional being zero is so great?

We always can do the same what we did before: **discretise the field with shape functions**, get element matrices linking external and internal integrals and get the solution. **It is that simple**…

The variational principle is

$$\boxed{\delta \Pi_{\mathrm{TPE}} = 0.}$$

Thank you