ELEC-E8116 Model-based control systems /exercises and solutions 8

1. Consider a simple integrator:

$$\dot{x}(t) = u(t)$$

Find an optimal control law that minimises a cost-function

$$J = \int_0^1 \left(x^2(t) + u^2(t) \right) dt$$

Further, consider the case, when the optimization horizon is infinite.

Solution: We can always define

$$J_1 = \frac{1}{2} J = \frac{1}{2} \int_0^1 \left(x^2(t) + u^2(t) \right) dt$$

without changing the solution (only the cost changes). Of course, the original cost could also be written as

$$J = \frac{1}{2} \int_0^1 \left(2x^2(t) + 2u^2(t) \right) dt$$

and continue from there. However, the first alternative is now used:

The Riccati equation:

$$-\dot{\mathbf{S}}(t) = \mathbf{A}^T \mathbf{S}(t) + \mathbf{S}(t)\mathbf{A}^T - \mathbf{S}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \mathbf{S}(t) + \mathbf{Q}, \quad \mathbf{S}(1) = 0$$

and the optimal control law is:

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}^T \mathbf{S}(t)\mathbf{x}^*(t)$$

Now for the given process we have:

$$A = 0; B = 1; R = 1; Q = 1$$

and

$$\dot{S}(t) = S^2(t) - 1,$$

which is a differential equation that should be solved with respect to time, Hence, by separating the variables

$$\frac{dS}{dt} = S^2 - 1 \Leftrightarrow \int \frac{1}{S^2-1} dS = \int dt$$
$$\Rightarrow \int \frac{1}{S^2-1} dS = t + C_1$$
$$\Rightarrow \int \frac{1}{S-1} dS - \int \frac{1}{S+1} dS = 2t + 2C_1$$
$$\Rightarrow \ln\left(\left|\frac{S-1}{S+1}\right|\right) = 2t + 2C_1$$
$$\Rightarrow \left|\frac{S-1}{S+1}\right| = \left|e^{2C_1}e^{2t}\right| \Leftrightarrow \frac{S-1}{S+1} = \pm e^{2C_1}e^{2t} = Ce^{2t}$$
$$\Rightarrow S(t) = \frac{1+Ce^{2t}}{1-Ce^{2t}}$$

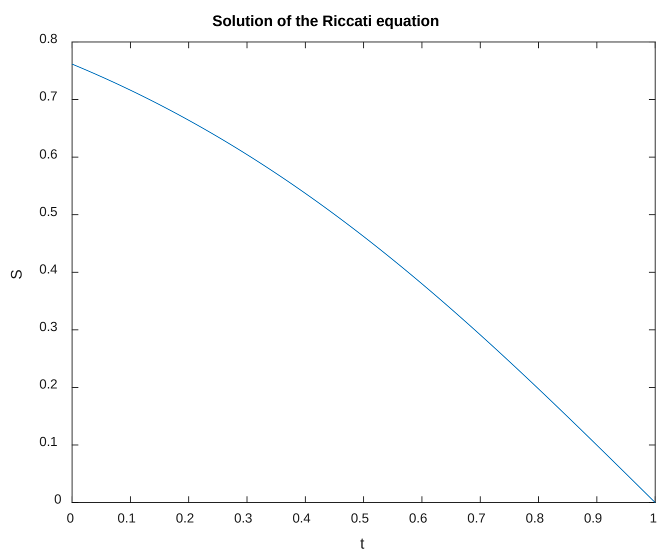
Now solve the unknown parameter C by using the fact $S(t_f) = S(1) = 0$, giving:

$$1 + Ce^2 = 0 \Leftrightarrow C = -e^{-2}$$

and

$$S(t) = \frac{1-e^{2(t-1)}}{1+e^{2(t-1)}}$$

Now the optimal control law is: $u^*(t) = -S(t)x(t)$



If the optimization horizon were infinite, the solution of the Riccati equation would be

$S(t) = \frac{1-e^{2(t-t_f)}}{1+e^{2(t-t_f)}} = \frac{e^{-2t}-e^{-2t_f}}{e^{-2t}+e^{-2t_f}} \to 1$ as $t_f \to \infty$. The same constant solution could have been obtained directly from $\dot{S}(t) = S^2(t) - 1$ by setting the derivative zero and taking the positive (positive definite) root of *S*.

2. Consider the 1. order process $G(s) = \dfrac{1}{s-a}$, which has a realization

$$\begin{aligned}\dot{x}(t) &= ax(t) + u(t)\\ y(t) &= x(t)\end{aligned}$$

so that the state is the measured variable. It is desired to find the control, which minimizes the criterion

$$J = \frac{1}{2}\int_0^\infty \left(x^2 + Ru^2\right)dt \quad (R > 0)$$

Calculate the control and investigate the properties of the resulting closed-loop system.

Solution:
The algebraic Riccati equation is

$$aX + Xa - XR^{-1}X + 1 = 0 \quad \Rightarrow \quad X^2 - 2aRX - R = 0$$

The solution must be positive semidefinite $X \geq 0$ so that

$$X = aR + \sqrt{(aR)^2 + R}$$

The optimal control law is thus

$$u = -K_r x \qquad \text{in which} \qquad K_r = X/R = a + \sqrt{a^2 + 1/R}$$

The closed-loop system is

$$\dot{x} = ax + u = -\sqrt{a^2 + 1/R}\; x$$

which has a pole at

$$s = -\sqrt{a^2 + 1/R} < 0$$

The root locus of this pole starts from $s = -|a|$ when $R = \infty$ (control has an infinite weight) and moves towards -∞, when *R* approaches zero. Note that the root locus is exactly the same in the stable (a < 0) process case as well as in the unstable (a > 0) case.

It is easily seen that for a small *R* the gain crossover frequency of the open loop transfer function

$$L = GK_r = K_r /(s-a)$$

is approximately

$$\omega_c \approx \sqrt{1/R}$$

and the gain drops 20 dB / decade in high frequencies, which is a general property of *LQ*-controllers. Moreover, the Nyquist curve does not in any frequency go inside the unit circle with the center at (-1,0). This means that

$$|S(i\omega)| = 1/|1+L(i\omega)| \le 1$$

for all frequencies. (Explanation: setting $L = x + iy$ gives

$$|S| = \frac{1}{|1+x+iy|} = \frac{1}{\sqrt{(1+x)^2+y^2}}$$

so that $|S| = 1$ in the circumference of the unit circle centered at (-1,0). Inside the circle $|S| > 1$ and outside $|S| < 1$.)

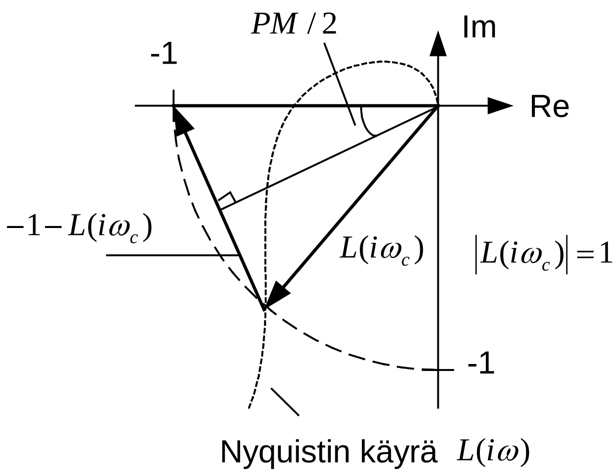
This property is clear for the stable process (a < 0), because K_r > 0 and the phase of $L(i\omega)$ changes from zero degrees (at the angular frequency 0) to –90 degrees (at the infinite angular frequency). It is remarkable that the property holds also in the case of unstable processes (a > 0), even though the phase of $L(i\omega)$ varies between -180°, -90°

3. Consider a SISO-system. The maximum values of the sensitivity and complementary functions are denoted M_S and M_T, respectively. Let the gain and phase margins of a closed-loop system be *GM* (gain margin) and *PM* (phase margin). Prove that

$$GM \geq \frac{M_S}{M_S - 1} \qquad PM \geq 2\arcsin\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S} \text{ [rad]}$$

$$GM \geq 1 + \frac{1}{M_T} \qquad PM \geq 2\arcsin\left(\frac{1}{2M_T}\right) \geq \frac{1}{M_T} \text{ [rad]}$$

Solution:
Start from the figure below, where the Nyquist diagram of the loop transfer function (*L*) has been presented.



Denote the phase crossover frequency by ω_{180} (then the phase of L is –180 degrees). By the definition of the gain margin

$$GM = \frac{1}{|L(i\omega_{180})|} \quad \Rightarrow \quad L(i\omega_{180}) = \frac{-1}{GM}$$

We obtain

$$T(i\omega_{180}) = \frac{L(i\omega_{180})}{1 + L(i\omega_{180})} = \frac{-1}{GM - 1}$$

$$S(i\omega_{180}) = \frac{1}{1 + L(i\omega_{180})} = \frac{1}{1 - \frac{1}{GM}}$$

Now use the abbreviations $M_T = \max_{\omega} |T(i\omega)|$, $\qquad M_S = \max_{\omega} |S(i\omega)|$

and it follows that

$$M_T \geq \frac{1}{|GM-1|}; \qquad M_S \geq \frac{1}{\left|1-\frac{1}{GM}\right|}$$

and the gain margin inequalities given in the problem follow easily. Let us calculate the first as an example.

$$M_T \geq \frac{1}{|GM-1|} \Rightarrow |GM-1| \geq \frac{1}{M_T} \Rightarrow GM-1 \geq \frac{1}{M_T} \Rightarrow GM \geq 1+\frac{1}{M_T}$$

The inequality related to M_S is derived correspondingly.

Considering the phase margin note that

$$|S(i\omega_c)| = \frac{1}{|1+L(i\omega_c)|} = \frac{1}{|-1-L(i\omega_c)|}$$

in which ω_c is the gain crossover frequency (the gain of *L* is one in this frequency). From the figure it can be seen that

$$|S(i\omega_c)| = |T(i\omega_c)| = \frac{1}{2\sin(PM/2)}$$

and the inequalities related to phase margin follow directly. (In the last form the following fact, obtained for example by the Taylor approximation, is used: when *x* is positive, arcsin(*x*) > *x*.)

The results show for example that if $M_T = 2$, then $GM \geq 1.5$, $PM \geq 29^\circ$.

Sometimes the maximum values (∞ - norms) M_S and M_T are used as alternatives to gain and phase margins. For example, demanding that $M_S < 2$, the often used ”rules of thumb” $GM > 2$, $PM > 30^\circ$ follow.