Statistical Mechanics
E0415

Fall 2019, lecture 7
Entropy

Take home… on snowflakes

"The problem argues that the six-fold symmetry of a snowflake results from the six-fold molecular crystal structure of ice. The growth of this snowflake with a six-fold symmetry is determined by its path through the clouds …The crystals experience an immense variety of different thermal and humidity variations, resulting in varying growing speeds and an immense variety of snowflake shapes."

“-- On the second site there's a photo of "identical-twin" snowflakes which were grown in a lab, but even in this controlled environment the snowflakes are still not precisely identical, reflecting the sensitivity of the process."

Why bother?

Says Sethna:

We shall see in this chapter that entropy has three related interpretations.[1] *Entropy measures the disorder in a system;* in Section 5.2 we will see this using the entropy of mixing and the residual entropy of glasses. *Entropy measures our ignorance about a system;* in Section 5.3 we will give examples from non-equilibrium systems and information theory. But we will start in Section 5.1 with the original interpretation, that grew out of the nineteenth century study of engines, refrigerators, and the end of the Universe. *Entropy measures the irreversible changes in a system.*

Irreversibility and the Carnot cycle

Four steps:

(ab): heat flow at T_1

(bc): expansion without heat transfer

(cd): gas compressed, heat flow at T_2

(da): compression, warm the gas back without heat transfer

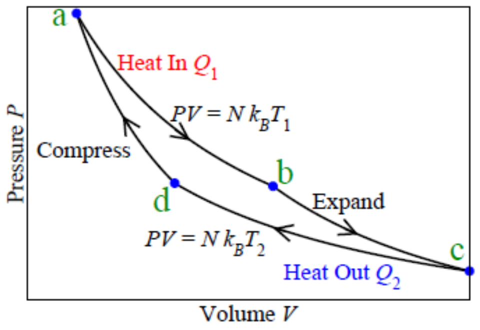
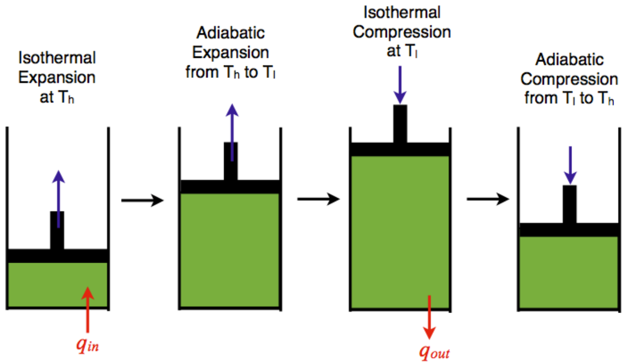


Fig. 5.3 Carnot cycle P–V diagram. The four steps in the Carnot cycle: a→b, heat in Q_1 at constant temperature T_1; b→c, expansion without heat flow; c→d, heat out Q_2 at constant temperature T_2; and d→a, compression without heat flow to the original volume and temperature.

Entropy, arrow of time (and irreversible heat machines)

$$\Delta S_{\text{thermo}} = \frac{Q}{T}.$$

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}.$$



Mixing entropy

Information and entropy via mixing: how much information there is in a configuration?

“Counting entropy”

Maxwell’s demon and entropy – and information.

$$S_{\text{unmixed}} = 2\,k_B \log[V^{N/2}/(N/2)!], \qquad S_{\text{mixed}} = 2k_B \log[(2V)^{N/2}/(N/2)!],$$

$$\Delta S_{\text{mixing}} = S_{\text{mixed}} - S_{\text{unmixed}} = k_B \log 2^N = N k_B \log 2.$$

$$S_{\text{counting}} = k_B \log(\text{number of configurations})$$

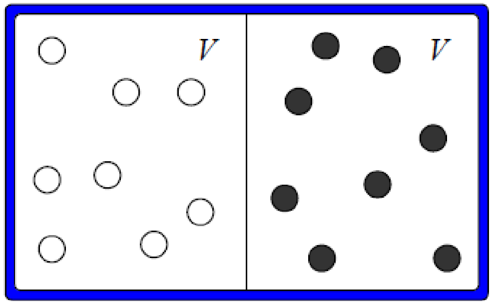


Fig. 5.4 Unmixed atoms. The premixed state: $N/2$ white atoms on one side, $N/2$ black atoms on the other.

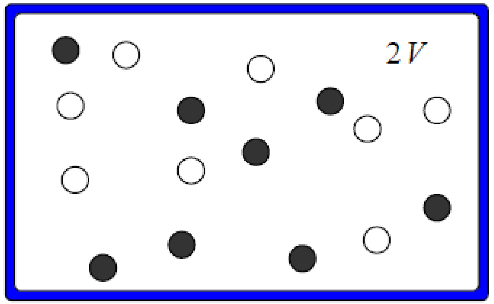


Fig. 5.5 Mixed atoms. The mixed state: $N/2$ white atoms and $N/2$ black atoms scattered through the volume $2V$.

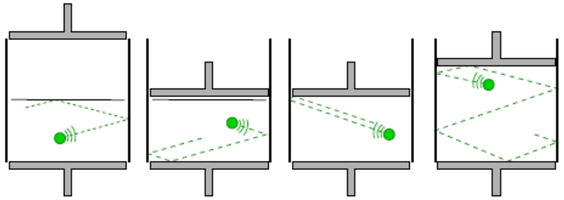
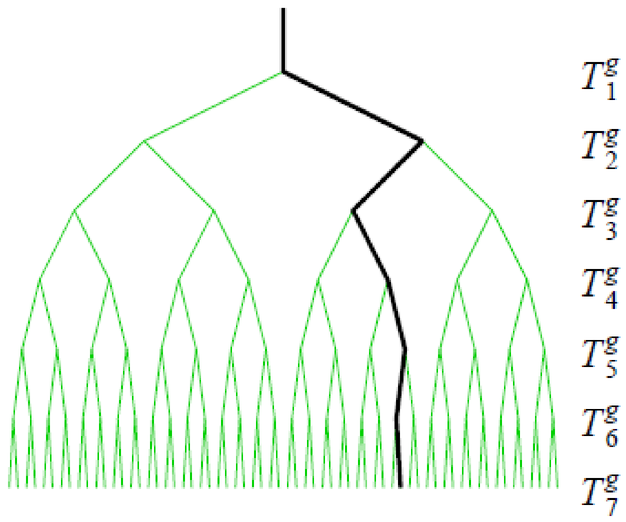


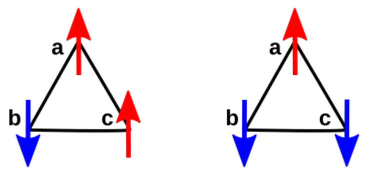
Fig. 5.11 Expanding piston. Extracting energy from a known bit is a three-step process: compress the empty half of the box, remove the partition, and retract the piston and extract $P\,dV$ work out of the ideal gas atom. (One may then restore the partition to return to an equivalent, but more ignorant, state.) In the process, one loses one bit of information (which side of the the partition is occupied).

Residual entropy of glasses

Argument: locally glasses are two-state systems (position of an atom in an amorphous system). Cool a glass from a liquid: freezing will lead to a random configuration, with a lot of frozen, metastable two-state configurations (extensive).

$$S_{\text{residual}} = S_{\text{liquid}}(T_\ell) - \int \frac{1}{T}\frac{\mathrm{d}Q}{\mathrm{d}t}\mathrm{d}t = S_{\text{liquid}}(T_\ell) - \int_0^{T_\ell} \frac{1}{T}\frac{\mathrm{d}Q}{\mathrm{d}T}\mathrm{d}T \quad ($$





Compare: Triangular Ising Antiferromagnet
(Residual T=0 entropy known)

Entropy: information, non-equilibrium

Various ways of considering the question, how random is a probability distribution (discrete, continuous, quantum statistical mechanics [density matrix – based], information theoretic).

$$S_{\mathrm{discrete}} = -k_B \langle \log p_i \rangle = -k_B \sum_i p_i \log p_i.$$

$$\begin{aligned} S_{\mathrm{nonequil}} &= -k_B \langle \log \rho \rangle = -k_B \int \rho \log \rho \\ &= -k_B \int_{E<\mathcal{H}(\mathbb{P},\mathbb{Q})<E+\delta E} \frac{\mathrm{d}\mathbb{P}\,\mathrm{d}\mathbb{Q}}{h^{3N}} \rho(\mathbb{P},\mathbb{Q}) \log \rho(\mathbb{P},\mathbb{Q}). \end{aligned}$$

$$S_{\mathrm{quantum}} = -k_B \mathrm{Tr}(\boldsymbol{\rho} \log \boldsymbol{\rho}).$$

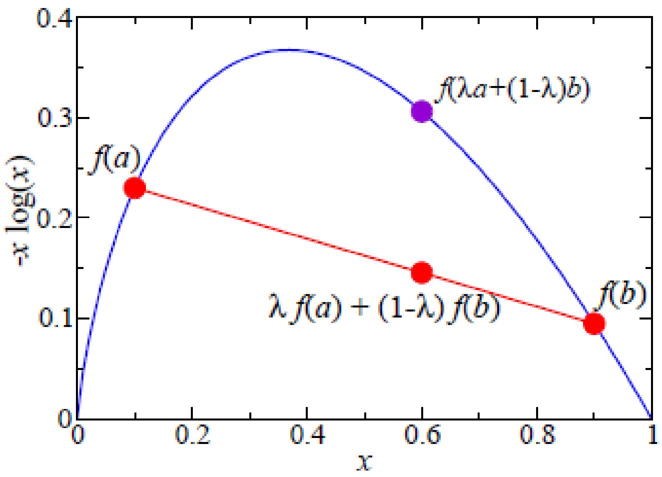
Shannon's entropy (base 2).

$$S_S = -k_S \sum_i p_i \log p_i = -\sum_i p_i \log_2 p_i,$$

Properties of entropy

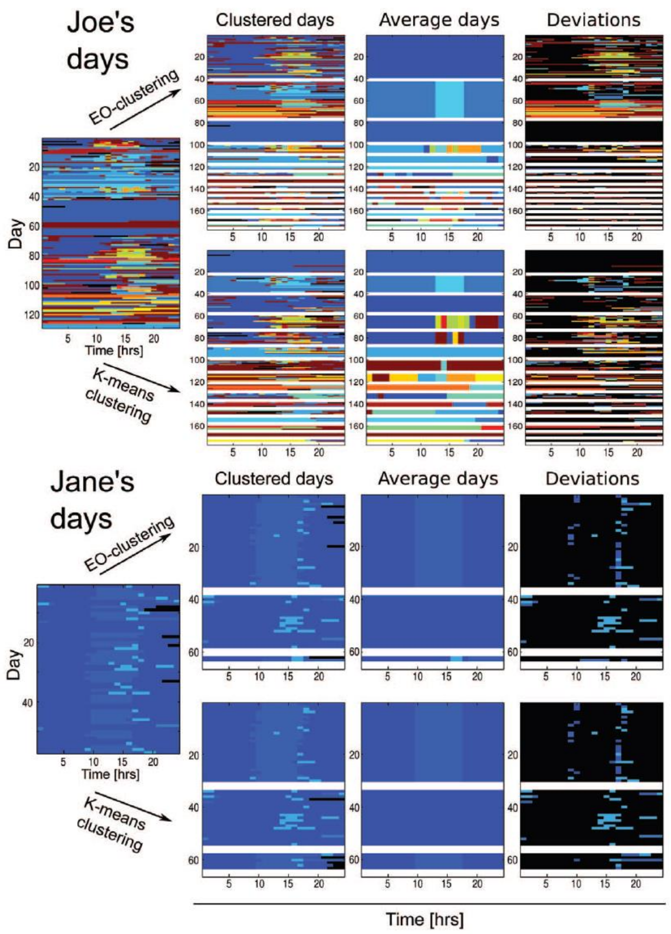
1) Maximum for equal probabilities.
2) Extra states with zero probability not important.
3) Entropy and conditional probabilities – ignorance is additive and entropy is extensive:

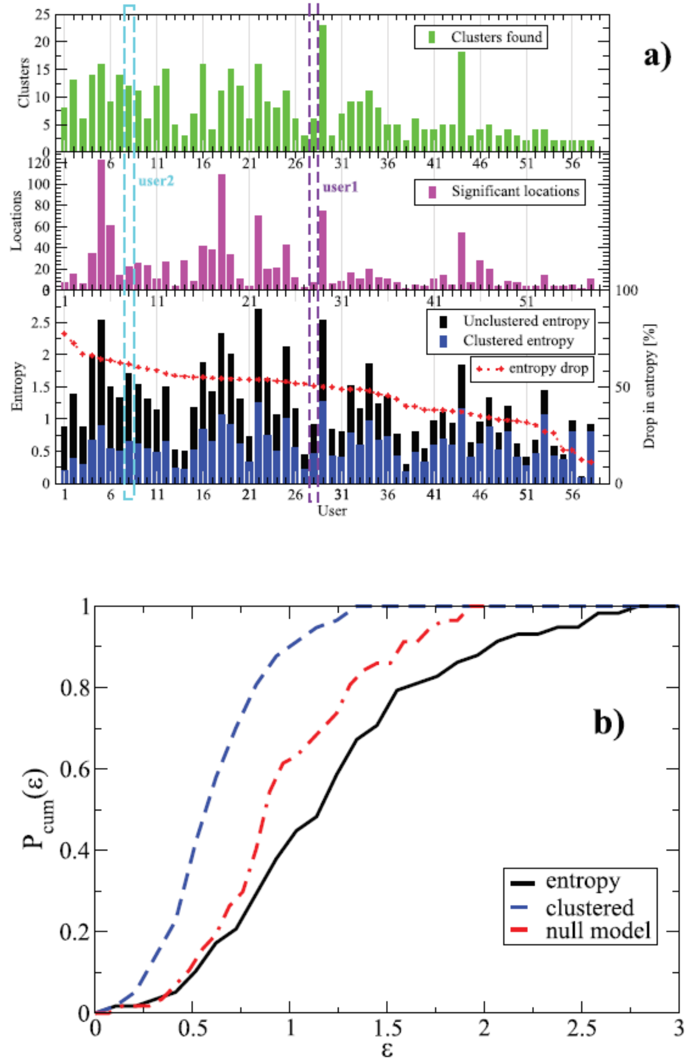
$$\langle S_I(A|B_\ell)\rangle_B = S_I(AB) - S_I(B).$$



Entropy is concave!

Applications: human life





Patterns, Entropy, and Predictability of Human Mobility and Life

Shao-Meng Qin[1], Hannu Verkasalo[2,3], Mikael Mohtaschemi[1], Tuomo Hartonen[1], Mikko Alava[1]*

Take home…

We now concentrate on Ch. 5 of Sethna (Entropy). The argument splits into three main points: role of entropy in classical thermodynamics, it as a measure of disorder, and finally entropy as a way to quantify information whether the system is in equilibrium or not. Check that you get the Carnot engine argument, and - referring to the last of these - what the (Shannon) entropy must have as its fundamental properties.

The take home quiz splits into two parts. We have two applications, one of which has to do with glasses (and their entropy) and the other one refers to the use of entropy outside of physics - brain science. Your task is now to pick one of these. After that, justify why you wanted that particular one, and read the article in question and summarize it with a few sentences. A target max length for your take home is 2+8 sentences.

And, the choice is between:

https://journals.aps.org/prresearch/abstract/10.1103/PhysRevResearch.2.013202 (glasses)

https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0089948 (brains and NMR)